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POLAR GRAPHS AND RAILWAY TRAFFIC

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At the Conference on Graph Theory at Štiřín in May 1972, F. Zítek [5] has introduced the concept of polar graph.

A polar graph (undirected, without loops and multiple edges) is an ordered quintuple $\langle V, E, P, \varkappa, \lambda \rangle$, where V, E, P are sets, $\varkappa$ and λ are mappings of the set V and E, respectively, into the set of unordered pairs of distinct elements of P such that the following conditions are satisfied:

(1) For each $u \in V$, $v \in V$, $u \neq v$, we have $\varkappa(u) \cap \varkappa(v) = \emptyset$.

(2) For each $e \in E, f \in E$, $e \neq f$, we have $\lambda(e) \neq \lambda(f)$.

(3) To each $p \in P$ there exists $v \in V$ so that $p \in \varkappa(v)$.

The elements of the set V, E and P are called vertices, edges and poles, respectively. If $p \in P$, $v \in V$, $p \in \varkappa(v)$, we say that the pole p belongs to the vertex v. If $p \in P$, $e \in E$, $p \in \lambda(e)$, we say that the edge e is incident with the pole p. If an edge e is incident with a pole p which belongs to a vertex v, we say that e is incident with v.

Thus a polar graph can be imagined as an undirected graph, at each of whose vertices the set of edges incident with this vertex is decomposed into two disjoint subsets (not necessarily non-empty). Each of these subsets is assigned a new element called pole. Thus each vertex has exactly two poles and each edge incident with this vertex is incident exactly with one of these poles. When drawing a polar graph we draw the vertices in the form of magnetic needles; the poles of such a needle denote the poles of the vertex.

Elementary properties of polar graphs are described in the papers [3] and [4].

At the above mentioned Conference at Štiřín J. Černý [1] suggested that polar graphs may be important for solving some problems of the railway traffic. The aim of this paper is to show such a solution. The terminology of the graph theory used in the paper is that of [2].

The paper describes a simplified mathematical model. It is no detailed study; the author has no sufficient special knowledge about the railway traffic.

Let us have a certain part of trackage and a certain number of trains on it. The task is to shunt these trains to other given places in the shortest time possible. We shall show how it is possible to use polar graphs for solving this problem.

First we construct a polar graph G_0. The vertices of this graph are switches in the given part of the trackage, the ends of the dead-end tracks and the points at which

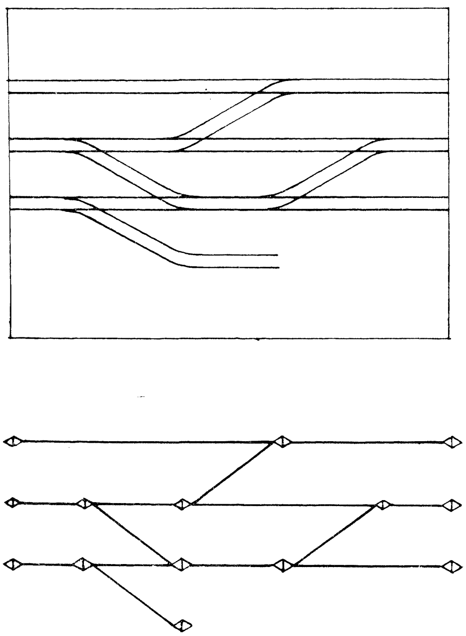

Fig. 1.

particular tracks leave this part. For the sake of simplicity let us suppose that in the given part of the trackage no turn tables, no slip switches and no normal track crossings occur. The edge joining two vertices of the graph G_0 is the part of the track between these two vertices which contains no switch. Two edges incident with the same vertex v denoting a switch are incident with distinct poles of this vertex if and only if a train can move from one of them to the other through the vertex (switch) v without changing the direction of its motion; in the opposite case such edges are incident with the same pole of the vertex v. If a vertex v is an end of a dead-end track or a point at which a track leaves our part of the trackage, it is incident only with one edge, therefore one of its poles is incident with this edge, the other is isolated, i.e., it is incident with no edge. (The part of the trackage is always delimited so that no switch lies exactly on its frontier.) In Fig. 1 a small part of a trackage and the corresponding graph G_0 is drawn.

We see that each vertex of the graph G_0 is incident either exactly with one edge (if it is an end of a dead-end track or a point in which a track leaves the delimited part of the trackage), or exactly with three edges so that one of its poles is incident with two edges, the other with one edges. Only in the case of a slip switch we should obtain a vertex incident with four edges, each of its poles being incident with two edges (Fig. 2). Now we do not consider this case for the sake of simplicity.

If no fly-over occurs, the graph G_0 is planar, in the opposite case it can be non-planar.

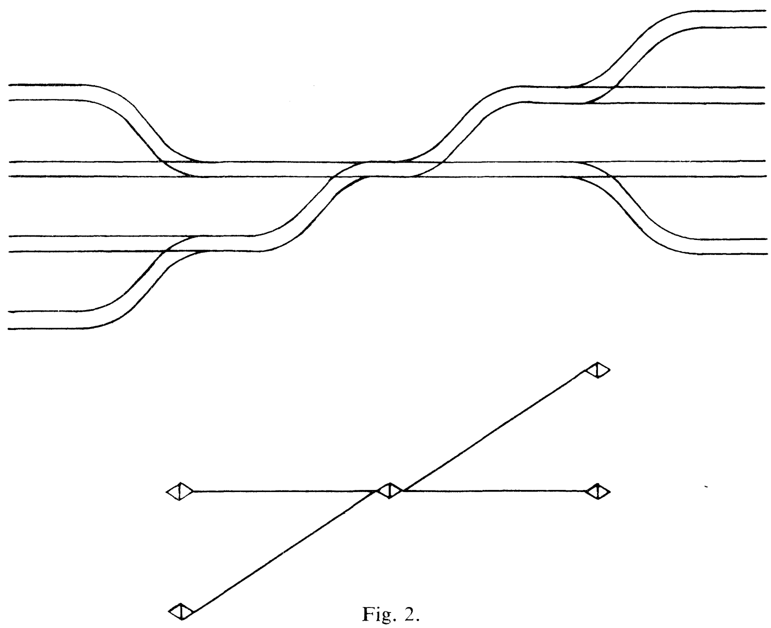

Fig. 2.

Now it is necessary to consider various positions in which a given train can occur. Evidently, theoretically their number is infinite. However, in practice it suffices to consider only a finite number of such positions. First we shall suppose that the trains are so short that each track segment between two vertices of the graph G_0 can be decomposed into a finite number of non-overlapping segments of approximatively equal length so that the position of a train is given by determining the segment where this (whole) train occurs. If the track segment between two vertices u, v of the graph G_0 is divided into d segments in this way, we replace this edge by a simple path of the length d connecting the vertices u and v; the poles of the inner vertices of this path will be chosen so that each of them is incident exactly with one edge (these inner vertices do not belong to the vertex set of the graph G_0 and each of them belongs only to one of the paths). If we do this with each edge of the graph G_0, we obtain a new polar graph G_1. The construction of this graph is shown in Fig. 3.

In this figure the graph G_0 is drawn and at each of its edges the number of segments into which the corresponding segment is divided is given; below this graph we see the corresponding graph G_1.

Then we find to the graph G_1 its line graph G_2. The concept of the line graph to a given graph was studied by various authors for undirected and directed graphs. Here we shall give its definition for polar graphs.

Let G be a polar graph. The line graph (or derived graph) ∂G of the graph G is the polar graph whose vertices are edges of the graph G and in which two vertices are joined by an edge if and only if the edges of the graph G corresponding to these vertices are incident with distinct poles of the same vertex of the graph G; if uv, uw are edges of the graph ∂G, then these edges are incident with the same pole of the

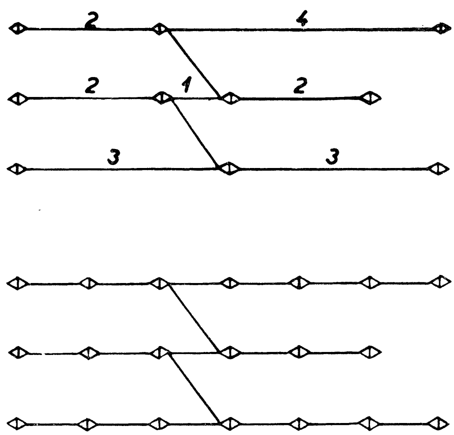

Fig. 3.

vertex u if and only if the corresponding edges v and w of the graph G are incident with the same end vertex of the edge u in G.

In Fig. 4, construction of the line graph to a given polar graph is shown.

Now we shall adapt G_2 into the non-polar directed graph G_3. First let us choose the time which a train needs to pass from one vertex of the graph G_2 into another vertex joined with it as the time unit. Further, for the sake of simplicity assume that the time t needed by a train to stop and take-off in the opposite direction is an integer in the chosen time unit. Now at each vertex u of the graph G_2, denote its poles by $p_1(u)$, $p_2(u)$; this notation is chosen arbitrarily at each vertex. To each vertex u of the graph G_2 a pair of distinct vertices $u(1)$ and $u(2)$ of the graph G_3 will correspond in a one-to-one manner. For each u of G_2 in G_3 we connect the vertices $u(1)$ and $u(2)$ by two vertex-disjoint directed arcs $D_1(u)$ and $D_2(u)$ from which one goes from $u(1)$ into $u(2)$, the other from $u(2)$ into $u(1)$ and which both have the same

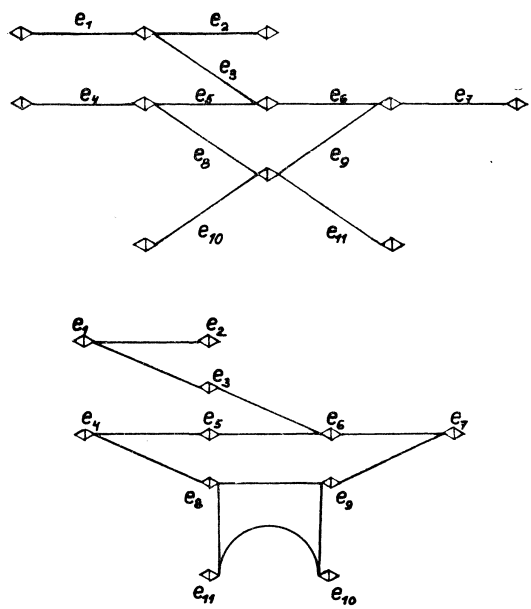
e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
e_9
e_{10}
e_{11}

Fig. 4.

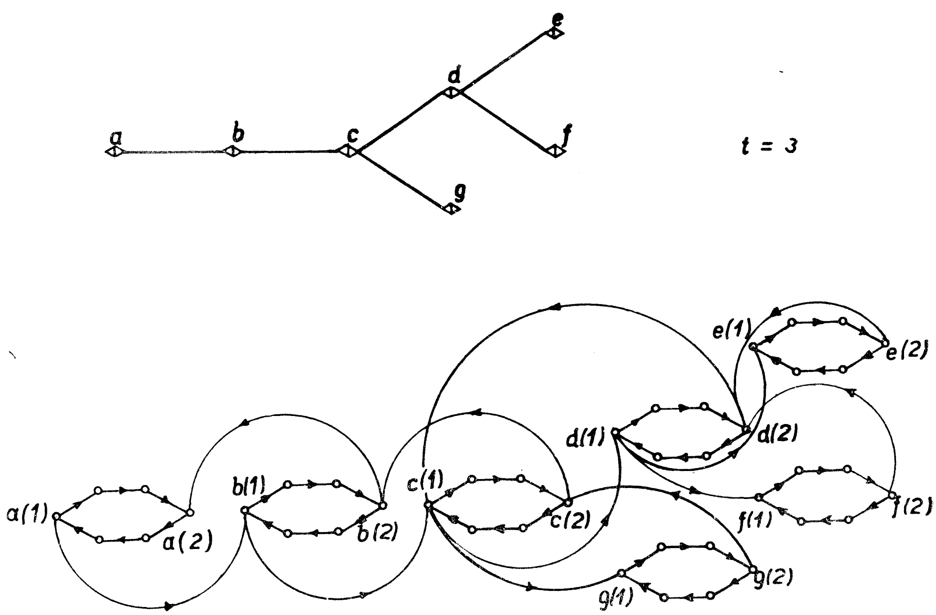
a
b
c
d
e
f
g
$t = 3$
a(1)
a(2)
b(1)
b(2)
c(1)
c(2)
d(1)
d(2)
e(1)
e(2)
f(1)
f(2)
g(1)
g(2)

Fig. 5.

length t. Further, if u, v are distinct vertices of the graph G_2, then a directed edge goes in G_3 from $u(i)$ into $v(j)$ (i, j are equal to 1 or 2) if and only if the poles $p_{1-i}(u)$, $p_j(v)$ are joined in G_2 by an edge. Obviously no pair of the above described directed arcs has a common inner vertex and their vertices are different from all $u(1)$ and $u(2)$.

Construction of the graph G_3 to a given graph G_2 is shown in Fig. 5.

The set of vertices of the graph G_3 consisting of the vertices $u(1)$, $u(2)$ and of all vertices of the directed arcs $D_1(u)$, $D_2(u)$ will be denoted by $M(u)$. If $x \in M(u)$, $y \in M(u)$ for the same u, we write $x\xi y$; the symbol ξ denotes a binary relation.

The vertices of the graph G_3 correspond again to the particular positions in which a train can occur; but now we distinguish also from which side the train has come to the given segment. All vertices of the set $M(u)$ correspond to the track segment which was denoted in G_2 as the vertex u. However, if we imagine that a train is in the vertex $u(1)$, it means either that it has just come into u along the edge incident with the pole $p_1(u)$ or it is just leaving u along the edge incident with the pole $p_2(u)$; analogously for $u(2)$. The "motion" of a train along the directed are $D_1(u)$ from $u(1)$ into $u(2)$ signifies here the fact that at the segment u the train must first stop and then take-off in the opposite direction; the time t needed for it is equal to the time needed for traversing t segments without changing the direction. Thus particular vertices of the graph G_3 correspond to various phases of the shunting, i.e., to situations in which a train occurs always after an integral number of time units.

From the above, we conclude

Theorem 1. *Let u, v be two track segments (corresponding to vertices of the graph G_2). The time needed by a train for traversing from the segment u to the segment v is equal to the length of the shortest possible directed path in the graph G_3 going from a vertex of the set $M(u)$ into a vertex of the set $M(v)$.*

(Here we neglect the initial take-off and the final stopping; but it is easy to introduce a necessary correction.)

If in the given part of the trackage there are n trains $\mathscr{V}_1, \ldots, \mathscr{V}_n$, $n \geqq 2$, which are to be shunted to given track segments, we use the graph G_4. The vertex set of the graph G_4 consists of all ordered n-tuples of vertices of the graph G_3 such that no two elements of an n-tuple are in the relation ξ. Now let $[x_1, \ldots, x_n]$, $[y_1, \ldots, y_n]$ be two vertices of the graph G_4. A directed edge goes from the vertex $[x_1, \ldots, x_n]$ into the vertex $[y_1, \ldots, y_n]$ in G_4 if and only if for each $i = 1, \ldots, n$ either $x_i = y_i$, or a directed edge goes in G_3 from x_i into y_i.

Now the vertex $[x_1, \ldots, x_n]$ denotes the shunting phase at which the i-th train is in the phase x_i for $i = 1, \ldots, n$. If a directed edge goes from $[x_1, \ldots, x_n]$ into $[y_1, \ldots, y_n]$, this means that it is possible to pass from the phase $[x_1, \ldots, x_n]$ to the phase $[y_1, \ldots, y_n]$ in a time unit. This is not quite exact, because we consider also transfers from one phase to another during which some trains do not move at all while we neglect the fact that the stopping and taking-off again in the same direction could need more than the time unit. Nevertheless, in practice it would be probably possible only to slow down instead of stopping for one or a few time units.

The construction of the graph G_4 implies the following.

Theorem 2. *Let there be n trains $\mathscr{V}_1, \ldots, \mathscr{V}_n$ in the trackage and let the train $\mathscr{V}_i$ for $i = 1, \ldots, n$ stand on the segment u_i and have to be shunted to the track segment v_i. The time needed for this shunting is equal to the length of the shortest possible directed path in G_4 going from the vertex $[x_1, \ldots, x_n]$ into the vertex $[y_1, \ldots, y_n]$, where $x_i \in M(u_i)$, $y_i \in M(v_i)$ for $i = 1, \ldots, n$.*

In this way the task is solved in the case when all trains are sufficiently short that the tracks can be divided into segments not containing switches so that no train is longer than one segment.

If this condition is not satisfied, the course of solution is more complicated, but analogous to the preceding one. Let again n trains $\mathscr{V}_1, \ldots, \mathscr{V}_n$ be given. Let us divide the tracks into segments as in the preceding case. We express the lengths of particular trains as integral multiples of the length of one segment; we round always upwards. Thus let l_i be the length of the train $\mathscr{V}_i$. The train $\mathscr{V}_i$ does not occupy only one edge in the graph G_1, but a whole heteropolar path of the length l_i. (A heteropolar path is a path in a polar graph at which each pole of an inner vertex is incident exactly with one edge of this path.) The terminal vertices of this path may coincide; then we have a heteropolar circuit of the length l_i.

Instead of the graph G_2 we construct polar graphs $H_1, \ldots, H_n$ so that for $i = 1, \ldots, n$ the vertex set of the graph H_i is the set of all heteropolar (simple) paths of the length l_i in the graph G_1 and two vertices are joined by an edge in H_i if and only if the intersection of the corresponding paths is a heteropolar path of the length $l_i - 1$. The vertices v and w are joined with the same pole of the vertex u if and only if both the corresponding intersections contain the same terminal vertex of the path corresponding to the vertex u.

To the graphs $H_1, \ldots, H_n$ we construct the graphs $H'_1, \ldots, H'_n$ analogously to the construction of the graph G_3 from the graph G_2. Instead of the graph G_4 we construct the graph G'_4 whose vertices are all possible ordered n-tuples $[x_1, \ldots, x_n]$, where x_i is a vertex of the graph H'_i for $i = 1, \ldots, n$ and the paths of the graph G_1 corresponding to different elements of an n-tuple have no edge in common. (Nevertheless, the same path of the graph G_1 may correspond to different vertices of the graph H'_i). The vertices are joined again analogously to the graph G_4 and a theorem analogous to Theorem 2 holds, which we are not going to introduce because of the complexity of its formulation.

The number of vertices of the graph G_4 is equal to $t^n r!/(r - n)!$, where r is the number of vertices of the graph G_2 (edges of the graph G_1). The number of vertices of the graph G'_4 depends on $l_1, \ldots, l_n$; it is evidently substantially greater, because the longer is a train, the greater is the number of positions in which it can occur.

It remains to add a remark on slip switches, normal track crossings and turn tables. If the trackage contains slip switches or normal track crossings, it is necessary in the graph G_4 or G_4' to omit the edges which correspond to transfers in the unit time from one shunting phase to another at which two trains would have to go simultaneously through the same slip swich or the same normal track crossing. Similarly one can proceed also in the case of turn tables; however, here it would be necessary to consider the time needed for the manipulation with the turn table. This could be made similarly as for the time needed for the change of direction by using directed paths of the corresponding length. These details will not be studied in this paper.

There exist also rules which forbid the shunted set of wagons to come even onto a track which is free (for example, if a train passing through the station can pass through it). If this is the case, this track is omitted in G_0 and all tracks having a common switch with it are considered as dead-end tracks.

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Souhrn

POLÁRNÍ GRAFY A ŽELEZNIČNÍ DOPRAVA

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V článku se ukazuje použití pojmu polárního grafu při řešení otázek souvisejících s přesunem vlaků v kolejišti. Pojem polárního grafu zavedl F. Zítek na konferenci o teorii grafů ve Štiříně v květnu 1972.

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