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ON THE SOLUTION OF THE DISPLACEMENT BOUNDARY-VALUE PROBLEM FOR ELASTIC-INELASTIC MATERIALS

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1. INTRODUCTION

The system of equations which describe the model with the so called internal state variables in continuum mechanics will be studied. The model itself was developed by Dillon and Kratochvíl in [1], [2]. Nečas and Kratochvíl gave the proof of existence and uniqueness of the solution of the traction problem ([3]). In this paper we study the displacement boundary-value problem using the method of [3]. At the same time, a simple technique due to Zacharias and Gajewski is applied which makes it possible to avoid the partition of the time interval in the course of solution.

Let the body before the deformation occupy a bounded domain Ω, $\Omega \subset R^3$. The body forces $F_i = F_i(x, t)$ as well as the displacement $\bar{u}_i = \bar{u}_i(x, t)$ on the boundary $\partial\Omega$ are given for $i = 1, 2, 3$, $t \in \langle 0, T \rangle$. Parameter t can be interpreted as time but not necessarily. We wish to determine the state in which the body finds itself after an elastic-inelastic deformation governed by the following equations:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0\,, \quad (i = 1, 2, 3) \tag{1}$$

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad (i, j = 1, 2, 3) \tag{2}$$

$$\varepsilon_{ij}^e = A_{ij}(\sigma, \alpha)\,, \quad (i, j = 1, 2, 3) \tag{3}$$

$$\varepsilon_{ij}^p(x, t) = \varepsilon_{ij}^{p,0}(x) + \int_0^t B_{ij}(\sigma(x, \tau), \alpha(x, \tau))\, d\tau\,, \quad (i, j = 1, 2, 3) \tag{4}$$

$$\alpha_l(x, t) = \alpha_l^0(x) + \int_0^t D_l(\sigma(x, \tau), \alpha(x, \tau))\, d\tau\,, \quad (l = 1, 2, \ldots, m) \tag{5}$$

$$u_i = \bar{u}_i \quad \text{on the frontier} \quad \partial\Omega \quad \text{of} \quad \Omega\,, \quad (i = 1, 2, 3)\,, \tag{6}$$

where $\varepsilon^{p,0}$ and α^0 are given functions.

Here $\sigma = \sigma(x, t)$ denotes the symmetric stress tensor, $\varepsilon = \varepsilon(x, t)$ is the symmetric small strain tensor composed of the plastic part ε^p and the elastic one ε^e. $u = u(x, t)$ is the displacement, $\alpha = \alpha(x, t)$ is an internal state variable, A, B and D are given functions of the variables σ and α connected with the physical properties of the material. (For the physical motivation see e.g. [1], [2].)

The relation (1) is the usual equilibrium equation, (2) is the condition of compatibility and (3) is a form of Hook's law. B and D in the system of equations (4) and (5) are the so called response functions which characterize the historical development of material.

The question is under what conditions on Ω, A, B, D, $\bar{u}$ and F there exist unique σ, u, ε^e, ε^p and α which solve the system of equations (1)–(6).

2. NOTATIONS AND PRELIMINARIES

a) Notations.

Let Ω be a bounded domain in R^3 with a generic point x and Lipschitzian boundary $\partial\Omega$.

For $\lambda \geqq 0$ and a Banach space X define $C_\lambda(\langle 0, T\rangle; X)$ to be the space of all continuous functions $\varphi : \langle 0, T\rangle \to X$ with the norm

(7) $$|||\varphi|||_{X,\lambda} = \sup_{t\in\langle 0,T\rangle} (e^{-\lambda t} \cdot \|\varphi[t]\|_X)$$

where $\|\cdot\|_X$ denotes the norm in the space X. (If it is necessary to express the dependence of an element $\varphi \in C_\lambda(\langle 0, T\rangle; X)$ on t, it will be done by means of square brackets). $C_\lambda(\langle 0, T\rangle; X)$ is a Banach space.

In case of $X = [L_2(\Omega)]^m = L_2(\Omega) \times \ldots \times L_2(\Omega)$ (m-times) we write simply $\|\cdot\|_m$ instead of $\|\cdot\|_{[L_2(\Omega)]^m}$, $|||\cdot|||_{m,\lambda}$ instead of $|||\cdot|||_{[L_2(\Omega)]^m,\lambda}$.

Remark 1. For any two non-negative numbers λ_1, λ_2 the norms $|||\cdot|||_{X,\lambda_1}$ and $|||\cdot|||_{X,\lambda_2}$ are equivalent. Nevertheless, the choice of λ will be helpful in the following. In case that the special choice of λ has no importance we omit the index λ in the expressions $C_\lambda(\langle 0, T\rangle; X)$ and $|||\cdot|||_{X,\lambda}$.

Let S be the Hilbert space of all symmetric tensor functions $\Theta = (\Theta_{ij})_{ij=1}^3$, $\Theta_{ij} = \Theta_{ji}$, $\Theta_{ij} \in L_2(\Omega)$ with the scalar product

$$(\omega, \Theta)_S = \sum_{i,j=1}^{3} (\omega_{ij}, \Theta_{ij})_{L_2(\Omega)} \,.$$

Further, denote $W = [W^{1,2}(\Omega)]^3$ and $W_0 = [W_0^{1,2}(\Omega)]^3$. For tensors and vectors we write often $|\alpha|$ instead of $\sum_{l=1}^{m} |\alpha_l|$ or instead of the equivalent norm $(\sum_{l=1}^{m} \alpha_l^2)^{1/2}$, $|\omega|$ instead of $\sum_{i,j=1}^{3} |\omega_{ij}|$ or $(\sum_{i,j=1}^{3} |\omega_{ij}|^2)^{1/2}$ etc.

b) Assumptions.

Suppose that

$$F \in C(\langle 0, T\rangle; [L_2(\Omega)]^3)\,, \tag{8}$$

$$\bar{u} \in C(\langle 0, T\rangle; W)\,. \tag{9}$$

The function

$$A : R^9 \times R^m \to R^9\,; \quad A : (\sigma, \alpha) \to A(\sigma, \alpha)\,, \quad A_{ij} = A_{ji}$$

is supposed to be continuous in its domain and such that there exists a function

$$P : R^9 \times R^m \to R^1$$

for which

$$A_{ij}(\sigma, \alpha) = \frac{\partial P(\sigma, \alpha)}{\partial \sigma_{ij}}\,, \quad (i, j = 1, 2, 3)\,, \tag{10}$$

$$\sum_{i,j,k,l=1}^{3} \left|\frac{\partial^2 P(\sigma, \alpha)}{\partial \sigma_{ij}\, \partial \sigma_{kl}}\right| + \sum_{i,j=1}^{3} \sum_{l=1}^{m} \left|\frac{\partial^2 P(\sigma, \alpha)}{\partial \sigma_{ij}\, \partial \alpha_l}\right| + \sum_{l,n=1}^{m} \left|\frac{\partial^2 P(\sigma, \alpha)}{\partial \alpha_l\, \partial \alpha_n}\right| \leqq c_1\,, \tag{11}$$

$$\frac{\partial^2 P(\sigma, \alpha)}{\partial \sigma_{ij}\, \partial \sigma_{kl}} \cdot \xi_{ij}\xi_{kl} \geqq c_2 \sum_{i,j=1}^{3} \xi_{ij}^2 \tag{12}$$

take place for all $\sigma \in R^9$, $\xi \in R^9$ and $\alpha \in R^m$. It means that A has potential P.

The symmetric tensor function $B : R^9 \times R^m \to R^9$ and the vector function $D : R^9 \times \times R^m \to R^m$ satisfy the following conditions:

$$\sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} \left|\frac{\partial B_{ij}(\sigma, \alpha)}{\partial \sigma_{kl}}\right| + \sum_{n=1}^{m} \left|\frac{\partial B_{ij}(\sigma, \alpha)}{\partial \alpha_n}\right| \right) \leqq c_3\,, \tag{13}$$

$$\sum_{n=1}^{m} \left(\sum_{k,l=1}^{3} \left|\frac{\partial D_n(\sigma, \alpha)}{\partial \sigma_{kl}}\right| + \sum_{k=1}^{m} \left|\frac{\partial D_n(\sigma, \alpha)}{\partial \alpha_k}\right| \right) \leqq c_3\,. \tag{14}$$

c) An auxiliary result.

Let $\beta : W \to S$ be defined by

$$\beta_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad (i, j = 1, 2, 3)\,. \tag{15}$$

Put $K_0 = \beta(W_0)$.

Lemma 1. K_0 *is a closed subspace of* S.

Proof. The inequalities

$$\|\beta(v)\|_S \leqq \|v\|_{W_0}, \quad \forall v \in W_0 \tag{16}$$

and

$$\|\beta(v)\|_S \geqq \frac{1}{\sqrt{2}} \|v\|_{W_0}, \quad \forall v \in W_0 \tag{17}$$

prove the fact that K_0 and W_0 are linearly homeomorphic. The estimate (16) is trivial. As for (17), we can either use the general result of Hlaváček and Nečas, [4], or prove our special case in the following way: Let $u \in [C_0^\infty(\Omega)]^3$. Using Green's formula and the fact that u vanishes on $\partial\Omega$ we can write

$$\int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \,\mathrm{d}x = - \int_\Omega u_i \frac{\partial^2 u_j}{\partial x_j \partial x_i} \,\mathrm{d}x = \int_\Omega \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} \,\mathrm{d}x .$$

Thus

$$\|\beta(u)\|_S^2 = \sum_{i,j=1}^{3} \frac{1}{2} \int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 \mathrm{d}x + \frac{1}{2} \int_\Omega \left(\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}\right)^2 \mathrm{d}x \geqq \frac{1}{2} \|u\|_{W_0}^2 .$$

This inequality together with the density of $[C_0^\infty(\Omega)]^3$ in W_0 yields (17).

3. MATHEMATICAL FORMULATION

Denote the resulting displacement by u and put

$$v = u - \bar{u} . \tag{18}$$

Then the boundary condition (6) can be written in the form

$$v \in C(\langle 0, T \rangle ; W_0) . \tag{19}$$

From (2), using (18) and (15), we get the compatibility equation in the form

$$\varepsilon^e + \varepsilon^p - \beta(\bar{u}) = \beta(v) . \tag{20}$$

Let now H_0 be an orthogonal complement to K_0 in S. From (18)–(20) and Lemma 1 we obtain that (20) can br written as

$$\int_\Omega \{\varepsilon_{ij}^e + \varepsilon_{ij}^p - \beta_{ij}(\bar{u})\} h_{ij} \,\mathrm{d}x = 0 , \quad \forall h \in H_0, t \in \langle 0, T \rangle . \tag{21}$$

The weak form of the equilibrium equation is

$$\int_\Omega \sigma_{ij}\,\beta_{ij}(\varphi)\,\mathrm{d}x = \int_\Omega F_i\varphi_i\,\mathrm{d}x\,, \quad \forall\varphi \in W_0,\ t \in \langle 0, T\rangle\,. \tag{22}$$

PROBLEM. *Find the elements* $v \in C(\langle 0, T\rangle; W_0)$, σ, ε^p, $\varepsilon^e \in C(\langle 0, T\rangle; S$ *and* $\alpha \in C(\langle 0, T\rangle; [L_2(\Omega)]^m)$ for which (21), (22), (3)–(5) take place.

To solve this problem we use the following process: For $\sigma \in C(\langle 0, T\rangle; S)$ we calculate α from (5), after that we get ε^p from (4). Denote these α, ε^p by $\alpha(\sigma)$, $\varepsilon^p(\sigma)$. Then we find such an $\omega \in C(\langle 0, T\rangle; S)$ that

(i) ω substituted for σ satisfies (22),

(ii) ω satisfies the condition

$$\int_\Omega \{A_{ij}(\omega, \alpha(\sigma)) + \varepsilon^p_{ij}(\sigma) - \beta_{ij}(\bar{u})\}\,h_{ij}\,\mathrm{d}x = 0\,, \quad \forall h \in H_0\,, \quad t \in \langle 0, T\rangle\,. \tag{21, a}$$

It will be proved that for each $\sigma \in C(\langle 0, T\rangle; S)$ there exists a unique $\omega = \omega(\sigma)$ such that the process defines an operator $\omega : C(\langle 0, T\rangle; S) \to C(\langle 0, T\rangle; S)$. Further, the operator ω is contractive as an operator from $C_{\bar{\lambda}}(\langle 0, T\rangle; S)$ to $C_{\bar{\lambda}}(\langle 0, T\rangle; S)$ for some $\bar{\lambda} \geqq 0$. Thus the existence and uniqueness of the solution of the PROBLEM follow easily from the Banach fixed-point theorem.

4. SOLUTION OF THE PROBLEM

Let us denote

$$\mathscr{D}(\sigma, \alpha)\,[t] = \int_0^t D(\sigma[\tau], \alpha[\tau])\,\mathrm{d}\tau\,, \tag{23}$$

$$\mathscr{B}(\sigma, \alpha)\,[t] = \int_0^t B(\sigma[\tau], \alpha[\tau])\,\mathrm{d}\tau\,. \tag{24}$$

Lemma 2.

$$\mathscr{D} : C(\langle 0, T\rangle; S) \times C(\langle 0, T\rangle; [L_2(\Omega)]^m) \to C(\langle 0, T\rangle; [L_2(\Omega)]^m)\,, \tag{25}$$

$$\mathscr{B} : C(\langle 0, T\rangle; S) \times C(\langle 0, T\rangle; [L_2(\Omega)]^m) \to C(\langle 0, T\rangle; S)\,. \tag{26}$$

Proof. From (14) it follows that for all $\sigma \in R^9$, $\alpha \in R^m$

$$|D(\sigma, \alpha)| \leqq c_4(1 + |\sigma| + |\alpha|)\,. \tag{27}$$

Hence for each $\sigma \in S$, $\alpha \in [L_2(\Omega)]^m$ we get $D(\sigma, \alpha) \in [L_2(\Omega)]^m$.

Again from (14) we obtain for each $\sigma \in C(\langle 0, T\rangle; S)$, $\alpha \in C(\langle 0, T\rangle; [L_2(\Omega)]^m)$, $t_1, t_2 \in \langle 0, T\rangle$,

$$\|D(\sigma[t_1], \alpha[t_1]) - D(\sigma[t_2], \alpha[t_2])\|_m \leqq \leqq c_5(\|\sigma[t_1] - \sigma[t_2]\|_S + \|\alpha[t_1] - \alpha[t_2]\|_m) . \tag{28}$$

This proves the fact that $D(\sigma, \alpha) \in C(\langle 0, T\rangle; [L_2(\Omega)]^m)$ and the integral in (23) exists in Riemann's sense.

Finally, using (27) again we can see immediately that

$$\mathscr{D}(\sigma, \alpha) \in C(\langle 0, T\rangle; [L_2(\Omega)]^m) .$$

and (25) is proved. (26) can be proved similarly.

Lemma 3. *There exists $\lambda > 0$ such that for each $\sigma \in C(\langle 0, T\rangle; S)$ the operator*

$$\mathscr{D}(\sigma, \cdot) : C_\lambda(\langle 0, T\rangle; [L_2(\Omega)]^m) \to C_\lambda(\langle 0, T\rangle; [L_2(\Omega)]^m)$$

is contractive.

Proof. Let $\sigma \in C(\langle 0, T\rangle; S)$, $\alpha^1, \alpha^2 \in C(\langle 0, T\rangle; [L_2(\Omega)]^m)$,

$$|||\mathscr{D}(\sigma, \alpha^1) - \mathscr{D}(\sigma, \alpha^2)|||_{m,\lambda} \leqq \sup_{t \in \langle 0,T\rangle} e^{-\lambda t} \cdot \int_0^t \|D(\sigma[\tau], \alpha^1[\tau]) - D(\sigma[\tau], \alpha^2[\tau])\| \, d\tau \leqq$$

$$\leqq [\text{using (28)}] \leqq c_5 \sup_{t \in \langle 0,T\rangle} \leqq e^{-\lambda t} \cdot \int_0^t \|\alpha^1[\tau] - \alpha^2[\tau]\| \, e^{-\lambda\tau} e^{\lambda\tau} \, d\tau \leqq$$

$$\leqq c_5 |||\alpha^1 - \alpha^2|||_{m,\lambda} \sup \left(e^{-\lambda t} \int_0^t e^{\lambda\tau} \, d\tau\right) \leqq$$

$$\leqq \frac{c_5}{\lambda} \cdot |||\alpha^1 - \alpha^2|||_{m,\lambda} .$$

Choosing now $\lambda = 2c_5$, we get

$$|||\mathscr{D}(\sigma, \alpha^1) - \mathscr{D}(\sigma, \alpha^2)|||_{m,\lambda} \leqq \tfrac{1}{2} |||\alpha^1 - \alpha^2|||_{m,\lambda} . \tag{29}$$

Assertion 1. *To each $\sigma \in C(\langle 0, T\rangle; S)$ there exists a unique $\alpha \in C(\langle 0, T\rangle; [L_2[\Omega)]^m)$ and $\varepsilon^p \in C(\langle 0, T\rangle; S)$ satisfying (4) and (5).*

Proof. Let σ be an arbitrary fixed element of $C(\langle 0, T\rangle; S)$. Using Lemma 3 we establish by the Banach fixed-point theorem that the operator equation

$$\alpha = \alpha^0 + \mathscr{D}(\sigma, \alpha) \tag{5, a}$$

has a unique solution $\alpha \in C(\langle 0, T\rangle; [L_2(\Omega)]^m)$. Denote it by $\alpha(\sigma)$. The corresponding $\varepsilon^p(\sigma)$ is then uniquely determined from the relation (4). Finally, $\varepsilon^p(\sigma) \in C(\langle 0, T\rangle; S)$ by Lemma 2.

Assertion 2. *To each* $\sigma \in C(\langle 0, T\rangle; S)$, $t \in \langle 0, T\rangle$ *fixed, there exists a unique* $\omega(\sigma)[t] \in S$ *for which*

(i) $$\int_\Omega \omega_{ij}(\sigma)[t]\, \beta_{ij}(\varphi)\, \mathrm{d}x = \int_\Omega F_i[t]\, \varphi_i\, \mathrm{d}x\,, \quad \forall \varphi \in W_0\,,$$

(ii) $$\int_\Omega \{A_{ij}(\omega(\sigma)[t], \alpha(\sigma)[t]) + \varepsilon^p_{ij}(\sigma)[t] - \beta_{ij}(\bar{u})[t]\}\, h_{ij}\, \mathrm{d}x = 0\,, \quad \forall h \in H_0\,,$$

(iii) $$\omega(\sigma) \in C(\langle 0, T\rangle; S)\,.$$

Proof. Let $\sigma \in C(\langle 0, T\rangle; S)$, $t \in \langle 0, T\rangle$ be fixed. We shall find ω in the form

(32) $$\omega(\sigma)[t] = \sigma^0 + w$$

where σ^0 is an arbitrary fixed element of S which satisfies (22). The demand that $\omega(\sigma)[t]$ satisfies (i) leads to the condition that $w \in H_0$.

From (10) and (32) it follows that the left-hand side in the equation (ii) is nothing else than a Gateaux differential Dif $\Phi(w, h)$ of the functional

(33) $$\Phi(w) = \int_\Omega \{P(\sigma^0 + w, \alpha(\sigma)[t]) + \varepsilon^p_{ij}(\sigma)[t]\, w_{ij} - \beta_{ij}(\bar{u})[t]\, w_{ij}\}\, \mathrm{d}x$$

defined on H_0. Rewriting (ii) as

(31, a) $$\mathrm{Dif}\, \Phi(w, h) = 0\,, \quad \forall h \in H_0$$

we see that the problem is to find all critical points of Φ.

If

(34) $$\lim_{\|w\|_S \to \infty} \Phi(w) = +\infty$$

and

(35) $$\mathrm{Dif}\, \Phi(w^2, w^2 - w^1) - \mathrm{Dif}\, \Phi(w^1, w^2 - w\gamma) \geqq C\|w^2 - w^1\|_S^2$$

take place (which will be proved below) then there exists a point of minimum of Φ in H_0 ([6] or [5], theorems 1.4.5, 1.6.3). This implies ([6] or [5], 1.6.2) that there exists a critical point of Φ. The uniqueness of such a point follows from (35). So w in (32) is uniquely determined. By an easy calculation we get the independence of $\omega(\sigma)[t]$ of the choice of σ^0 which completes the proof of its existence and uniqueness.

To show that (34) holds we write (omitting for the sake of brevity the arguments in α, ε^p, β):

(36) $$P(\sigma^0 + w, \alpha) = P(0, 0) + \frac{\partial P(0, 0)}{\partial \sigma_{ij}}(w_{ij} + \sigma^0_{ij}) +$$

$$+ \frac{\partial P(0, 0)}{\partial \alpha_l} \alpha_l + \frac{\partial^2 P(\bar{\sigma}, \bar{\alpha})}{\partial \sigma_{ij} \partial \sigma_{kl}} (\sigma_{ij}^0 + w_{ij}) (\sigma_{kl}^0 + w_{kl}) +$$

$$+ \frac{\partial^2 P(\bar{\sigma}, \bar{\alpha})}{\partial \sigma_{ij} \partial \alpha_l} (\sigma_{ij}^0 + w_{ij}) \alpha_l + \frac{\partial^2 P(\bar{\sigma}, \bar{\alpha})}{\partial \alpha_l \partial \alpha_n} \alpha_l \alpha_n .$$

Using (11), (12) and the inequality

$$|ab| \leqq \varepsilon \frac{a^2}{2} + \frac{1}{\varepsilon} \frac{b^2}{2} , \quad \varepsilon > 0 \tag{37}$$

we can estimate

$$P(\sigma^0 + w, \alpha) \geqq - M_1 - M_2 |\sigma|^2 - M_3 |\alpha|^2 + M_4 |w|^2 , \tag{38}$$

where M_1, M_2, M_3 and M_4 are positive constants.

Estimating in the analogous way

$$\varepsilon_{ij}^p w_{ij} \geqq - M_5 |\varepsilon^p|^2 - \frac{M_4}{4} . |w|^2 , \tag{39}$$

$$- \beta_{ij} w_{ij} \geqq - M_6 |\beta|^2 - \frac{M_4}{4} . |w|^2 \tag{40}$$

we get after substituting (38)–(40) into Φ:

$$\Phi(w) \geqq - M + \frac{M_4}{2} \int_\Omega |w|^2 \, dx \tag{41}$$

where the constant M appears as a consequence of the integration of the members independent of w over Ω. (41) implies (34) immediately.

Using the definition of Dif $\Phi(w, h)$, the Lagrange meanvalue theorem and (12), we can write

$$\text{Dif } \Phi(w^2, w^2 - w^1) - \text{Dif } \Phi(w^1, w^2 - w^1) = \int_\Omega \left\{ \frac{\partial P(\sigma^0 + w^2, \alpha)}{\partial \sigma_{ij}} - \right.$$

$$\left. - \frac{\partial P(\sigma^0 + w^1, \alpha)}{\partial \sigma_{ij}} \right\} (w_{ij}^2 - w_{ij}^1) \, dx = \int_\Omega \frac{\partial^2 P(\bar{\sigma}, \bar{\alpha})}{\partial \sigma_{ij} \partial \sigma_{kl}} (w_{kl}^2 - w_{kl}^1) (w_{ij}^2 - w_{ij}^1) \, dx \geqq$$

$$\geqq c_2 \int_\Omega |w^2 - w^1|^2 \, dx = c_2 \|w^2 - w^1\|_S^2$$

which is the inequality (35).

It remains to prove (iii). Suppose that σ^0 is chosen in such a way that

$\sigma^0 \in C(\langle 0, T\rangle; S)$. Let $t_1, t_2 \in \langle 0, T\rangle$. We have from (ii)

$$(42) \quad \int_\Omega \left\{\frac{\partial P(\omega[t_k], \alpha[t_k])}{\partial \sigma_{ij}} + \varepsilon_{ij}[t_k] - \beta_{ij}[t_k]\right\} h_{ij}\, dx = 0\,, \quad k = 1, 2, h \in H_0\,.$$

Put in (42)

$$(43) \qquad h = (\omega[t_1] - \omega[t_2]) - (\sigma^0[t_1] - \sigma^0[t_2])\,.$$

After a rearrangement of members we get

$$(44) \quad 0 = \int_\Omega \left\{\frac{\partial P(\omega[t_1], \alpha[t_1])}{\partial \sigma_{ij}} - \frac{\partial P(\omega[t_2], \alpha[t_1])}{\partial \sigma_{ij}}\right\} \{(\omega_{ij}[t_1] - \omega_{ij}[t_2]) -$$
$$- (\sigma^0_{ij}[t_1] - \sigma^0_{ij}[t_2])\}\, dx + \int_\Omega \left\{\frac{\partial P(\omega[t_2], \alpha[t_1])}{\partial \sigma_{ij}} - \frac{\partial P(\omega[t_2], \alpha[t_2])}{\partial \sigma_{ij}} + \right.$$
$$\left. + \varepsilon^p_{ij}[t_1] - \varepsilon^p_{ij}[t_2] - \beta_{ij}[t_1] + \beta_{ij}[t_2]\right\} \{(\omega_{ij}[t_1] - \omega_{ij}[t_2]) -$$
$$- (\sigma^0_{ij}[t_1] - \sigma^0_{ij}[t_2])\}\, dx\,.$$

By means of the same technique as in the proofs of (34), (35) we obtain finally from (44)

$$(45) \qquad \|\omega(\sigma)[t_2] - \omega(\sigma)[t_1]\|_S \leqq M\{\|\sigma^0[t_2] - \sigma^0[t_1]\|_S +$$
$$+ \|\alpha(\sigma)[t_1] - \alpha(\sigma)[t_2]\|_m + \|\varepsilon^p(\sigma)[t_1] - \varepsilon^p(\sigma)[t_2]\|_S\}\,.$$

The continuity of $\omega(\sigma)$ in $\langle 0, T\rangle$ follows then from (45) and from Assertion 1.

Now we want to prove that the mapping $\omega : \sigma \to \omega(\sigma)$ from $C_{\bar\lambda}(\langle 0, T\rangle; S)$ to $C_{\bar\lambda}(\langle 0, T\rangle; S)$ is for some $\bar\lambda > 0$ contractive. It will be done in Lemmas 4–7.

Lemma 4. *Let ω be a mapping defined in Assertion 2. There exists a constant $\bar c > 0$ such that for each $\lambda \geqq 0$, $\sigma^1, \sigma^2 \in C(\langle 0, T\rangle; S)$,*

$$(46) \quad |||\omega(\sigma^1) - \omega(\sigma^2)|||_{S,\lambda} \leqq \bar c(|||(\sigma^1) - \alpha(\sigma^2)|||_{m,\lambda} + |||\varepsilon^p(\sigma^1) - \varepsilon^p(\sigma^2)|||_{S,\lambda})\,.$$

Proof. Let $\sigma^1, \sigma^2 \in C(\langle 0, T\rangle; S)$. Assertion 2 implies

$$(47) \quad \int_\Omega \left\{\frac{\partial P(\omega(\sigma^k)[t], \alpha(\sigma^k)[t])}{\partial \delta_{ij}} + \varepsilon^p_{ij}(\sigma^k)[t] - \beta_{ij}(\bar u)[t]\right\} h_{ij}\, dx = 0\,,$$
$$h \in H_0,\ t \in \langle 0, T\rangle,\ k = 1, 2\,.$$

Put in (47)

$$(48) \qquad h = \omega(\sigma^1) - \omega(\sigma^2)$$

for both $k = 1$, $k = 2$ and substract these two identities. We get after an obvious transformation (omitting the argument again)

$$\int_\Omega \left\{ \frac{\partial P(\omega(\sigma^1), \alpha(\sigma^1))}{\partial \sigma_{ij}} - \frac{\partial P(\omega(\sigma^2), \alpha(\sigma^1))}{\partial \sigma_{ij}} \right\} (\omega_{ij}(\sigma^1) - \omega_{ij}(\sigma^2))\, \mathrm{d}x +$$
$$+ \int_\Omega \left\{ \frac{\partial P(\omega(\sigma^2), \alpha(\delta^1))}{\partial \sigma_{ij}} - \frac{\partial P(\omega(\sigma^2), \alpha(\sigma^2))}{\partial \sigma_{ij}} + \varepsilon^p_{ij}(\sigma^1) - \varepsilon^p_{ij}(\sigma^2) \right\} (\omega_{ij}(\sigma^1) -$$
$$- \omega_{ij}(\sigma^2))\, \mathrm{d}x = 0 .$$

Hence using the same estimates as in the proof of Assertion 2 we get

$$\|\omega(\sigma^1)[t] - \omega(\sigma^2)[t]\|_S \leqq \bar{c}(\|\varepsilon^p(\sigma^1)[t] - \varepsilon^p(\sigma^2)[t]\|_S + \|\alpha(\sigma_1)[t] - \alpha(\sigma^2)[t]\|_m), \quad \forall t \in \langle 0, T\rangle . \tag{49}$$

Let now $\lambda \geqq 0$. From (7) and (49), (46) follows immediately.

Lemma 5. *Let $\bar{c}$ be the constant from the previous Lemma. There exists $\lambda_1 \geqq 0$ such that for each $\lambda \geqq \lambda_1$:*

$$\sigma^1, \sigma^2 \in C(\langle 0, T\rangle; S) \Rightarrow |||\alpha(\sigma^1) - \alpha(\sigma^2)|||_{m,\lambda} \leqq \frac{1}{4\bar{c}} |||\sigma^1 - \sigma^2|||_{S,\lambda} . \tag{50}$$

Proof. Using (14), (5) we get for each $\sigma^1, \sigma^2 \in C(\langle 0, T\rangle; S)$, $t \in \langle 0, T\rangle$

$$\|\alpha(\sigma^1)[t] - \alpha(\sigma^2)[t]\|_m \leqq c_3 \int_0^t (\|\sigma^1[\tau] - \sigma^2[\tau]\|_S + \|\alpha(\sigma^1)[\tau] - \alpha(\sigma^2)[\tau]\|_m)\, \mathrm{d}\tau.$$

By the same method as in the proof of Lemma 3, we get for $\lambda > 0$

$$|||\alpha(\sigma^1) - \alpha(\sigma^2)|||_{m,\lambda} \leqq \frac{c_3}{\lambda} (|||\sigma^1 - \sigma^2|||_{S,\lambda} + |||\alpha(\sigma^1) - \alpha(\sigma^2)|||_{m,\lambda}) .$$

Putting $\lambda_1 = c_3(4\bar{c} + 1)$ we obtain the estimate (50).

Lemma 6. *Let $\bar{c}$ be the constant from Lemma 4. There exists $\lambda_2 \geqq 0$ such that for each $\lambda \geqq \lambda_2$,*

$$\sigma^1, \sigma^2 \in C(\langle 0, T\rangle; S) \Rightarrow |||\varepsilon^p(\sigma^1) - \varepsilon^p(\sigma^2)|||_{S,\lambda} \leqq \frac{1}{4\bar{c}} |||\sigma^1 - \sigma^2|||_{S,\lambda} . \tag{51}$$

Proof is quite similar to that of Lemma 5.

Lemma 7. Let $\bar{\lambda} = \max(\lambda_1, \lambda_2)$. *Then the mapping ω is contractive from $C_{\bar{\lambda}}(\langle 0, T\rangle; S)$ to $C_{\bar{\lambda}}(\langle 0, T\rangle; S)$ with the constant $\frac{1}{2}$.*

Proof follows immediately from Lemmas 4, 5, 6.

Theorem. *There exists a unique solution of the* PROBLEM.

Proof. Let $\bar{\sigma}$ be the fixed point of the mapping ω. Using Lemma 3 we calculate $\alpha(\bar{\sigma})$ satisfying (5). From (3) and (4) we get $\varepsilon^p(\bar{\sigma})$, $\varepsilon^e(\bar{\sigma})$. It follows from the definition of the mapping ω (Assertion 2) that the equilibrium equation as well as the compatibility condition (21) are satisfied. Thanks to that, the displacement u can be calculated in the unique manner.

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Souhrn

O ŘEŠENÍ DRUHÉHO OKRAJOVÉHO PROBLÉMU PRO ELASTICKO-INELASTICKÉ MATERIÁLY

OLDŘICH JOHN

V článku je zkoumána druhá okrajová úloha (jsou zadána posunutí na hranici oblasti) pro soustavu rovnic mechaniky kontinua, popisujících model s vnitřními parametry. Kombinací teorie monotonních operátorů a Banachova principu kontrakce je dokázána existence a jednoznačnost slabého řešení úlohy.

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