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INHOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR THE VON KÁRMÁN EQUATIONS, I

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1. INTRODUCTION

The present paper deals with the existence of a solution for the von Kármán equations governing the equilibrium of a thin elastic plate, which is subjected to combined perpendicular and edge loading. The boundary conditions under consideration correspond with a plate whose edge is partially clamped, supported and elastically clamped, and partially subjected to a transversal loading and a moment distribution, being also elastically supported and clamped. Moreover, the boundary is permitted to have corners.

In Part I we consider only configurations of boundary conditions such that any motion of the rigid plate, the elastic clampings and supports of which become also rigid, is eliminated. In the case of a plate which is loaded by tension forces in its plane, we prove the existence of a solution to the "full" problem, i.e., for any magnitude of loads. In the remaining cases however, we are forced to restrict the compressive forces to sufficiently small magnitudes. Moreover, for sufficiently small perpendicular and edge loading, we prove the uniqueness of the solution.

Our approach consists in restating the boundary value problem considered as a system of integral identities in suitably chosen Hilbert function spaces. These integral identities admit an equivalent operator formulation to which recent abstract existence results for solutions of nonlinear operator equations apply.

In [1], Berger and Fife have proved the existence of buckled states of a thin elastic plate subjected only to compressive forces. Related studies for a thin elastic shallow shell are carried out by Naumann [10][2]). A global existence theorem for the von Kármán equations of a plate which is subjected to a combined perpendicular

[1]) The paper was written during the stay of the second named author at the Department of Mathematics, Charles University, Prague.

[2]) Note that in [10] the case of compression is included in an inequality of the type of Condition (+) (cf. Remark 4.3) but with the opposite sign.

and edge loading and clamped along the whole boundary, has been established by Knightly [6]. In this paper, no restrictions upon the magnitude of the compressive forces are made. Some results estimating the number of solutions for the same problem, are presented by Knightly and Sather [7]. The problem of uniform tension studied by Fife [3], is included in our discussion as a particular case. Finally, Morozov [8] has considered the von Kármán equations under certain homogeneous boundary conditions which belong also to the class of our boundary conditions.

The existence of a solution for the von Kármán equations of a thin elastic plate, the edge of which is completely free of forces and supports (and the rigid plate motions are admitted), has been proved by Naumann [9]. A more general systematic treatment of this type of edge conditions will be presented in Part II of our paper.

Section 2 contains the formulation of the boundary value problem investigated. Moreover, a short discussion of the mechanical meaning of the boundary conditions is given. The following Section 3 is devoted to some preliminaries and the definition of the notion of a variational solution of our boundary value problem. The structure of the equations enables us to reduce the inhomogeneous boundary conditions upon the stress function into homogeneous ones. This is carried out in Section 4 where our main result is also presented. Section 5 contains its proof and a corollary concerning the uniqueness of the solution.

2. SETTING OF THE BOUNDARY VALUE PROBLEM

Let Ω be a bounded, simply connected domain in the x, y-plane, representing the shape of the plate. Throughout the paper we assume that each point (x, y) of the boundary Γ of Ω can be represented in the form $x = x(s)$, $y = y(s)$ where the functions $x(s)$, $y(s)$ are continuous and piecewise three-times continuously differentiable (here s denotes the arc length). Thus, we can write

$$\Gamma = \bigcup_{j=1}^{l} S_j \quad (1 \leqq l < \infty)$$

where each S_j is a smooth simple arc, and the angles of the tangents at the corners (if any) between the adjacent arcs are positive[1]).

We consider a thin elastic plate (whose midsurface is identified with Ω) which is subjected both to a perpendicular load q and to forces acting along Γ. Then equilibrium states of the plate are characterized as solutions of the following system of equations:

$$\Delta^2 w = [\Phi, w] + q \quad \text{in} \quad \Omega\,, \tag{2.1}$$

$$\Delta^2 \Phi = -[w, w] \quad \text{in} \quad \Omega \tag{2.2}$$

[1]) The conditions imposed upon Γ enable us to apply the results of [5]. On the other hand, these conditions include the important cases of rectangular and polygonal domains.

(the equations (2.1), (2.2) are usually called the von Kármán equations of a thin elastic plate). Here $w = w(x, y)$ represents the deflection of the plate while $\Phi = \Phi(x, y)$ is the Airy stress function. Δ^2 is the biharmonic operator, and

$$[u, v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy} .$$

In order to specify the boundary conditions for (2.1), (2.2), let Γ consist of three mutually disjoint parts Γ_i,

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

where each Γ_i $(i = 1, 2, 3)$ is either empty or possesses a positive measure (length) and does not contain isolated points[2]). Moreover, we assume that $mes(\Gamma_2 \cup \Gamma_3) > 0$[3]).

We shall consider the following boundary conditions:

$$\begin{aligned} &(2.3) \qquad && w = w_n = 0 && && \text{on} \quad \Gamma_1 , \\ & && w = 0 , && M(w) + k_2 w_n = m_2 && \text{on} \quad \Gamma_2 , \\ & && M(w) = k_{31} w_n = m_3 , && T(w) + k_{32} w = t_3 && \text{on} \quad \Gamma_3 , \end{aligned}$$

where

$$w_n = \frac{\partial w}{\partial n} ,$$

$$M(w) = \mu \, \Delta w + (1 - \mu)\left(w_{xx}n_x^2 + 2w_{xy}n_x n_y + w_{yy}n_y^2\right) ,$$

$$T(w) = - \frac{\partial}{\partial n} \Delta w + (1 - \mu) \frac{\partial}{\partial s} \left[w_{xx}n_x n_y - w_{xy}(n_x^2 - n_y^2) - w_{yy}n_x n_y\right] + $$
$$+ X w_x + Y w_y ,$$

$n = (n_x, n_y)$ being the unit outward normal with respect to Ω, $\mu = \text{const}$ $(0 < \mu < \frac{1}{2})$ is the Poisson ratio of the plate material and m_2, m_3, t_3, X, Y are prescribed functions (cf. (3.2), (3.3), (3.4) below).

The functions k_2, k_{3j} $(j = 1, 2)$ satisfy the following conditions:

(2.3′) $\qquad k_2 \in L^p(\Gamma_2)$[1]), $\quad k_2 \geqq 0$ a.e. on Γ_2 ,

(2.3″) $\qquad k_{31} \in L^p(\Gamma_3)$, $\quad k_{32} \in L^1(\Gamma_3)$,
$\qquad k_{3j} \geqq 0$ a.e. on Γ_3 $(j = 1, 2)$

where $1 < p < \infty$.

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[1]) Here we do not give the definition of the spaces $L^p(\Gamma)$ $(1 \leqq p < \infty)$. For details we refer to the book [11].

[2]) With respect to our approach we cannot consider boundary conditions at isolated points.

[3]) For the case $\Gamma_2 \cup \Gamma_3 = \emptyset$ we refer to Knightly [6].

In the presence of corners $(x(s_i), y(s_i))$, $i = 1, \ldots, r$ $(r \leqq l)$ in the interior of Γ_3, (2.3) have to be completed by the conditions

$$(2.4) \qquad H(w(s_i^+), n(s_i^+)) - H(w(s_i^-), n(s_i^-)) = h_i\,, \quad i = 1, \ldots, r$$

where we have set

$$\chi(s_i^+) = \lim_{s \to s_i + 0} \chi(s)\,, \quad \chi(s_i^-) = \lim_{s \to s_i - 0} \chi(s)$$

for any piecewise continuous function $\chi = \chi(s)$,

$$H(w, n) = (1 - \mu)\left[w_{xx}n_x n_y - w_{xy}(n_x^2 - n_y^2) - w_{yy}n_x n_y\right]$$

while h_i $(i = 1, \ldots, r)$ are prescribed constants.

In the present Part I of our paper we consider (2.3) under the assumption that at least one of the following five conditions holds:

1° $\operatorname{mes}(\Gamma_1) > 0$;

2° $\operatorname{mes}(\Gamma_2) > 0$, and Γ_2 is not a segment of a straight line;

3° there exists a subset $\Gamma_2' \subset \Gamma_2$ such that:
  a) Γ_2' is a segment of a straight line,
  b) $\int_{\Gamma_2'} k_2 \, ds > 0$;

4° there exists a subset $\Gamma_3' \subset \Gamma_3$ such that:
  a) $\operatorname{mes}(\Gamma_3') > 0$,
  b) Γ_3' is not a segment of a straight line,
  c) $k_{32} > 0$ a.e. on Γ_3';

5° there exists a subset $\Gamma_3'' \subset \Gamma_3$ such that:
  a) Γ_3'' is a segment of a straight line,
  b) $k_{32} > 0$ a.e. on Γ_3'',
  c) either $\int_{\Gamma_3} k_{31} \sin^2(n, t) \, ds > 0$ (where t denotes the direction of Γ_3'') or: Γ_2 is a segment of a straight line which does not coincide with the straight line containing Γ_3'', and $k_2 = 0$ a.e. on Γ_2.

Let us briefly explain the mechanical sense of the boundary conditions above. According to (2.3) the plate is clamped along Γ_1, supported and elastically clamped (if $k_2 > 0$) on Γ_2 or loaded only by a moment distribution (if $k_2 = 0$). In particular, it is simply supported on Γ_2 if $k_2 = m_2 = 0$. On Γ_3 elastic supports (if $k_{32} > 0$) and elastic clamping (if $k_{31} > 0$) or a transversal load and a moment distribution only (if $k_{31} = 0$ and $k_{32} = 0$, respectively) are prescribed.

The inequalities in (2.3′), (2.3″) are based on the fact that the deformation energy of elastic supports cannot be negative.

At the corners lying inside Γ_3, the jumps of the twisting moment are given according to (2.4).

The conditions $1^\circ - 5^\circ$ guarantee that if the elastic bending energy of the plate and of the elastic clamping and supports vanish (i.e. if the plate and the supports become "rigid"), and if the geometrical boundary conditions are satisfied, then the plate cannot move in the direction of w.

Finally, we impose the following boundary conditions upon Φ:

$$\Phi = \varphi_0 \,, \quad \Phi_n = \varphi_1 \quad \text{on} \quad \Gamma \,, \tag{2.5}$$

$$\Phi_{yy} n_x - \Phi_{xy} n_y = X \,, \quad \Phi_{xx} n_y - \Phi_{xy} n_x = Y \text{ on } \Gamma_3 \,, \tag{2.5'}$$

where φ_0, φ_1, X, Y are given functions (cf. (3.4), (4.1) below).

Note that φ_0, φ_1 depend on X, Y (if $\Gamma_3 \neq \emptyset$). In fact, let $\bar{X}(s)$, $\bar{Y}(s)$ be the lateral tractions acting along the boundary Γ in the x- and y-direction, respectively. The midsurface stresses σ_x, σ_y, τ can be expressed by means of the stress function Φ:

$$\sigma_x = Dh^{-1}\Phi_{yy} \,, \quad \sigma_y = Dh^{-1}\Phi_{xx} \,, \quad \tau = -Dh^{-1}\Phi_{xy}$$

where h is the (constant) thickness of the plate and D is the cylindrical rigidity,

$$D = \frac{Eh^3}{12(1-\mu^2)} \qquad (E = \text{Young's modulus}) \,.$$

The lateral tractions equal to

$$\bar{X} = h(\sigma_x n_x + \tau n_y), \quad \bar{Y} = h(\tau n_x + \sigma_y n_y) \quad \text{on} \quad \Gamma \,.$$

Observing (2.5') we get

$$D^{-1}\bar{X} = X = \frac{\mathrm{d}}{\mathrm{d}s}\Phi_y \,, \quad D^{-1}\bar{Y} = Y = -\frac{\mathrm{d}}{\mathrm{d}s}\Phi_x \,.$$

Thus,

$$\Phi = A + Bx + Cy + \int_0^s \mathrm{d}t \left[n_y \int_0^t Y \,\mathrm{d}u + n_x \int_0^t X \,\mathrm{d}u \right] , \tag{2.6}$$

$$\Phi_n = Bn_x + Cn_y - n_x \int_0^s Y \,\mathrm{d}u + n_y \int_0^s X \,\mathrm{d}u$$

where A, B, C are arbitrary constants.

Thus on the whole Γ the given (reduced) tractions X and Y will be transformed into φ_0, φ_1 according to (2.6). Besides, on Γ_3 we have to prescribe also X and Y directly by means of (2.5').

All assumptions formulated above are assumed to be satisfied throughout the present paper. The system (2.1), (2.2) with the boundary conditions (2.3), (2.4) and (2.5), (2.5') will be referred to as boundary value problem I.

We denote by $L^p(\Omega)$ $(1 \leqq p < \infty)$ the space of all real functions which are integrable with power p on Ω (with respect to the Lebesgue measure $\mathrm{d}x\,\mathrm{d}y$).

Using the notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\,\partial y^{\alpha_2}}\,, \qquad |\alpha| = \alpha_1 + \alpha_2\,,$$

we define for any integer $m \geqq 1$

$$W^{m,p}(\Omega) = \{u \mid u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leqq m\}$$

where the derivatives are to be understood in the sense of distributions. $W^{m,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left\{\int\!\!\int_\Omega |u|^p \,\mathrm{d}x\,\mathrm{d}y + \sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p \,\mathrm{d}x\,\mathrm{d}y\right\}^{1/p}.$$

In particular, the scalar product

$$(u, v)_{W^{2,2}(\Omega)} = \int_\Omega uv \,\mathrm{d}x\,\mathrm{d}y + \sum_{|\alpha|=2} \int_\Omega D^\alpha u\, D^\alpha v \,\mathrm{d}x\,\mathrm{d}y$$

turns $W^{2,2}(\Omega)$ into a Hilbert space.

Remark 3.1. It is easy to see that our assumptions on the boundary Γ imply that Γ is Lipschitzian (see [11] for details). Thus the imbedding and trace theorems for the spaces $W^{m,p}(\Omega)$ hold.

Let $C^\infty(\bar{\Omega})$ denote the space of all infinitely continuously differentiable functions in Ω which together with all their derivatives can be continuously extended onto $\bar{\Omega}$,

We then set

$$\mathscr{V} = \{u \mid u \in C^\infty(\bar{\Omega}),\ u = u_n = 0 \text{ on } \Gamma_1,\ u = 0 \text{ on } \Gamma_2\}\,,$$

and define

$$V = \text{closure of } \mathscr{V} \text{ in } W^{2,2}(\Omega)\,.$$

Obviously, V is a Hilbert space with respect to the scalar product $(.,.)_{W^{2,2}(\Omega)}$. Note that in the case $\Gamma_1 \cup \Gamma_2 = \varnothing$ we have $\mathscr{V} = C^\infty(\bar{\Omega})$ and $V = W^{2,2}(\Omega)$ (see [11]). Sobolev's Imbedding Theorem and the trace theorem imply, for any $u \in V$,

$$\begin{aligned} u &= 0 \quad \text{pointwise on} \quad && \Gamma_1 \cup \Gamma_2\,, \\ u_n &= 0 \quad \text{in the trace sense on} \quad && \Gamma_1 \end{aligned}$$

(see [11]).

For our discussion below it is convenient to introduce another scalar product on V. To this end, we define for $u, v \in W^{2,2}(\Omega)$ the bilinear forms

$$A(u, v) \equiv \int_\Omega [u_{xx}v_{xx} + 2(1 - \mu)\, u_{xy}v_{xy} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx})]\, \mathrm{d}x\, \mathrm{d}y$$

and

$$a(u, v) \equiv \int_{\Gamma_2} k_2 u_n v_n \, \mathrm{d}s + \int_{\Gamma_3} k_{31} u_n v_n \, \mathrm{d}s + \int_{\Gamma_3} k_{32} uv \, \mathrm{d}s .$$

By virtue of the trace theorem and Sobolev's Imbedding Theorem (see [11]), the boundary integrals above exist so that $a(u, v)$ makes sense for the functions u, v under consideration.

Observing the conditions $1° - 5°$ (Section 2) it is readily seen that, for any $u \in V$, $A(u, u) + a(u, u) = 0$ implies $u \equiv 0$.

Lemma 3.1. *There exist positive constants c_1, c_2 such that*

$$c_1 \|u\|^2_{W^{2,2}(\Omega)} \leqq A(u, u) + a(u, u) \leqq c_2 \|u\|^2_{W^{2,2}(\Omega)} \tag{3.1}$$

holds for all $u \in V$.

Proof. The first inequality in (3.1) follows immediately from [4; Theorem 2.1]. Using the estimates

$$\int_{\Gamma_2} k_2 u_n^2 \, \mathrm{d}s \leqq \text{const}\, \|k_2\|_{L^p(\Gamma_2)} \|u\|^2_{W^{2,2}(\Omega)} ,$$

$$\int_{\Gamma_3} k_{31} u_n^2 \, \mathrm{d}s \leqq \text{const}\, \|k_{31}\|_{L^p(\Gamma_3)} \|u\|^2_{W^{2,2}(\Omega)} ,$$

$$\int_{\Gamma_3} k_{32} u^2 \, \mathrm{d}s \leqq \|k_{32}\|_{L^1(\Gamma_3)} (\max |u(s)|)^2 \leqq$$
$$\leqq \text{const}\, \|k_{32}\|_{L^1(\Gamma_3)} \|u\|^2_{W^{2,2}(\Omega)} ,$$

the second inequality in (3.1) is easily seen.

Thus, setting

$$(u, v)_V = A(u, v) + a(u, v)$$

for $u, v \in V$, Lemma 3.1 implies that V with the scalar product $(.,.)_V$ forms a Hilbert space. Henceforth, V will be understood as provided with this scalar product.

We denote by $C_c^\infty(\Omega)$ the space of all real infinitely differentiable functions having their support in Ω. Let $W_0^{2,2}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $W^{2,2}(\Omega)$. Setting

$$(u, v)_{W_0^{2,2}(\Omega)} = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy})\, \mathrm{d}x\, \mathrm{d}y$$

for $u, v \in W^{2,2}(\Omega)$, the norms $\| \;\|_{W^{2,2}(\Omega)}$ and $\| \;\|_{W_0^{2,2}(\Omega)} = (.,.)^{1/2}_{W_0^{2,2}(\Omega)}$ are equivalent on $W_0^{2,2}(\Omega)$. Thus, in what follows, we consider $W_0^{2,2}(\Omega)$ as provided with the scalar product $(.,.)_{W_0^{2,2}(\Omega)}$, and we get

$$W_0^{2,2}(\Omega) \subset V \subset W^{2,2}(\Omega)$$

where each injection is continuous.

In order to characterize the right hand sides q (see equation (2.1)) we denote by $[C(\bar{\Omega})]'$ the dual space of $C(\bar{\Omega})$ (the space of functions which are continuous in Ω and have continuous extensions onto $\bar{\Omega}$), and by $\langle m, \varphi \rangle$ the dual pairing between $m \in [C(\bar{\Omega})]'$ and $\varphi \in C(\bar{\Omega})$. Taking q in $[C(\bar{\Omega})]'$, as right hand side any (Radon) measure is admitted.

We briefly discuss some particular cases for q which are included in our treatment of boundary value problem I. First, let Ω_0 be a measurable subset of Ω, and let $q \in L^1(\Omega_0)$. Identifying q with the measure

$$\varphi \to \int_{\Omega_0} q(x, y)\, \varphi(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

we get

$$\langle q, \varphi \rangle = \int_{\Omega_0} q(x, y)\, \varphi(x, y)\, \mathrm{d}x\, \mathrm{d}y\,.$$

The function q describes a load distribution on Ω_0. The same argument applies to the measure

$$\varphi \to \int_{\gamma} q(s)\, \varphi(s)\, \mathrm{d}s$$

where γ is a (rectificable) curve in Ω, and $q \in L^1(\gamma)$. Here q represents a load distribution along γ. Further, setting $q = \alpha \delta_{(x_0,y_0)}$ where $\delta_{(x_0,y_0)}$ is the Dirac measure at the point $(x_0, y_0) \in \bar{\Omega}$ ($\alpha = \text{const}$), q may be interpreted as a concentrated load at the single point (x_0, y_0).

Finally, we impose the following conditions upon the boundary data in (2.3), (2.5'):

$$m_2 \in L^p(\Gamma_2)\,, \tag{3.2}$$

$$m_3 \in L^p(\Gamma_3)\,, \quad t_3 \in L^1(\Gamma_3)\,, \tag{3.3}$$

$$X, Y \in L^p(\Gamma_3) \tag{3.4}$$

where $1 < p < \infty$.

Setting

$$B(\varphi, \psi; \zeta) \equiv \int_{\Omega} [(\varphi_{xy}\psi_y - \varphi_{yy}\psi_x)\, \zeta_x + (\varphi_{xy}\psi_x - \varphi_{xx}\psi_y)\, \zeta_y]\, \mathrm{d}x\, \mathrm{d}y^{1)}$$

for $\varphi, \psi, \zeta \in W^{2,2}(\Omega)$, we give the

[1]) Note that the existence of the integral under consideration is easily verified by the aid of Sobolev's Imbedding Theorem (see also the proof of Proposition 5.1 below).

Definition 3.1. *The pair w, Φ is called a variational solution of boundary value problem I if:*

1° $w \in V$,

2° $\Phi \in W^{2,2}(\Omega)$; $\Phi = \varphi_0$, $\Phi_n = \varphi_1$ *on* Γ[1]),

3° *the identity*

$$A(w, \varphi) + a(w, \varphi) = B(\Phi, w; \varphi) + \int_{\Gamma_2} m_2 \varphi_n \, ds + \int_{\Gamma_3} m_3 \varphi_n \, ds + \\ + \int_{\Gamma_3} t_3 \varphi \, ds + \sum_{i=1}^{r} h_i \, \varphi(x(s_i), y(s_i)) + \langle q, \varphi \rangle \tag{3.5}$$

holds for all $\varphi \in V$,

4° *the identity*

$$(\Phi, \psi)_{W_0^{2,2}(\Omega)} = -B(w, w; \psi) \tag{3.6}$$

holds for all $\psi \in W_0^{2,2}(\Omega)$.

Remark 3.2. The integral identities (3.5), (3.6) can be obtained formally, multiplying (2.1) and (2.2) by the test functions φ and ψ, respectively, integrating by parts and making use of the boundary conditions (2.3), (2.4), (2.5').

Remark 3.3. In the present paper we do not discuss the sense in which the equations (2.1), (2.2) and the boundary conditions (2.3) are satisfied by a variational solution. This will be done elsewhere.

4. REFORMULATION. STATEMENT OF THE MAIN RESULT

In this section we give a modified definition of the notion of a variational solution of boundary value problem I. Based on this modification we are able to apply abstract operator methods for proving the existence of a solution of boundary value problem I.

We impose the following conditions upon the data φ_0 and φ_1:

$$\begin{aligned} &\varphi_0 \in W^{3/2,2}(S_j)\,, \quad \varphi_1 \in W^{1/2,2}(S_j) \quad (j = 1, \ldots, l)\,, \\ &\varphi_0 \in W^{1,2}(\Gamma)\,, \\ &\varphi_{01} \equiv -\, n_y \frac{d\varphi_0}{ds} + n_x \varphi_1 \in W^{1/2,2}(\Gamma)\,, \\ &\varphi_{10} \equiv n_x \frac{d\varphi_0}{ds} + n_y \varphi_1 \in W^{1/2,2}(\Gamma)\,.^{2)} \end{aligned} \tag{4.1}$$

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[1]) The trace equalities make sense if we suppose $\varphi_0 \in W^{1,r}(\Gamma)$, $\varphi_1 \in L^r(\Gamma)$ $(1 \leqq r < \infty)$. However, for the proof of the existence of a variational solution we have to sharpen these conditions (cf. (4.1) below).

[2]) Since the spaces $W^{\alpha,2}(\Gamma)$ (α real) and their properties are not explicitely used in the sequel, we dipense with their definition. For details we refer to the book [11].

Then there exists a function $g \in W^{2,2}(\Omega)$ such that

$$(4.2) \qquad g = \varphi_0\,, \quad g_n = \varphi_1 \quad \text{in the trace sense on } \Gamma\,,$$
$$\|g\|_{W^{2,2}(\Omega)} \leqq \text{const}\; T_1(\varphi_0, \varphi_1)$$

where

$$T_1(\varphi_0, \varphi_1) \equiv \sum_{j=1}^{l} [\|\varphi_0\|_{W^{3/2,2}(S_j)} + \|\varphi_1\|_{W^{1/2,2}(S_j)}] + \|\varphi_{01}\|_{W^{1/2,2}(\Gamma)} + \|\varphi_{10}\|_{W^{1/2,2}(\Gamma)}$$

(see [5]).

Remark 4.1. If Γ is sufficiently smooth (e.g. if $\Omega \in \mathfrak{N}^{(2),1}$ [1]), which in particular means the absence of corners on Γ) then (4.1) reduces to

$$(4.1') \qquad \varphi_0 \in W^{3/2,2}(\Gamma)\,, \quad \varphi_1 \in W^{1/2,2}(\Gamma)\,.$$

Then, analogously to the above, there exists a $g \in W^{2,2}(\Omega)$ such that

$$(4.2') \qquad g = \varphi_0\,, \quad g_n = \varphi_1 \quad \text{in the trace sense on } \Gamma\,,$$
$$\|g\|_{W^{2,2}(\Omega)} \leqq \text{const}\,[\|\varphi_0\|_{W^{3/2,2}(\Gamma)} + \|\varphi_1\|_{W^{1/2,2}(\Gamma)}]$$

(see [11]).

Remark 4.2. If $\varphi_0 = \varphi_1 = 0$ on Γ, the reformulation carried out below becomes superfluous, and in this case the boundary Γ is required to be only Lipschitzian. Moreover, the case of an only Lipschitzian boundary can be handled if φ_0, φ_1 in (2.5) are given a-priori as traces of a function in $W^{2,2}(\Omega)$. Then, by the trace theorem, $\varphi_0 \in W^{1,q}(\Gamma)$, $\varphi_1 \in L^q(\Gamma)$ for $1 \leqq q < \infty$.

Proposition 4.1. *Suppose* (4.1) (*or* (4.1')) *is satisfied. Then there exists a unique function* $F \in W^{2,2}(\Omega)$ *such that*:

1. $F - g \in W_0^{2,2}(\Omega)$,
2. $(F, \psi)_{W_0^{2,2}(\Omega)} = 0$ *holds for all* $\psi \in W_0^{2,2}(\Omega)$.

Moreover, it holds the estimate

$$(4.3) \qquad \|F\|_{W^{2,2}(\Omega)} \leqq \text{const}\,\|g\|_{W^{2,2}(\Omega)}\,.$$

The proof of Proposition 4.1 is readily obtained from [11]. The estimate in (4.2) (or (4.2')) and (4.3) yield

$$(4.4) \qquad \|F\|_{W^{2,2}(\Omega)} \leqq \text{const}\; T_1(\varphi_0, \varphi_1)\,.$$

Remark 4.3. The function F represents the Airy stress function of the associated linear plane stress problem.

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[1]) See [11] for the definition of the class $\mathfrak{N}^{(2),1}$.

One can state the following condition upon F:

$$(+)\qquad \int_{\Omega} \left[(F_{xy}u_y - F_{yy}u_x)\, u_x + (F_{xy}u_x - F_{xx}u_y)\, u_y \right] \mathrm{d}x\, \mathrm{d}y \leqq 0 \quad \text{for all} \quad u \in V\,.$$

It is easy to see that $(+)$ is satisfied in the case of a "unilateral" tension, i.e., if

$$F_{xx} = F_{xy} = 0\,, \quad F_{yy} > 0 \quad \text{a.e. in } \Omega\,,$$

or in the case of a "bilateral" tension, i.e., if

$$F_{xx}m_y^2 + F_{yy}m_x^2 - 2F_{xy}m_xm_y \geqq 0 \quad \text{a.e. in } \Omega$$
$$\text{for any unit vector } m = (m_x, m_y)$$

(cf. Naumann [10] for the case of compression). Note that the uniform tension studied by Fife [3] ($\Gamma_3 = \varnothing$) corresponds with $F_{xx} = F_{yy} = \text{const} > 0$, $F_{xy} = 0$.

We set $\Phi = F + f$ where $f \in W_0^{2,2}(\Omega)$. Clearly, Φ satisfies the boundary conditions (2.5), and it remains to determine the function f.

Assuming (4.1) (or (4.1')), Definition 3.1 now takes the following form.

Definition 3.1'. *The pair w, f is called an excess variational solution of boundary value problem I if*:

1° $w \in V$, $f \in W_0^{2,2}(\Omega)$,

2° *the identity*

$$(4.5)\qquad A(w, \varphi) + a(w, \varphi) = B(F, w; \varphi) + B(f, w; \varphi) + \\ + \int_{\Gamma_2} m_2\varphi_n \, \mathrm{d}s + \int_{\Gamma_3} m_3\varphi_n \, \mathrm{d}s + \int_{\Gamma_3} t_3\varphi \, \mathrm{d}s + \\ + \sum_{i=1}^{r} h_i\, \varphi(x(s_i), y(s_i)) + \langle q, \varphi \rangle$$

holds for all $\varphi \in V$ (F according to Proposition 4.1),

3° *the identity*

$$(4.6)\qquad (f, \psi)_{W_0^{2,2}(\Omega)} = -B(w, w; \psi)$$

holds for all $\psi \in W_0^{2,2}(\Omega)$.

If the pair w, f is an excess variational solution of boundary value problem I, the pair w, Φ, in which $\Phi = F + f$ (F according to Proposition 4.1), satisfies Definition 3.1, thus representing a variational solution of boundary value problem I.

The following theorem presents the main result of our paper.

Theorem. *Suppose the data m_2 and m_3, t_3 satisfy* (3.2) *and* (3.3), *respectively, and the right hand side q is in $[C(\overline{\Omega})]'$. Further, let the data φ_0, φ_1 and X, Y satisfy* (4.1) (or (4.1')) *and* (3.4), *respectively.*

Moreover, let the Condition $(+)$ *hold, or* $T_1(\varphi_0, \varphi_1)$ *be sufficiently small.*[1]). *Then boundary value problem I possesses at least one excess variational solution.*

5. PROOF OF THE THEOREM

We begin with the equivalent operator formulation of the variational setting of boundary value problem I.

Proposition 5.1.[2]) *The integral identities* (4.5), (4.6) *are equivalent to the system of operator equations*

$$w = Lw + C_1(f, w) + q^* \text{ in } V, \tag{5.1}$$

$$f = C_2(w, w) \text{ in } W_0^{2,2}(\Omega) \tag{5.2}$$

in which $w \in V$, $f \in W_0^{2,2}(\Omega)$.

Here C_1 *is a bilinear mapping* $W_0^{2,2}(\Omega) \times V \to V$, *and* L *a linear mapping* $V \to V$; q^* *is a fixed element in* V. *Further,* C_2 *is a bilinear mapping* $V \times V \to W_0^{2,2}(\Omega)$.

Proof. We first consider the identity (4.5).

Turning to the form $B(F, u; \varphi)$, let $u \in V$ be arbitrary. By Sobolev's Imbedding Theorem (see [11] and Hölder's inequality, one gets, for any $\varphi \in V$,

$$\left| \int_\Omega F_{xy} u_y \varphi_x \, dx \, dy \right| \leqq \left(\text{const} \left\|F\right\|_{W^{2,2}(\Omega)} \left\|u\right\|_{W^{1,4}(\Omega)}\right) \left\|\varphi\right\|_V{}^{3})$$

(note that (3.1) is used). Obviously, the same argument applies to the remaining integrals in $B(F, u; \varphi)$. Thus, the estimate

$$\left|B(F, u; \varphi)\right| \leqq \left(\text{const} \left\|F\right\|_{W^{2,2}(\Omega)} \left\|u\right\|_{W^{1,4}(\Omega)}\right) \left\|\varphi\right\|_V \tag{5.3}$$

holds for all $\varphi \in V$. By Riesz representation theorem for linear functionals, there exists a (uniquely determined) element $Lu \in V$ such that

$$(Lu, \varphi)_V = B(F, u; \varphi) \quad \text{for all} \quad \varphi \in V. \tag{5.4}$$

Let $v \in W_0^{2,2}(\Omega)$, $u \in V$. Proceeding as above, the estimate

$$\left|B(v, u; \varphi)\right| \leqq \left(\text{const} \left\|v\right\|_{W_0^{2,2}(\Omega)} \left\|u\right\|_{W^{1,4}(\Omega)}\right) \left\|\varphi\right\|_V \tag{5.5}$$

[1]) We do not give the numerical value of the bound restricting $T_1(\varphi_0, \varphi_1)$ since it involves the constant in (4.4).

[2]) Without particularly referring to it, the assumptions of our Theorem are assumed to be satisfied in deriving of all auxiliary material needed.

[3]) $\|u\|_V = (u, u)_V^{1/2}$, $u \in V$.

is readily verified for any $\varphi \in V$ so that we can conclude the existence of a unique $C_1(v, u) \in V$ such that

$$(C_1(v, u), \varphi)_V = B(v, u; \varphi) \quad \text{for all} \quad \varphi \in V. \tag{5.6}$$

Using again the trace theorem and Sobolev's Imbedding Theorem, we get for the remaining terms in (4.5) the estimate

$$\left| \int_{\Gamma_2} m_2 \varphi_n \,\mathrm{d}s + \int_{\Gamma_3} m_3 \varphi_n \,\mathrm{d}s + \int_{\Gamma_3} t_3 \varphi \,\mathrm{d}s + \sum_{i=1}^{r} h_i \,\varphi(x(s_i), y(s_i)) + \langle q, \varphi \rangle \right| \leqq \tag{5.7}$$
$$\leqq \text{const}\, K(m_2; m_3; t_3; h_1, \ldots, h_r; q) \, \|\varphi\|_V$$

where we have denoted

$$K(m_2; m_3; t_3; h_1, \ldots, h_r; q) \equiv \|m_2\|_{L^p(\Gamma_2)} + \|m_3\|_{L^p(\Gamma_3)} +$$
$$+ \|t_3\|_{L^1(\Gamma_3)} + \sum_{i=1} |h_i| + \|q\|_{[C(\bar{\Omega})]'} \,.$$

Thus, there exists a unique element $q^* \in V$ such that

$$(q^*, \varphi)_V = \int_{\Gamma_2} m_2 \varphi_n \,\mathrm{d}s + \int_{\Gamma_3} m_3 \varphi_n \,\mathrm{d}s + \int_{\Gamma_3} t_3 \varphi \,\mathrm{d}s + \tag{5.8}$$
$$+ \sum_{i=1}^{r} h_i \,\varphi(x(s_i), y(s_i)) + \langle q, \varphi \rangle$$

for all $\varphi \in V$.

Observing now the defining relations (5.4), (5.6), (5.8), the equivalence of the identity (4.5) to the operator equation (5.1) is readily seen.

Passing to the identity (4.6), let $u, \bar{u} \in V$ be arbitrary. Arguing as above, we get, for any $\psi \in W_0^{2,2}(\Omega)$,

$$\left| \int_{\Omega} u_{xy} \bar{u} \psi_x \,\mathrm{d}x \,\mathrm{d}y \right| \leqq \left(\text{const} \|u\|_V \|\bar{u}\|_{W^{1,4}(\Omega)}\right) \|\psi\|_{W_0^{2,2}(\Omega)}$$

Hence,

$$|B(u, \bar{u}; \psi)| \leqq \left(\text{const} \|u\|_V \|\bar{u}\|_{W^{1,4}(\Omega)}\right) \|\psi\|_{W_0^{2,2}(\Omega)} \tag{5.9}$$

for all $\psi \in W_0^{2,2}(\Omega)$ which implies the existence of a unique element $C_2(u, \bar{u}) \in W_0^{2,2}(\Omega)$ such that

$$(C_2(u, \bar{u}), \psi)_{W_0^{2,2}(\Omega)} = -B(u, \bar{u}; \psi) \quad \text{for all} \quad \psi \in W_0^{2,2}(\Omega)\,. \tag{5.10}$$

By (5.10), the equivalence of (4.6) to (5.2) is obvious.

P r o o f o f t h e T h e o r e m c o m p l e t e d. We introduce the (nonlinear) mapping $C : V \to V$, defined as follows

$$C(u) = -Lu - C_1(C_2(u, u), u)\,, \quad u \in V. \tag{5.11}$$

The system of operator equations (5.1), (5.2) *is equivalent to the* (*single*) *equation*

$$(*) \qquad u + C(u) = q^* \quad in \quad V.$$

In fact, if the pair w, f is a solution of (5.1), (5.2), by substitution, $u = w$ is a solution of (*). Conversely, if u satisfies (*), the pair w, f in which $w = u, f = C_2(u, u)$, is a solution of (5.1), (5.2). Thus, by Proposition 5.1, our Theorem will be proved if we establish the existence of a solution of (*).

To this end we use the following abstract theorem (see e.g. [2]):

Let H be a (*separable*) *Hilbert space* (*with the scalar product* $(\ ,\ )$ *and the norm* $\|\ \| = (\ ,\ )^{1/2}$), *and let C be a completely continuous*[1]) *mapping of H into H. Moreover, let*

$$\lim_{\|u\| \to \infty} \frac{((I + C)\, u, u)}{\|u\|} = +\infty\,.$$

Then $I + C$ maps H onto H.

We set $H = V$ and verify the conditions of the abstract theorem above for the operator C defined by (5.11).

a) *Complete continuity of C.* The estimate

$$(5.12) \qquad \|L(u - v)\|_V \leqq \text{const}\ \|F\|_{W^{2,2}(\Omega)}\ \|u - v\|_{W^{1,4}(\Omega)}$$

holding for all $u, v \in V$, follows immediately from (5.3). Next, the identity

$$B(u, v; \psi) = B(v, u; \psi)$$

in which $u, v \in V$, $\psi \in W_0^{2,2}(\Omega)$ are arbitrary, is easily obtained by integration by parts (cf. [10]). Hence,

$$C_2(u, v) = C_2(v, u) \quad \text{for all} \quad u, v \in V.$$

Using this symmetry property, we get by virtue of (5.5) and (5.9)

$$(5.13) \qquad \begin{aligned} &\|C_1(C_2(u, u), u) - C_1(C_2(v, v), v)\|_V \leqq \\ &\leqq \text{const}\,(\|u\|_V^2 + \|v\|_V^2)\ \|u - v\|_{W^{1,4}(\Omega)} \end{aligned}$$

for all $u, v \in V$.

Using Sobolev's Imbedding Theorem and the trace theorem, the continuity of C follows from (5.12)–(5.13).

Let $\{u_j\}$ be any bounded sequence in V. Since the imbedding $W^{2,2}(\Omega) \subset W^{1,4}(\Omega)$ is compact(see[11]), there exists a subsequence $\{u_{j_n}\}$ of $\{u_j\}$ such that $u_{j_n} \to u$ strongly in $W^{1,4}(\Omega)$ as $n \to \infty$. By (5.12)–(5.13), $C(u_{j_n}) \to C(u)$ strongly in V as $n \to \infty$.[2])

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[1]) A mapping of H into H is said to be completely continuous if it is continuous and maps each bounded set of H into a compact set.

[2]) In fact we have got a slightly stronger property for C than the compactness: If $\{u_j\}$ is any sequence lying in a bounded subset of V then there exists a subsequence $\{u_{j_n}\}$ of $\{u_j\}$ and $u \in V$ such that $C(u_{j_n}) \to C(u)$ strongly in V as $n \to \infty$.

b) Coerciveness. Observing assumption (i) we get by virtue of Condition (+)

$$(Lu, u)_V = B(F, u; u) \leqq 0 \quad \text{for all} \quad u \in V .$$

Dispensing with Condition (+) we find an $\varepsilon_1 > 0$ such that $\|Lu\|_V \leqq \frac{1}{2}\|u\|_V$ for all $u \in V$ and $T_1(\varphi_0, \varphi_1) \leqq \varepsilon_1$ (cf. (5.3); estimate (4.4) is used). Thus,

$$(Lu, u)_V \leqq \tfrac{1}{2}\|u\|_V^2 \quad \text{for all} \quad u \in V$$

if Condition (+) holds, or if $T_1(\varphi_0, \varphi_1) \leqq \varepsilon_1$.

An integration by parts leads to the identity

$$B(v, u; \varphi) = B(\varphi, u; v)$$

which is valid for all $v \in W_0^{2,2}(\Omega)$, $u, \varphi \in V$ (cf. [10]). With the use of (5.10) we obtain

$$(5.15) \qquad (C_1(C_2(u, u), u), u)_V \leqq 0 \quad \text{for all} \quad u \in V .$$

Gathering (5.14)–(5.15) one gets

$$(5.16) \qquad ((I + C)\, u, u)_V \geqq \tfrac{1}{2}\|u\|_V^2 \quad \text{for all} \quad u \in V .$$

Taking into account Proposition 5.1, the existence of an excess variational solution of boundary value problem I follows by application of the abstract theorem to (*). The proof of our Theorem is complete.

Corollary. *Suppose the assumptions of the Theorem are fulfilled. Then:*

(i) *Each excess variational solution of the boundary value problem I satisfies the estimates*

$$(5.17) \qquad \|w\|_V \leqq \text{const}\, K(m_2; m_3; t_3; h_1, \ldots, h_r; q) ,$$
$$\|f\|_{W_0^{2,2}(\Omega)} \leqq \text{const}\, [K(m_2; m_3; t_3; h_1, \ldots, h_r; q)]^2 .$$

(ii) *For sufficiently small $K(m_2; m_3; t_3; h_1, \ldots, h_r; q)$, the excess variational solution of the boundary value problem I is unique.*

Proof. (i) The estimate (5.16) yields $\|u\|_V \leqq 2\|q^*\|_V$ for each solution u of (*). By (5.7),

$$\|u\|_V \leqq \text{const}\, K(m_2; m_3; t_3; h_1, \ldots, h_r; q) .$$

Setting $w = u$, $f = C_2(u, u)$ and using (5.9), the estimates asserted are readily seen.

(ii) Using (5.12), (5.14) as well as (5.13) one obtains

$$\|(I + C)\, u - (I + C)\, v\|_V \leqq [\tfrac{1}{2} - \text{const}\, (\|u\|_V^2 + \|v\|_V^2)]\, \|u - v\|_V$$

for all $u, v \in V$. This implies that the solution of (*) is unique in a ball with sufficiently small radius, centered at the origin. By virtue of (5.17) the assertion (ii) is verified.

Using (4.4) we get for the stress function Φ the estimate

$$\|\Phi\|_{W_0^{2,2}(\Omega)} \leqq \text{const}\, \{T_1(\varphi_0, \varphi_1) + [K(m_2; m_3; t_3; h_1, \ldots, h_r; q)]^2\} .$$

Remark 5.1. The uniqueness of a variational solution of boundary value problem I for large $\|w\|_V$ remains unsettled. However, results about the bifurcation of non-trivial solutions in the case of compression (cf. [1], [7], [10]) show that uniqueness in general cannot be expected.

Remark 5.2. A careful analysis of the technique of the proof developed above shows that the system

$$\Delta^2 w = [\Phi, w] + k_1 \Phi_{xx} + k_2 \Phi_{yy} + q \quad \text{in} \quad \Omega\,,$$

$$\Delta^2 \Phi = -\tfrac{1}{2}[w, w] - k_1 w_{xx} - k_2 w_{yy} \quad \text{in} \quad \Omega$$

(which describes the equilibrium of a thin elastic shallow shell with curvatures k_1, k_2) with the boundary conditions considered above, possesses a variational solution, provided the curvatures are sufficiently small.

Remark 5.3. It is easy to realize that also the boundary conditions

$$w_n = 0\,, \quad T(w) + k_4 w = t_4 \quad \text{on} \quad \Gamma_4$$

could be included without difficulties. Conditions of this type will be considered in Part II.

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Souhrn

NEHOMOGENNÍ OKRAJOVÉ ÚLOHY PRO KÁRMÁNOVY ROVNICE I.

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V článku se dokazuje existence řešení okrajové úlohy pružné tenké desky pro jistou třídu kombinovaných okrajových podmínek. Deska může být zčásti vetknutá, zčásti podepřená a zčásti pružně vetknutá a pružně podepřená nebo volná. Tato volná část okraje a část podepřená je obecně zatížena momenty a volná část i posouvajícími silami. Na celém okraji desky, který může mít rohy, působí též zatížení v rovině desky. Okrajové podmínky pro průhyb vylučují však vždy možnost pohybu desky jako tuhého celku kolmo k její rovině.

Variační formulace problému je převedena na operátorovou rovnici, pro kterou platí abstraktní existenční věta. Řešení existuje, když okrajové zatížení, působící v rovině desky, vyvolává v desce tahové namáhání nebo je dostatečně malé. Řešení je jediné, když příčné zatížení desky je dostatečně malé.

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