Aplikace matematiky

Jan Ámos Víšek

On properties of binary random numbers

Aplikace matematiky, Vol. 19 (1974), No. 6, 375–385

Persistent URL: `http://dml.cz/dmlcz/103555`

Terms of use:

© Institute of Mathematics AS CR, 1974

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

ON PROPERTIES OF BINARY RANDOM NUMBERS

JAN ÁMOS VÍŠEK

(Received April 30, 1973)

1. This paper was stimulated by investigating properties of random numbers produced by physical generators (see [1]). The output of an ideal generator is a sequence of zeros and ones which can be viewed as independent relaizations of random variables (rv) taking on values 0 and 1 with equal probability $\frac{1}{2}$. Considered as a binary expansion of a number from the interval $[0, 1]$, such a sequence represents a rv uniformly distributed over $[0, 1]$.

A real physical generator, however, produces a sequence in which the probabilities of zeros and ones are only approximately equal and whose terms are only approximately independent. Moreover, a real generator produces finite expansions only. For these reasons, the produced random numbers are only approximately uniformly distributed. The purpose of this paper is to investigate theoretically the consequences of violation of just one postulate of an ideal generator; namely, the consequences of unequal probabilities $p \neq q$ of zero and one. Particularly, the behaviour of sums of such random numbers is studied, because by means of those sums normally distributed rvs are usually simulated.

Some results (Proposition (i)) are known; others were obtained by applying known general theorems to our situation (Propositions (vi) and (vii); also Proposition (ii) can be deduced from a known more general result, but here a simple direct proof is given); some other results are new, although not very difficult to prove by standard tools of probability theory.

2. Denote by $\mathcal{N}$ the set of all positive integers. Let $\{X_k\}_{k=1}^{\infty}$, $\{X_k^{(j)}\}_{k=1}^{\infty}$, $j \in \mathcal{N}$ and $\{X_k^{(m,n)}\}_{k=1}^{\infty}$, $1 \leqq m \leqq n$, $n \in \mathcal{N}$, be sequences of independent zero-one rvs with $P(X_k = 1) = p$, $P(X_k^{(j)} = 1) = p$; $P(X_k^{m,n} = 1) = p_n$ for all $k \in \mathcal{N}$; $j \in \mathcal{N}$, $1 \leqq m, \leqq$ $\leqq n$, $n \in \mathcal{N}$ and for some $0 < p < 1$, $0 < p_n < 1$; denote $\Delta = p - \frac{1}{2}$, $\Delta_n = p_n - \frac{1}{2}$. Put $Y = \sum_{k=1}^{\infty} X_k \,.\, 2^{-k}$; $Y_j = \sum_{k=1}^{\infty} X_k^{(j)} \,.\, 2^{-k}$, $Y_{mn} = \sum_{k=1}^{\infty} X_k^{(m,n)} \,.\, 2^{-k}$, $S_n = (\sum_{j=1}^{n} Y_j - n \mathsf{E} Y_1) :$ $: \sqrt{(n \operatorname{var} Y_1)}$ and $S_{nn} = (\sum_{m=1}^{n} Y_{mn} - n \mathsf{E} Y_{1n}) / \sqrt{(n \operatorname{var} Y_{1n})}$. Let $G_n(x)$, $G_{nn}(x)$ be distribu-

tion functions (df) of S_n and S_{nn} respectively. Denote by Φ the normal df with zero mean and unit variance. A df is said to be singular if it is continuuous and if the corresponding probability distribution is concentrated on a set of Lebesgue measure zero.

For rvs defined above, the following propositions hold true:

(i) *Y is uniformly distributed over* $[0; 1]$ *if* $p = \frac{1}{2}$.

(ii) *The df of Y is singular and increasing on* $[0, 1]$ *if* $p \neq \frac{1}{2}$.

(iii) *Y has the following central moments and cumulants*:

$$\mu_1 = \varkappa_1 = p\,,$$
$$\mu_2 = \varkappa_2 = \tfrac{1}{3}p(1 - p)\,,$$
$$\mu_3 = \varkappa_3 = \tfrac{1}{7}\,p(1 - 3p + 2p^2)\,,$$
$$\mu_4 = \tfrac{1}{15}\,p(1 - 2p + 2p^2 - p^3)\,,$$
$$\varkappa_4 = \tfrac{1}{15}\,p(1 - 7p + 12p^2 - 6p^3)\,,$$
$$\mu_5 = \tfrac{1}{651}\,p(21 - 625p + 2290p^2 - 2810p^3 + 1124p^4)\,,$$
$$\varkappa_5 = \tfrac{1}{31}\,p(1 - 15p + 50p^2 - 60p^3 + 24p^4)\,,$$
$$\mu_6 = \tfrac{1}{441}\,p(7 + 20p - 211p^2 + 498p^3 - 471p^4 + 157p^5)\,,$$
$$\varkappa_6 = \tfrac{1}{63}p(1 - 31p + 180p^2 - 390p^3 + 360p^4 - 120p^5)\;;$$
$$\mu_1 = \varkappa_1 = \tfrac{1}{2} + \Delta\,,$$
$$\mu_2 = \varkappa_2 = \tfrac{1}{3}(\tfrac{1}{4} - \Delta^2)\,,$$
$$\mu_3 = \varkappa_3 = \tfrac{2}{7}\Delta(\tfrac{1}{4} - \Delta^2)\,,$$
$$\mu_4 = \tfrac{1}{15}(\tfrac{1}{4} - \Delta^2)(\tfrac{3}{4} + \Delta^2)\,,$$
$$\varkappa_4 = \tfrac{1}{15}(\tfrac{1}{4} - \Delta^2)(6\Delta^2 - \tfrac{1}{2})\,,$$
$$\mu_5 = \tfrac{1}{651}\Delta(\tfrac{1}{4} - \Delta^2)(116\Delta^2 - 71)\,,$$
$$\varkappa_5 = \tfrac{4}{31}\Delta(\tfrac{1}{4} - \Delta^2)(1 - 6\Delta^2)\,,$$
$$\mu_6 = \tfrac{1}{7056}(\tfrac{1}{4} - \Delta^2)(63 + 104\Delta^2 - 2512\Delta^4)\,,$$
$$\varkappa_6 = \tfrac{1}{63}(\tfrac{1}{4} - \Delta^2)(1 - 30\Delta^2 + 120\Delta^4)\,.$$

(iv) *In the case $p \neq \frac{1}{2}$ the distribution function of the random variable S_n is singular for all $n \geqq 1$.*

(v) *For the characteristic function of the random variable Y we have*

$$\limsup_{|t|\to\infty} |\varphi_Y(t)| < 1\,.$$

(vi) $G_n(x)$ *can be approximated as follows:*

$$G_n(x) = \Phi(x) + \exp\left(-\tfrac{1}{2}x^2\right)(2\pi)^{-1/2}\left[(1 - x^2)\frac{p(1 - 3p + 2p^2)}{42n^{1/2}}\,.\right.$$

$$\cdot\sqrt{\left(\frac{27}{p^3(1-p)^3}\right)} + x(3 - x^2)\frac{1 - 7p + 12p^2 - 6p^3}{40p(1-p)^2\,n} +$$

$$\left. + (x^5 - 10x^3 + 15x)\frac{3(1 - 3p + 2p^2)^2}{392p(1-p)^3\,n}\right] + O(n^{-3/2})$$

uniformly in x; *particularly, for* $p = \frac{1}{2}$

$$G_n(x) = \Phi(x) + \exp(-x^2/2)(2\pi)^{-1/2}\frac{x(x^2 - 3)}{20n} + O(n^{-3/2}).$$

(vii) *Let* $0 < a < p_n < b < 1$, $n \in \mathcal{N}$. *Then*

(1)
$$G_{nn}(x) = \Phi(x) +$$

$$+ \exp\left(-\frac{x^2}{2}\right)(2\pi)^{-1/2}\left[(15x^5 - 588x^3 - 30x^2 + 1689x - 30)\frac{1}{980n} +\right.$$

$$+ (\Delta + 6\Delta^3 + 30\Delta^5 + 140\Delta^7 + \ldots)(1 - x^2)\,.\sqrt{\tfrac{3}{14n}} +$$

$$+ (\Delta^2 + 4\Delta^4 + 16\Delta^6 + 67\Delta^8 + \ldots)(15x^5 - 588x^3 + 166x^2 + 1689x - 618)\,.$$

$$\left.\cdot\frac{1}{980n}\right] + O(n^{-3/2}),$$

where – after disclosing the parentheses – only those terms A_n *are preserved for which the relation*

(2)
$$\lim_{n\to\infty} |A_n|\, n^{3/2} = \infty$$

holds true.
Particularly, for $\Delta_n = dn^{-1/2}$ *we have*

$$G_n(x) = \Phi(x) + \exp(-x^2/2)(2\pi)^{-1/2}\,.$$

$$.\left[15x^5 - 588x^3 - 30x^2 + 1689x - 30)\frac{1}{980n} + \frac{d}{n}(1 - x^2)\sqrt{\frac{3}{14}}\right] + O(n^{-3/2})\,.$$

(viii) *Let* $\{h_n\}_{n=1}^{\infty}$ *be a sequence satisfying*

(3)
$$0 < \liminf_{n\to\infty} h_n n^s \leqq \limsup_{n\to\infty} h_n n^s < \infty$$

for some $s > 0$. Then we have

$$\lim_{n\to\infty} \frac{G_n(x + h_n) - G_n(x)}{h_n} = \frac{\exp(-x^2/2)}{\sqrt{2\pi}}$$

uniformly in $x \in [-N, N], N \in \mathcal{N}$.

(ix) *Let $\{h_n\}_{n=1}^{\infty}$ be a sequence satisfying* (3). *Let $r = [s]$, $r < s < r + 1$.*
Then we have

$$\frac{G_n(x + h_n) - G_n(x)}{h_n} = \frac{\exp(-x^2/2)}{\sqrt{2\pi}} + \sum_{v=1}^{2r} \frac{q_v(x)}{n^{v/2}} + o(n^{-r})$$

uniformly in $x \in [-N, N]$, $N \in \mathcal{N}$, where $q_v(x)$ are polynomials of degree $3v - 2$, defined by (12).

3. In this section, the proofs of proposition just stated will be given.

Propositions (i) and (ii). Let φ_{X_k} be the chf of X_k. Then

$$\varphi_{X_k}(t) = q + p \exp\{it\},$$

where $q = 1 - p$. Hence

$$\varphi_Y = \prod_{k=1}^{\infty} (q + p \exp\{it \,.\, 2^{-k}\}). \tag{4}$$

Now for $p = \frac{1}{2}$ we get

$$\varphi_Y = \prod_{h=1}^{\infty} \tfrac{1}{2}(1 + \exp\{it \,.\, 2^{-k}\}) =$$

$$= \prod_{k=1}^{\infty} \tfrac{1}{2}[\exp\{- it \,.\, 2^{-(k+1)}\} + \exp\{it \,.\, 2^{-(k+1)}\}] \exp\{it \,.\, 2^{-(k+1)}\} =$$

$$= \exp\{\tfrac{1}{2}it\} \prod_{k=1}^{\infty} \cos(t \,.\, 2^{-(k+1)}) \,.$$

Using the Viet formula

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos(x \,.\, 2^{-k})$$

we obtain

$$\varphi_Y = \frac{\sin t/2}{t/2} \exp\{i \,.\, t/2\} = \frac{\exp\{it\} - 1}{it} \,.$$

I.e., Y is uniformly distributed over $[0, 1]$.

Now for $p \neq \frac{1}{2}$ let us define new rvs $Z_n = \sum_{k=1}^{n} X_k \,.\, 2^{-k}$ and denote by $F_n(x)$ their dfs. It follows from the definition of Z_n that $F_n(x) = 0$ for $x \leqq 0$, $F_n(x) = 1$ for $x > 1$ and $F_n(x)$ is constant over every interval $(j \,.\, 2^{-n}, (j + 1)\, 2^{-n}]$, $0 \leqq j \leqq 2^n - 1$. As all rvs X_k are equally distributed and independent, the discontinuity jumps of $F_n(x)$ will be of size

$$p^{n-l}q^l \tag{5}$$

for some $l \in \{0, 1, \ldots, n\}$. Hence

$$\forall(\varepsilon > 0)\, \exists(n_0 \in \mathcal{N}), \quad \forall(n \geqq n_0, \quad n \in \mathcal{N}), \quad \forall(j \in \{0, 1, \ldots, 2^n - 1\}):$$

$$F_n(j \,.\, 2^{-n} + 0) - F_n(j \,.\, 2^{-n}) < \varepsilon,$$

As

$$F_n(j \,.\, 2^{-n} + 0) = F_n((j + 1) \,.\, 2^{-n}),$$

we have also

$$F_n((j + 1) \,.\, 2^{-n}) - F_n(j \,.\, 2^{-n}) < \varepsilon. \tag{6}$$

Notice that

$$\forall(n \in \mathcal{N})\, \forall(m \geqq n, n \in \mathcal{N})\, \forall(j \in \{0, 1, 2, \ldots, 2^n - 1\})$$

$$F_m(j \,.\, 2^{-n}) = F_n(j, 2^{-n}). \tag{7}$$

As (7) holds we can define a function F^* on the set

$$A = \{x\colon\ x \in [0, 1],\ x = j \,.\, 2^{-n},\ n \in \mathcal{N},\ j \in \{0, 1, 2, \ldots, 2^n - 1\}\}$$

by the relation

$$F^*(j \,.\, 2^{-n}) = F_n(j \,.\, 2^{-n}).$$

F^* is continuous and increasing on A as follows from (5) and (6). Let us define a continuous function F on $[0, 1]$ such that $F(x) = F^*(x)$ on A. The sequence $\{F_n(x)\}_{n=1}^{\infty}$ converges uniformly to F on $[0, 1]$ as can be seen from (6). Thus we have proved that the sequence $\{Z_n\}_{n=1}^{\infty}$ converges in law. Consequently, F is the df of $Y = \sum_{k=1}^{\infty} X_k \,.\, 2^{-k}$. Denote by $m_n(t)$ the number of ones among the first n digits in the binary expansion of the number t. Obviously the binary expansion of the number X is $0, X_1 X_2 \ldots$. As all rvs X_k have the same distribution with expectation p, it follows from the strong law of large numbers that

$$P\left(\left\{\omega\colon \lim_{n \to \infty} \frac{m_n(X(\omega))}{n} = p\right\}\right) = 1$$

and hence

$$P_F\left(\left\{x\colon\ x \in [0, 1],\ \lim_{n \to \infty} \frac{m_n(x)}{n} = p\right\}\right) = 1.$$

On the other hand,

$$\lambda\left(\left\{x\colon\ x \in [0, 1],\ \lim_{n\to\infty} \frac{m_n(x)}{n} = \frac{1}{2}\right\}\right) = 1$$

(where λ is the Lebesgue measure), which proves that F is singular.

Proposition (iii). The central moments and cumulants were obtained by direct calculation from the chf of Y.

Proposition (iv). First, we shall quote a definition and a theorem (see [3]) that will be used in the proof.

Definition. *A rv X is said to have a distribution of pure type if either*

i) *there is a countable set D such that $P(X \in D) = 1$ or*

ii) *$P(X = x) = 0$ for every $x \in R$, but there is a set $D \in \mathscr{B}$ of Lebesgue measure zero such that $P(X \in D) = 1$, or*

iii) *$P(X \in \mathrm{d}x) \ll \lambda(\mathrm{d}x)$ (λ Lebesgue measure).*

Recall that $\mu \ll \nu$ denotes that μ is absolutely continuous with respect to ν; R denotes the real line and $\mathscr{B}$ the set of all Borel sets.

Theorem (Jessen-Wintner). Let $U_1, U_2, \ldots,$ be independent rvs such that

i) $\sum_{k=1}^{\infty} U_k \to U$ *a.s.,*

ii) *for each k, there is a countable set F_k such that $P(U_k \in F_k) = 1$. Then the distribution of U is of pure type.*

Now we shall show that Jessen-Wintner theorem applies to rvs S_n:

$$S_n = [(\sum_{j=1}^{n} Y_j) - n\,\mathsf{E}\,Y]\,[n \operatorname{var} Y]^{-1/2} =$$

$$= [(\sum_{j=1}^{n} \sum_{k=1}^{\infty} X_k^{(j)} \,.\, 2^{-k}) - n\,\mathsf{E}\,Y]\,[n \operatorname{var} Y]^{-1/2} =$$

$$= [(\sum_{k=1}^{\infty} \sum_{j=1}^{n} X_k^{(j)} \,.\, 2^{-k}) - n\,\mathsf{E}\,Y]\,[n \operatorname{var} Y]^{-1/2}\,.$$

Evidently, S_n satisfy both conditions i) and ii). Thus S_n is a rv of pure type. S_n cannot be a discrete rv as its df is the n-th convolution power of the df of $(Y_1 - \mathsf{E}\,Y_1) : : \sqrt{(n \operatorname{var} Y_1)}$ and thus continuous. We shall show that the df of S_n cannot be absolutely continuous, either. As the chf of $S_n\sqrt{(\operatorname{var} Y_1)} + n\,\mathsf{E}\,Y_1$ is

$$[\prod_{j=1}^{\infty} (q + p \exp\{it \,.\, 2^{-j}\})]^n$$

we must find that

$$\limsup_{|t|\to\infty} |[\prod_{j=1}^{\infty} (q + p \exp\{it \cdot 2^{-j}\})]^n| > 0$$

and then apply the Riemann-Lebesgue lemma. Let us estimate the absolute value of this chf at the point $2^k \cdot 2\pi$ for some k.

$$(8) \qquad |[\prod_{j=1}^{\infty} (q + p \exp\{it \cdot 2^{-j}\})]^n| =$$

$$= [\prod_{j=1}^{\infty} |q + p \exp\{it \cdot 2^{-j}\}|]^n .$$

For $\quad 1 \leqq j \leqq k, \quad q + p \exp\{it \cdot 2\pi \cdot 2^{-j}\} = 1,$

for $\quad j = k + 1, \quad q + p \exp\{it \cdot 2\pi \cdot 2^{-j}\} = q - p$

and the remainder of the product (8) will be

$$R = [\prod_{r=1}^{\infty} |q + p \exp\{\pi \cdot 2^{-r}\}|]^n .$$

Let us compare R with the absolute value of chf φ_n of a sum on n rvs distributed uniformly on $[0, 1]$, at the point π. We have

$$|\varphi_n(\pi)| = \left|\frac{\exp\{i\pi\} - 1}{i\pi}\right|^n = \left(\frac{2}{\pi}\right)^n$$

and at the same time

$$|\varphi_n(\pi)| = [\prod_{r=1}^{\infty} |\tfrac{1}{2} + \tfrac{1}{2}\exp\{i\pi \cdot 2^{-r}\}|]^n .$$

Let us compare first

$$(9) \qquad \prod_{r=1}^{\infty} |q + p \exp\{i\pi \cdot 2^{-r}\}|$$

and

$$(10) \qquad \prod_{r=1}^{\infty} |\tfrac{1}{2} + \tfrac{1}{2}\exp\{i\pi \cdot 2^{-r}\}),$$

the n-th factor in (9) being

$$a_n = |q + p \exp\{i\pi \cdot 2^{-n}\}| ,$$

the n-th factor in (10)

$$b_n = |\tfrac{1}{2} + \tfrac{1}{2}\exp\{i\pi \cdot 2^{-n}\}| .$$

We find easily that $a_n = \sqrt{(1 - 2pq(1 - \cos \pi \cdot 2^{-n}))}$. Let us minimize a_n with respect to p. The minimum is achieved at $p = \frac{1}{2}$ and is equal to b_n. Thus we obtain

$$a_n \geqq b_n ,$$

hence

$$R \geqq \left(\frac{2}{\pi}\right)^n$$

for every $k \in \mathcal{N}$. Finally we have

$$\limsup_{|t| \to \infty} |[\prod_{j=1}^{\infty} (q + p \exp\{it . 2^{-j}\}]^n| \neq 0 .$$

Proposition (v). Let us assume $\varepsilon \in (0, 2\pi)$. We shall show that $|\varphi(t)| < c_\varepsilon < 1$ for all $|t| \geqq \varepsilon$. Let $t > 0$, t fixed. Let us take the largest k_0 such that

$$t . 2^{-k_0} \geqq \frac{\varepsilon}{2} .$$

As $t . 2^{-(k_0+1)} < \frac{1}{2}\varepsilon$, we have $\frac{1}{2}\varepsilon \leqq t . 2^{-k_0} < \varepsilon$. As the k_0-th factor in product (8) has the form

$$a_{k_0} = \sqrt{(1 - 2pq(1 - \cos t . 2^{-k_0}))} ,$$

we have $a_{k_0} < c_\varepsilon$, where $c_\varepsilon = \sqrt{(1 - 2pq(1 - \cos \frac{1}{2}\varepsilon))} < 1$, because $(1 - \cos \frac{1}{2}\varepsilon) > 0$.

Proposition (vi). The following theorem is stated in (4). *Let $\{Z_i\}_{i=1}^{\infty}$ be a sequence of equally distributed rvs with df $V(x)$, $\mathsf{E}\, Z_i = 0$, $\mathsf{E}\, Z_i^2 = \sigma^2 > 0$. Let $v(t)$ be the chf of Z_1 and $H_n(z) = P(\{(\sigma \sqrt{n})^{-1} \sum_{i=1}^{n} Z_i < z\})$. If $\mathsf{E}|Z_i|^k < \infty$ for some $k \geqq 3$, then for all z and n*

$$\left|H_n(z) - \Phi(z) - \sum_{v=1}^{k-2} \frac{Q_v(z)}{n^{v/2}}\right| \leqq$$

$$\leqq c(k) \{\sigma^{-k} n^{-(k-2)/2}(1 + |z|)^{-k} \int_{|y| \geqq \sigma\sqrt{n}(1+|z|)} |y|^k \,\mathrm{d}V(y) +$$

$$+ \sigma^{-k-1} n^{-(k-1)/2}(1 + |z|)^{-k-1} \int_{|y| < \sigma\sqrt{n}(1+|z|)} |y|^{k+1} \,\mathrm{d}V(y) +$$

$$+ (\sup_{|t| \geqq \delta} |v(t)| + \tfrac{1}{2n})^n n^{k(k+1)/2}(1 + |z|)^{-k-1} ,$$

where $\delta = \sigma^2/(12\, \mathsf{E}|Z_1|^3)$ and $c(k)$ is a positive constant depending on k only. The functions $Q_v(z)$ are defined by

$$Q_v(z) = -\frac{\exp(-z^2/2)}{\sqrt{2\pi}} \sum H_{v+2s-1}(z) \prod_{m=1}^{v} \frac{1}{k_m!} \left(\frac{\varkappa_{m+2}}{(m+2)!\, \sigma^{m+2}}\right)^{k_m} ,$$

where the summation runs through all nonnegative solutions of the equations

$$s = k_1 + k_2 + \ldots + k_v ,$$

$$v = k_1 + 2k_2 + \ldots + vk_v$$

and where $H_j(z)$ is the j-th Hermite polynomial. Inserting proper values into the relations defining $Q_v(z)$ we obtain Proposition (vi), if we put $k = 4$ and if we estimate the remainder in the following way. There is an n_0 such that for all $n \geqq n_0$, the set $\{y : |y| \geqq (1 + |z|)\,\sigma\sqrt{n}\}$ is of measure zero. Thus the first term of the remainder is annihilated. For the same reason, the integral in the second term equals to the $(k + 1)$ -st absolute moment for all $n \geqq n_0$, the upper bound for the second term being thus

$$\sigma^{-5/2}n^{-3/2}\,\mathsf{E}|Y_1|^5 = [\tfrac{1}{3}p(1-p)]^{-5/2}\,n^{-3/2}\,\mathsf{E}|Y_1|^5 .$$

As $\sup_{|t| \geqq \delta} |\varphi_{Y_1}(t)| < 1$, we have also $(\sup_{|t| \geqq \delta} |\varphi_{Y_1}(t)| + 1/2n) \leqq w < 1$ for all n sufficiently large; hence $w^n n^{10}$ is a upper bound for the third term of the remainder. Now it is easy to see that the whole remainder is of order $O(n^{-3/2})$.

Proposition (vii). To prove Proposition (vii) we shall make use of Proposition (vi). We have only to check that (vi) remains true also for $p = p_n$. To this aim it is sufficient to show that the set $\{|y| \geqq (1 + |z|)\,\sigma_n\sqrt{n}\}$ is null set with respect to $V_n(y)$ for every $n \geqq n_0$ and some n_0 and that $\sigma_n^{-5/2}n^{-3/2}\,\mathsf{E}|Y_{1n}|^5 = O(n^{-3/2})$. However both these requirements are fulfilled owing to the fact that $\sigma_n = p_n(1 - p_n)$ are bounded away from zero. Finally $\sup_{|t| \geqq \delta_n} |\varphi_{Y_{1n}}(t)| < c < 1$ with some c not depending on n, for the same reason. Now we may put $p = \frac{1}{2} + \Delta$ in the formula given in Proposition (vi) and expand the obtained expressions into power series in Δ. We shall get

$$G_n(x) = \Phi(x) + \tag{11}$$

$$+ \exp(-x^2/2)(2\pi)^{-1/2}\left\{\Delta\left(1 + \left(\sum_{k=1}^{\infty}\frac{\frac{3}{2}(\frac{3}{2}+1)\colon\ldots.(\frac{3}{2}+k-1)}{k!}(4\Delta^2)^k(1-x^2)\right)\right.\right. .$$

$$\cdot\sqrt{\frac{3}{14n}} - [\Delta^2(\sum_{k=0}^{\infty}(4\Delta^2)^k)(15x^5 - 588x^3 - 30x^2 + 1689x - 30) +$$

$$\left. + \sum_{k=0}^{\infty}(4\Delta^2)(49x^2 - 147)]\frac{1}{980n}\right\} + O(n^{-3/2}),$$

and after a rearrangement, the relation (1).

Proposition (viii). To prove Proposition (viii) we shall make use of the following theorem proved in (4):

Let $\{Z_i\}_{i=1}^{\infty}$ be a sequence of independent equally distributed rvs and let H_n be the df of $[\sum_{i=1}^{\infty}(Z_i - \mathsf{E}\,Z_1)]/\sqrt{(n \operatorname{var} Z_1)}$. If $\sup_{|t| > \delta}|v(t)| < 1$, $\mathsf{E}|Z_1|^k < \infty$ for some $k \geqq 3$ then

$$H_n(z) = \Phi(z) + \sum_{v=1}^{k-2}\frac{Q_v(z)}{n^{v/2}} + o(n^{-(k-2)/2})$$

uniformly in $z(-\infty < z < \infty)$. We can write $s = [s] + \lambda$, $0 \leqq \lambda < 1$. Let us use the theorem just stated with $k = 2[s] + 4$. Then

$$\lim_{n\to\infty} \frac{G_n(x + h_n) - G_n(x)}{h_n} = \lim_{n\to\infty} \left\{\frac{\Phi(x + h_n) - \Phi(x)}{h_n} + \right.$$

$$\left. + \sum_{v=1}^{2[s]+2} \frac{Q_v(x + h_n) - Q_v(x)}{n^{v/2} h_n} + \frac{1}{h_n} o(n^{-([s]+1)})\right\} =$$

$$= \exp\left(-\frac{x^2}{2}\right)(2\pi)^{-1/2} + \lim_{n\to\infty}\left[\sum_{v=1}^{2[s]+2} \frac{Q_v'(\xi_n)}{n^{v/2}} + \frac{1}{h_m} o(n^{-([s]+1)})\right];$$

where $\xi_n \in (x, x + h_n)$. The last member converges to zero, because

$$\lim_{n\to\infty} \frac{1}{h_n} o(n^{-([s]+1)}) = \lim_{n\to\infty} \frac{n^{[s]+1} o(n^{-([s]+1)})}{h_n n^{[s]+1}} = 0 .$$

As

$$\frac{d}{dx}\left[\exp\left(-\frac{x^2}{2}\right) H_k(x)\right] = -\exp\left(-\frac{x^2}{2}\right) H_{k+1}(x),$$

$Q_v'(\xi_n)$ is bounded by a constant depending on v and N only. Thus

$$\lim_{n\to\infty} \sum_{v=1}^{2[s]+2} \frac{Q_v'(\xi_n)}{n^{v/2}} = 0 .$$

Proposition (ix). The proof of Proposition (ix) is based on the theorem quoted in the proof of Proposition (viii), where we put $k = 4r + 4$:

$$\frac{G_n(x + h_n) - G_n(x)}{h_n} = \frac{\Phi(x + h_n) - \Phi(x)}{h_n} + \tag{12}$$

$$+ \sum_{v=1}^{4r+4} \frac{Q_v(x + h_n) - Q_v(x)}{n^{v/2} h_n} + \frac{1}{h_n} o(n^{-(2r+1)}),$$

$$q_v(x) = \frac{d}{dx} Q_v(x) .$$

The last term is of order $o(n^{-r})$, as

$$\lim_{n\to\infty} n^r \frac{1}{h_n} o(n^{-2r-1}) = 0 .$$

The use of Taylor formula gives

$$\frac{\Phi(x + h_n) - \Phi(x)}{h_n} = \Phi'(x) + \frac{\Phi''(\xi)}{2!} h_n ,$$

where $\xi \in (x, x + h_n)$. As $\lim_{n \to \infty} h_n n^r = 0$, $\Phi''(\xi) h_n = o(n^{-r})$. Analogous considerations are to be made for further terms and the members possessing only Q_v with $v > 2r$ may be omitted, because they have order $o(n^{-r})$.

I wish to express may gratitude to Prof. RNDr. Václav Dupač, CSc., under whose guidance the paper was written.

References

[1] *Г. А. Козлов:* О распределении случайных чисел, вырабатываемых последовательными физическими датчиками. Теория вероятностней 16 (1971) 370.

[2] *M. Kac:* Statistical Independence in Probability Analysis and Number Theory. Carus Mathematical Monograph, No. 12 The Mathematical Association of America, 1959.

[3] *Jessen-Wintner*: Distributions and the Riemann Zeta functions. Trans. Amer. Math. Soc. 38, 48—88 (1935).

[4] *В. В. Петров:* Суммы независимых случайных величин. Москва 1972.

Souhrn

O VLASTNOSTECH DVOJKOVÝCH NÁHODNÝCH ČÍSEL

JAN ÁMOS VÍŠEK

Nechť $\{X_r\}_{k=1}^{\infty}$ je posloupnost nezávislých nula- jedničkových náhodných veličin. Potom dvojkovým náhodným číslem (dnč) rozumíme $Y = \sum_{k=1}^{\infty} X_k \,.\, 2^{-k}$. Nechť $P(X_k = 1) = \frac{1}{2} + \Delta$, $-\frac{1}{2} < \Delta < \frac{1}{2}$. V článku je ukázáno, že Y má rovnoměrné rozdělení na $[0, 1]$ pro $\Delta = 0$ a singulární v ostatních případech. Dále je pomocí Edgeworthova rozvoje studována rychlost konvergence normovaných sum dnč k normálnímu rozdílení závislosti na Δ a to i v případě $\Delta = \Delta_k$.

Author's address: Jan Ámos Víšek, Matematicko-fysikální fakulta KU, Sokolovská 83, 186 00 Praha 8.