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ON INTEGRATION OF DIFFERENTIAL EQUATIONS IN ELASTOSTATICS THROUGH DETERMINATION OF THE MEAN STRESS

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The paper presents an analytical method of integration of differential equations in elastostatics. The differential equations are formulated in terms of the displacement components (u, v, w) and the mean normal stress (p) for any real value of Poisson's ratio in the interval $-1 < \nu \leqq 1/2$. By this means we obtain in a three-dimensional case a set of four equations in four unknown functions, and in a plane (two-dimensional) case – a set of three equations in three, both sets unlike Navier's equations being valid for incompressible bodies $(\nu = 1/2)$ as well. Derivation of their solution is based on the condition of single-valuedness of the mean normal stress at any point of the body, with displacements presented by means of the Stokes-Helmholtz resolution of vector fields. Integration of this type of differential equations was first studied by the author in works on the deformation of nonhomogeneous elastic incompressible bodies, and carried further in works on elastostatic problems of incompressible and compressible bodies, both homogeneous and non-homogeneous.

1. INTRODUCTION

Navier's equilibrium equations, in terms of displacement, for isotropic homogeneous linear elastic bodies, do not hold for the case of an incompressible body (Poisson's ratio $\nu = 1/2$) which has to be considered separately. The deformation of elastic incompressible bodies was first studied by F. and C. Neumann (1859–60) in connection with their work on light-wave propagation [44]. Later Thomson (1864), in his work on an application of the elasticity theory to geophysics, studied the behaviour of an elastic incompressible liquid [42, 43]. He defined a pressure p by the expression.

$$p = -(\lambda + \tfrac{2}{3}\mu)\,\theta\,, \tag{1.1}$$

transformed Navier's equations into the form

$$\mu(\nabla^2 u, \nabla^2 v, \nabla^2 w) - \frac{\lambda + \mu}{\lambda + \frac{2}{3}\mu}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) + \varrho(X, Y, Z) = 0 \tag{1.2}$$

and, on the assumption $\lambda \to \infty$, $\theta = 0$, obtained a set of four equations in four unknown functions, u, v, w, p:

$$\mu(\nabla^2 u, \nabla^2 v, \nabla^2 w) - \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) + \varrho(X, Y, Z) = 0\,, \tag{1.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,.$$

Love (1892–93), in his work on the deformation of an incompressible sphere in the gravitational field [36], derived more precisely formulated basic equations, assuming;

$$\lim_{\substack{\lambda \to \infty \\ \theta \to 0}} \lambda\theta = -p\,. \tag{1.4}$$

These basic relations were also derived independently by A. and L. Föppl (1920), [5]. An important stage in this field is due to Westergaard (1940–50), who adapted Galerkin's procedure for an incompressible body, and developed a new procedure based on the solution for a specific value of Poisson's ratio (e.g. $\nu = 1/2$), with correction for the actual value of ν for the compressible case[1]); [7], [47–48]. Problems of rock and soil mechanics induced the author to study deformation of nonhomogeneous isotropic incompressible elastic bodies (1958), and the papers which followed [9–11], deal with two-dimensional elastostatic problems by combined use of the displacement-function and stress function approaches, with the solution worked out for simply- and multiply-connected regions. These results encouraged the author to investigate also the deformation of homogeneous isotropic elastic bodies (incompressible and compressible) in two- and three-dimensional cases (1959–62); [12–16], [41]. In the course of this (1961–62), modified elastostatic and elastodynamic Navier's equations were formulated and the first draft was presented of an analytical method for their integration with the aid of a linear function (q) of the mean normal stress, in which the coefficient of linearity is a material constant dependent on the elastic parameters ν and G. The most recent years (1968–71) were devoted by the author to specific problems (nonhomogeneity described by variable Poisson's ratio), for which he derived a number of boundary-value solutions with the aid of an original method of integration [17–19].

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[1]) The correction is estimated by a so-called "twinned-gradient" method; the shear modulus G is assumed fixed.

A special chapter in the development of homogeneous elastostatics concerns the design of rubber elements. In this context, first credit is due to Biderman (1953 to 1958), who derived a number of particular solutions of engineering interest with the aid of the linear theory of deformation of elastic incompressible bodies and worked out approximate design methods, mainly variational [2]. These were developed further by Lavendel in the course of his work on elastic isotropic incompressible bodies (1959–67), in which he adapted the Neuber-Papkovich solution, also using variational principles [31–34]. In 1967–69, some additional works on approximate methods in elastostatics of incompressible bodies appeared as a result of investigations on the deformation of linear polymers [4], [35].

In the U.S., work on deformation of linear elastic homogeneous isotropic incompressible bodies began only a few years ago. Gehman (1964) treated a rubber plate as an incompressible body, under the assumptions of the classical theory of elasticity [8]. Independently, the problem was taken up in the aircraft industry in connection with rubber, rubber-like compounds (such as solid propellants), and other materials obeying a linear viscoelastic model. The analogy between elastostatics of incompressible bodies and the process of steady slow flow of viscoelastic bodies (apparently first observed and commented on by Goodier, [26]) recalled attention to the theory of elasticity. Thus, Herrmann and Toms (1964) rederived Navier's equations in a modified form and presented a numerical solution for a pressurized infinite cylinder with any admissible value of Poisson's ratio ($0 < \nu \leqq 1/2$), [29]; later Herrmann applied variational principles to incompressible and nearly-incompressible materials and worked out numerical methods for boundary-value problems [28]. An extensive review of Herrmann's work and related studies, including unpublished material, was given by Yeh (1967), [50]. Mention should also be made of Wallis's numerical approach to determination of the axisymmetric stress in elastic incompressible solids of revolution (1969), [45].

2. MEAN NORMAL STRESS APPROACH

Introducing Love's vector notation, we present the basic elastostatic equations as follows [20]:

(2.1)
$$\sigma_i = 2G\varepsilon_i + \frac{3\nu}{1+\nu}\,p\,, \quad p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)\,;$$

$$\tau_{ij} = 2G\gamma_{ij}\,, \quad i, j = x, y, z\,; \quad i \neq j\,.$$

(2.2a)
$$G\,\nabla^2(u, v, w) + \frac{3}{2(1+\nu)}\,\mathrm{grad}\,p + (X, Y, Z) = 0\,,$$

(2.2b)
$$\mathrm{div}\,(u, v, w) = \frac{3(1-2\nu)}{E}\,p = \frac{3}{2G}\,\frac{1-2\nu}{1+\nu}\,p\,.$$

For integration of the set (2.2a – b), we express the displacements in terms of functions Ψ, Φ, F and f in the form:

$$(u, v, w) = \operatorname{rot}(\Psi, \Phi, F) + \operatorname{grad} f \tag{2.3}$$

assuming only that Ψ, Φ. F components of a vector potential are of the C^4 class and f (a scalar potential) is at least of the C^3 class. Formula (2.3), which represents the so-called Stokes-Helmholtz theorem on the resolution of vector fields, is valid for any vector field in finite regions and for one subject to additional conditions (e.g. with regard to regularity, or the mode of vanishing, at infinity) in infinite regions [37], [39], [40]. It was shown that for an arbitrary displacement field (u, v, w) the vector field (Ψ, Φ, F) and the function f are obtainable, accordingly, with accuracy up to the gradient of a certain function ω and å constant[2]), [3]. This apparent incompleteness of the representation (2.3) with respect to the vector field (Ψ, Φ, F) may be remedied by stipulating the so-called side or gauge condition: $\operatorname{div}(\Psi, \Phi, F) = 0$, whereby a unique definition of the function ω is ensured[3]). The last "classical" formulation, originally introduced in magnetostatics [37], [49], has a number of applications also in the theory of elasticity. Love used it for strain analysis [36] and Mindlin and Gurtin in studying the completeness of the Neuber-Papkovich solution for finite and infinite regions; [38] and [27]. In the application proposed below, the starting point is the question: what kind of restrictions are imposed on functions Ψ, Φ, F, f in the light of the equations (2.1a – b) ? In order to find the answer to this question, we substitute (2.3) in (2.2a – b), obtaining

$$G\,\nabla^2 \operatorname{rot}(\Psi, \Phi, F) + G \operatorname{grad} \nabla^2 f + \frac{3}{2(1+\nu)} \operatorname{grad} p + (X, Y, Z) = 0\,; \tag{2.4a}$$

$$\nabla^2 f = \frac{3}{2G}\,\frac{1-2\nu}{1+\nu}\,p\,. \tag{2.4b}$$

Substitution of (2.4b) in (2.4a) yields a set of three partial differential equations:

$$G\,\nabla^2 \operatorname{rot}(\Psi, \Phi, F) + \frac{3(1-\nu)}{1+\nu} \operatorname{grad} p + (X, Y, Z) = 0\,. \tag{2.5}$$

Hence

$$\operatorname{grad} p = -\frac{(1+\nu)\,G}{3(1-\nu)}\left[\operatorname{rot}(\nabla^2 \Psi, \nabla^2 \Phi, \nabla^2 F) + \frac{1}{G}(X, Y, Z)\right]; \tag{2.6a}$$

$$p = p_0 - \frac{(1+\nu)\,G}{3(1-\nu)} \int_{\breve{M_0 M}} \left[\operatorname{rot}(\nabla^2 \Psi, \nabla^2 \Phi, \nabla^2 F) + \frac{1}{G}(X, Y, Z)\right] . \, d\bar{r}\,, \tag{2.6b}$$

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[2]) Immaterial for our purpose.

[3]) In this case the field (Ψ, Φ, F) is termed "solenoidal" or "solenoidal vector-potential function".

where the line integral in (2.6b) is taken over a path between points M_0 and M in the region under consideration, and p_0 denotes the value of p at M_0.

Next, we determine f; substitution of (2.6b) in (2.4b) yields:

$$(2.7) \qquad \nabla^2 f = \frac{3(1-2\nu)}{2(1+\nu)G} p_0 - $$
$$- \frac{1-2\nu}{2(1-\nu)} \int_{\overset{\smile}{M_0 M}} \left[\operatorname{rot}(\nabla^2 \Psi, \nabla^2 \Phi, \nabla^2 F) + \frac{1}{G}(X, Y, Z) \right] . d\bar{r} .$$

In addition, we derive further differential relations for p and f. Differentiating each equation in (2.6a) with respect to x, y, and z, and summing the resulting equations, we arrive at

$$(2.8) \qquad \nabla^2 p = -\frac{1}{3}\frac{1+\nu}{1-\nu} \operatorname{div}(X, Y, Z) .$$

Bearing in mind this last relationship and assuming that f is a function of the C^4 class, we obtain from (2.4b):

$$(2.9) \qquad \nabla^2\nabla^2 f = \frac{3}{2G}\frac{1-2\nu}{1+\nu}\nabla^2 p = -\frac{1}{2G}\frac{1-2\nu}{1-\nu}\operatorname{div}(X, Y, Z).$$

Let us consider now the problem of single-valued determination of the function p. For bodies in the form of simply-connected regions, the necessary and sufficient conditions for single-valued determination of both p and f, irrespective of the choice of path of integration, read as follows:

$$(2.10) \qquad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\nabla^2 F - \frac{\partial}{\partial z}\nabla^2\Phi + \frac{X}{G}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\nabla^2\Psi - \frac{\partial}{\partial x}\nabla^2 F + \frac{Y}{G}\right),$$
$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\nabla^2\Psi - \frac{\partial}{\partial x}\nabla^2 F + \frac{Y}{G}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\nabla^2\Phi - \frac{\partial}{\partial y}\nabla^2\Psi + \frac{Z}{G}\right),$$
$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\nabla^2\Phi - \frac{\partial}{\partial y}\nabla^2\Psi + \frac{Z}{G}\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\nabla^2 F - \frac{\partial}{\partial z}\nabla^2\Phi + \frac{X}{G}\right).$$

Adding to both sides of each equation in (2.10) the components

$$\frac{\partial^2}{\partial z^2}\nabla^2 F, \quad \frac{\partial^2}{\partial x^2}\nabla^2\Psi, \quad \frac{\partial^2}{\partial y^2}\nabla^2\Phi$$

we obtain, after transformations,

$$(2.11) \qquad \nabla^2\nabla^2(\Psi, \Phi, F) = \operatorname{grad}\operatorname{div}(\nabla^2\Psi, \nabla^2\Phi, \nabla^2 F) + \frac{1}{G}\operatorname{rot}(X, Y, Z) .$$

Turning now the question of how to uncouple the set (2.11), we note first that by the previous assumptions concerning the class of functions Ψ, Φ and F, the function:

$$H = H(x, y, z) = \operatorname{div}(\nabla^2\Psi, \nabla^2\Phi, \nabla^2 F) \tag{2.12a}$$

is arbitrary and of the C^1 class. This kind of function can be defined by the differential equation:

$$\nabla^2\nabla^2 h = H\,, \tag{2.12b}$$

where $h = h(x, y, z)$ is an arbitrary function of the C^5 class. Substituting (2.12b) in (2.11), we have:

$$\nabla^2\nabla^2(\Psi^*, \Phi^*, F^*) = \frac{1}{G}\operatorname{rot}(X, Y, Z)\,, \tag{2.13}$$

where: $(\Psi^*, \Phi^*, F^*) = (\Psi, \Phi, F) - \operatorname{grad} h$.
It follows that functions Ψ, Φ, F can be represented in the form:

$$(\Psi, \Phi, F) = (\Psi^*, \Phi^*, F^*) + \operatorname{grad} h\,. \tag{2.14}$$

Thus, the definition (2.12a–b) yields the condition[4]):

$$\operatorname{div}(\nabla^2\Psi^*, \nabla^2\Phi^*, \nabla^2 F^*) = 0\,. \tag{2.15}$$

If it is assumed that:

$$\Psi^* = \Psi_b + \Psi_p\,, \quad \Phi^* = \Phi_b + \Phi_p\,, \quad F^* = F_b + F_p\,, \tag{2.16}$$

where Ψ_p, Φ_p, F_p are particular integrals of (2.13), we have that Ψ_b, Φ_b, F_b are biharmonic functions satisfying – by (2.15) the condition:

$$\operatorname{div}(\nabla^2\Psi_b, \nabla^2\Phi_b, \nabla^2 F_b) = -\operatorname{div}(\nabla^2\Psi_p, \nabla^2\Phi_p, \nabla^2 F_p)\,. \tag{2.17}$$

In the particular case when the body force-field is a potential field then:

$$\Psi_p = \Phi_p = F_p = 0\,, \quad \Psi^* = \Psi_b\,, \quad \Phi^* = \Phi_b\,, \quad F^* = F_b \tag{2.18a}$$

and the condition (2.15) takes the form:

$$\operatorname{div}(\nabla^2\Psi_b, \nabla^2\Phi_b, \nabla^2 F_b) = 0\,. \tag{2.18b}$$

It remains to consider the influence of h on the solution of Navier's equations in their modified form (2.2a–b). Analysis of (2.3), (2.6a–b), (2.7) shows that h may be assumed constant without loss of generality, which yields the definition:

$$\Psi = \Psi^*\,. \quad \Phi = \Phi^*\,, \quad F = F^*\,. \tag{2.19}$$

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[4]) An identical condition can be derived in the particular case where h is a biharmonic function (and of the C^4 class only).

In addition, we point out that the condition (2.15) can be written in the form:

$$\nabla^2 \operatorname{div}(\Psi, \Phi, F) = 0 \tag{2.20}$$

and is satisfied identically if the field (Ψ, Φ, F) is assumed solenoidal[5]), i.e.

$$\operatorname{div}(\Psi, \Phi, F) = 0 . \tag{2.21}$$

To sum up, equations (2.3), (2.6b), (2.7) and also (2.13), (2.15) and (2.19), determine a general solution of the differential equations (2.2a–b) in simply-connected regions.

In this solution, the formulae (2.6b), (2.7) contain the following integral expression:

$$I = \int_{\breve{M_0 M}} \left[\operatorname{rot}(\nabla^2\Psi, \nabla^2\Phi, \nabla^2 F) + \frac{1}{G}(X, Y, Z) \right] . \mathrm{d}\bar{r} , \tag{2.22}$$

which deserves special attention. Recalling (2.16) and (2.19), (2.22) takes the form:

$$I = \alpha + \int_{\breve{M_0 M}} \left[\operatorname{rot}(\nabla^2\Psi_b, \nabla^2\Phi_b, \nabla^2 F_b) \right] . \mathrm{d}\bar{r} , \tag{2.23a}$$

where in the general case:

$$\alpha = \int_{\breve{M_0 M}} \left[\operatorname{rot}(\nabla^2\Psi_p, \nabla^2\Phi_p, \nabla^2 F_p) + \frac{1}{G}(X, Y, Z) \right] . \mathrm{d}\bar{r} \tag{2.23b}$$

and is, for a known body-force field (X, Y, Z), a single-valued function in the variables x, y, z. The second integral term in formulae (2.23a) – which contains biharmonic functions – can be transformed into a differential expression. For this purpose, Almansi's theorem can be used and the biharmonic functions expressed in terms of harmonic ones [1], [46]. This procedure also transforms the differential-integral equations (2.7) into differential form. Such a method was successfully applied by the author in his investigations on displacement and traction boundary-value problems for an elastic half-plane with variable Poisson's ratio[6]), [17–19]. We note, moreover, that in the case of incompressible bodies (the particular case $\nu = 1/2$) equation (2.7) automatically transforms into differential form – the harmonic equation:

$$\nabla^2 f = 0 .$$

[5]) As in the classical formulation of the Stokes-Helmholtz theorem.

[6]) Recently, the author obtained new results in two-dimensional homogeneous elasticity — research reports [21—22].

This fact considerably simplified the solution, but at the same time introduces an indeterminacy of magnitude p_0 in displacement boundary-value problems; the result which was remarked upon and discussed in ref. [6], [13], [25].

When $\nu \neq 1/2$ $(-1 < \nu < 1/2)$ the general solution of (2.7) can be presented as a sum of two solutions:

$$f = f^* + f_p^* , \tag{2.25}$$

where f^* is the general solution of Poisson's equation:

$$\nabla^2 f^* = \frac{3(1-2\nu)}{2(1+\nu)G} p_0 \tag{2.26}$$

and f_p^* — a particular solution of the equation:

$$\nabla^2 f_p^* = -\frac{1-2\nu}{2(1-\nu)} \int_{\overset{\smile}{M_0 M}} \left[\operatorname{rot}(\nabla^2 \Psi, \nabla^2 \Phi, \nabla^2 F) + \frac{1}{G}(X, Y, Z) \right] . \mathrm{d}\bar{r} , \tag{2.27}$$

which is defined for the known functions Ψ, Φ, F. In turn, it is possible to present the general solution of Poisson's equation (2.26) as follows:

$$f^* = f_0 + 1/2(ax^2 + by^2 + cz^2), \quad \nabla^2 f_0 = 0 , \tag{2.28a}$$

where the coefficients a, b, c are constants linked with p_0 by the equation:

$$a + b + c = \frac{3(1-2\nu)}{2(1+\nu)G} p_0 . \tag{2.28b}$$

Finally, the solution of (2.7) takes the form of a sum of three solutions: the general harmonic solution and two particular ones, the first which depends on Ψ, Φ, F and the second on p_0.

3. SOLUTION FOR BODIES IN THE FORM OF MULTIPLY-CONNECTED REGIONS

In multiply-connected regions expressions (2.10) [or, equivalently, (2.13) and (2.15)] constitute necessary (but not sufficient) conditions for single-valued determination of the mean normal stress p. The many-valuedness of p is due to the appearance of an additional constant in the expression (2.6b). The difference between function p and p_0 can be written as follows:

$$p - p_0 = C_D - \frac{(1+\nu)G}{3(1-\nu)} \int_{\overset{\smile}{M_0 M}} \left[\operatorname{rot}(\nabla^2 \Psi, \nabla^2 \Phi, \nabla^2 F) + \frac{1}{G}(X, Y, Z) \right] . \mathrm{d}\bar{r} \tag{3.1}$$

where C_D is a constant defined for the region under consideration, and $\overset{\smile}{M_0 M}$ — an open curve inside the region. Tho explain this problem, we consider an arbitrary

n-connected region D (Fig. 1). By $n - 1$ cuts $a_k b_k$, D is transformed into a simply-connected one D^*, whose boundary is more complex, being bounded by the positive and negative sides of the cuts $a_k b_k$ (Fig. 2). If we choose in D a path of integration $\breve{M_0 M}$ which does not cross the boundary $a_k b_k$, then, as in a simply-connected region, the integral expression I [formula (2.23) and (3.1)] is path-independent and therefore single-valued. On the other hand, if the path crosses the cut $a_k b_k$ (equivalent to circling the boundary L_k at least once) the expression I is dependent on the contour integral over an arbitrary closed curve L'_k (Figs. 1, 2), i.e. on the integral

$$(3.2) \qquad I_k = \oint_{L_k'} \left[\operatorname{rot}(\nabla^2 \Psi, \nabla^2 \Phi, \nabla^2 F) + \frac{1}{G}(X, Y, Z) \right] . \, d\bar{r} \, ; \quad k = 1, 2, \ldots, n - 1 \, ,$$

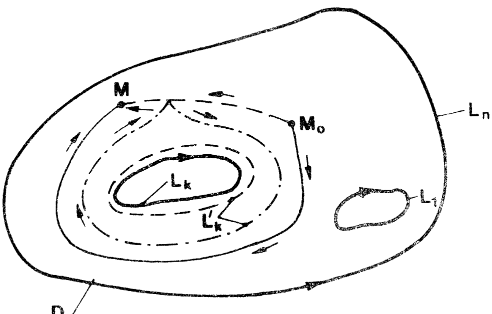

Fig. 1.

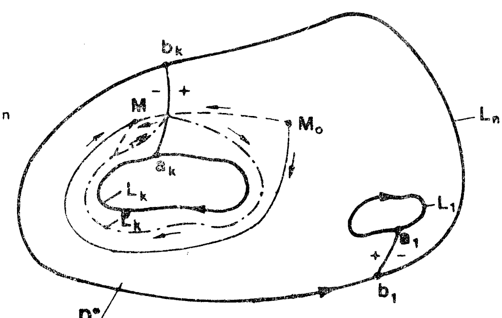

Fig. 2.

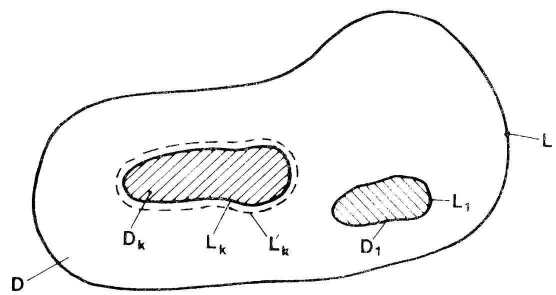

Fig. 3.

which generally is constant and non-zero. Integral I_k is identically zero when the conditions of the continuity imposed on the integrands can be extended to include the region D_k bounded by the curve L_k, and may differ from zero – when the integrands are singular in D_k (Fig. 3). Hence, the formula for C_D can be presented in the form:

$$(3.3a) \qquad C_D = \sum_{k=1}^{n-1} \sum_{j=1}^{m} n_{kj} I_k \, ,$$

n_{kj} denoting the number of circlings[7]):

(3.3b) $$n_{kj} = \pm j \quad (j = 1, 2, \ldots, m) .$$

The many-valuedness which occurs in the determination of the mean normal stress p — in view of expressions (2.4b), (2.3) and (2.1) — cause many-valuedness of the displacement and stresses[8])[9]). Therefore, in the solution for multiply-connected regions, the functions Ψ, Φ and F must be subject to the additional conditions unless satisfied identically[10]):

(3.4) $$I_k = 0 , \quad k = 1, 2, \ldots, n - 1 .$$

APPENDIX

In the classical formulation of elastostatic equations, we have to deal with the magnitude

$$\frac{1}{1 - 2\nu} e \quad \left[e = \operatorname{div}(u, v, w)\right] ,$$

which is indeterminate when $\nu = 1/2$ and $e = 0$ (the case of incompressible bodies). In order to extend the classical formulation we have to prove that there exists the limit of the expression $e/1 - 2\nu$ when e tends to zero and ν tends to $1/2$. From the equation (2.2b) we derive

$$\lim_{\substack{e \to 0 \\ \nu \to 1/2}} \frac{e}{1 - 2\nu} = \lim_{\nu = 1/2} \frac{3}{2G(1 + \nu)} p = \frac{1}{G|_{\nu = 1/2}} \times \lim_{\nu \to 1/2} p \quad \left(G|_{\nu = 1/2} = \frac{1}{3} E\right) .$$

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[7]) The path of integration may cross the cut $a_k b_k$ n_{kj} times in the positive and negative sense.

[8]) The displacement field is single-valued in the case of incompressible bodies. On the other hand, to a known displacement field, there correspond, generally, many-valued normal stresses differing by a constant.

[9]) Many-valuedness of stress concerns only the normal stresses.

[10]) Cf. also [24].

Bearing in mind that the function p is defined by the equation (2.6b), we obtain, finally[11]):

$$\lim_{\substack{e \to 0 \\ \nu = 1/2}} \frac{e}{1 - 2\nu} = \frac{1}{G|_{\nu = 1/2}} \times p_0 - \int_{M \breve{o} M} \left[\operatorname{rot} (\nabla^2 \Psi, \nabla^2 \Phi, \nabla^2 F) + \frac{1}{G|_{\nu = 1/2}} (X, Y, Z) \right] . \mathrm{d}\bar{r} .$$

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Souhrn

INTEGRACE DIFERENCIÁLNÍCH ROVNIC ELASTOSTATIKY URČENÍM STŘEDNÍHO NAPĚTÍ

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Uvedená metoda integrace diferenciálních rovnic elastostatiky určením středního napětí dává řešení závislé na parametrech pružnosti a na topologii tělesa, které je proto přímo ovlivněno Poissonovou konstantou. Například, předpokládáme-li nestlačitelnost $\left(\nu = \frac{1}{2}\right)$, přejde příslušná Poissonova rovnice ve složkách v harmonickou rovnici. Řešení na vícenásobně souvislé oblasti musí kromě toho vyhovovat podmínkám, které mj. závisí na geometrii dané oblasti. Tyto podmínky zaručují jednoznačnost středního kolmého napětí.

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