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AN EXISTENCE THEOREM FOR THE v. KÁRMÁN EQUATIONS UNDER THE CONDITION OF FREE BOUNDARY

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1. INTRODUCTION

It is the purpose of this paper to treat by functional analysis methods a boundary value problem for a system of nonlinear partial differential equations governing the bending of a thin elastic plate which is free along its edge and subject to a perpendicular load. Using the divergence structure of the partial differential equations considered, one can replace this boundary value problem (in its variational formulation) by an equivalent operator equation in a suitably chosen Hilbert function space such that the original problem is reduced to the investigation of an abstract nonlinear operator equation.

Let Ω be a bounded domain in the x, y-plane, constituting the middle plane of the undeflected plate, and let $\partial\Omega$ be the boundary of Ω. We then consider the following version of the v. Kármán equations:

$$\begin{aligned} \Delta^2 w &= [f, w] + q \quad \text{in} \quad \Omega, \\ \Delta^2 f &= -[w, w] \quad \text{in} \quad \Omega. \end{aligned} \tag{1.1}$$

Here $w = w(x, y)$ represents the deflection of the plate, $f = f(x, y)$ is the stress function, and $q = q(x, y)$ denotes the perpendicular load. Furthermore, $[\,,\,]$ is defined to be

$$[u, v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}\,.$$

We now introduce the boundary operators

$$B_1 w \equiv \nu\Delta w + (1 - \nu)\left(n_x^2 w_{xx} + 2n_x n_y w_{xy} + n_y^2 w_{yy}\right),$$

$$B_2 w \equiv -\frac{\partial}{\partial n}\Delta w + (1 - \nu)\frac{\partial}{\partial s}\left(n_x n_y w_{xx} - \left(n_x^2 - n_y^2\right) w_{xy} - n_x n_y w_{yy}\right)$$

where $n = (n_x, n_y)$ is the outer normal of Ω, $s = (-n_y, n_x)$ is the corresponding tangent, and ν is the Poisson ratio of the plate material.

The boundary conditions imposed on (1.1) are then the following ones:

$$B_1 w = B_2 w = 0 \quad \text{on} \quad \partial\Omega\,,^{1)} \tag{1.2}$$

$$f = \frac{\partial f}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\,. \tag{1.3}$$

The first condition in (1.2) expresses that the bending moment vanishes along $\partial\Omega$, while the second one may be interpreted as vanishing shearing force. Condition (1.3) implies in a certain sense that the boundary of the plate is free of stresses. Thus, (1.1)–(1.3) describes the equilibrium of a thin elastic plate under the condition of free boundary.

Considering the buckling problem for (1.1), Berger and Fife [1] deal with mixed boundary conditions on w in which (1.2) is required only on a part of $\partial\Omega$. In [3], Knightly has proved by a technique which is completely different from ours, an existence theorem for (1.1) under Dirichlet conditions both for w and f in the case of combined normal and edge forces. Detailed results about the bifurcation of nontrivial solutions for (1.1) have been presented in [4]. Various other boundary value problems for (1.1) are treated in [2], [5], [7].

In Section 2 we put (1.1)–(1.3) into the framework of elliptic boundary value problems and introduce the definition of the notion of variational solution. Our main results, Theorem 1 and Theorem 2, are then presented in the following section. Preparing their proofs, the abstract operator formulation of boundary value problem (1.1)–(1.3) in an appropriate Hilbert space is stated in Section 4. Moreover, some properties of the occurring operators which are necessary for the application of an abstract existence theorem are also given in this section. They enable us to prove our results quite easily, which is done in the last section.

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2. TERMINOLOGY

Let Ω be a bounded domain in the x, y-plane whose boundary $\partial\Omega$ is of type $\mathfrak{N}^{(0),1}$ (cf. [6] for details). $L^p(\Omega)$ will denote the space of all real functions which are integrable with power $1 \leqq p < \infty$ on Ω (with respect to the Lebesgue measure $dx\,dy$).

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1) In the presence of corners, boundary conditions of the type

$$B_3 w \equiv (1 - \nu)\,[n_x n_y w_{xx} - (n_x^2 - n_y^2)\,w_{xy} - n_x n_y w_{yy}]_-^+ = 0$$

are added at the corners, i.e., the jump of the twisting moment vanishes.

Using the usual notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\,\partial y^{\alpha_2}}, \quad |\alpha| = \alpha_1 + \alpha_2,$$

we define for an integer $m \geqq 1$

$$W^{m,p}(\Omega) = \{u \mid u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leqq m\}$$

(the derivatives are taken in the sense of distributions). $W^{m,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{Wm,p} = \left\{\int_\Omega |u|^p \,\mathrm{d}x\,\mathrm{d}y + \sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p \,\mathrm{d}x\,\mathrm{d}y\right\}^{1/p}.$$

In particular, the scalar product

$$(u, v)_{W2,2} = \int_\Omega uv \,\mathrm{d}x\,\mathrm{d}y + \sum_{|\alpha|=2} \int_\Omega D^\alpha u D^\alpha v \,\mathrm{d}x\,\mathrm{d}y$$

turns $W^{2,2}(\Omega)$ into a Hilbert space.

By $W_0^{2,2}(\Omega)$ we denote the closure of $\mathscr{D}(\Omega)$ in the norm $\|\ \|_{W^2,2}$ where $\mathscr{D}(\Omega)$ is the space of all in Ω infinitely continuously differentiable functions with support in Ω. In what follows the space $W_0^{2,2}(\Omega)$ will be regarded as equipped with the scalar product

$$(u, v)_{W_0{}^2,2} = \sum_{|\alpha|=2} \int_\Omega D^\alpha u D^\alpha v \,\mathrm{d}x\,\mathrm{d}y.$$

On the space $W_0^{2,2}(\Omega)$ the norms $\|\ \|_{W^2,2}$ and $\|\ \|_{W_0{}^2,2} = (\ ,\ )^{1/2}_{W_0{}^2,2}$ are equivalent.

Let P_1 be the space of all polynomials of the first degree in x and y; P_1 is a closed subspace of $W^{2,2}(\Omega)$.

We now form the orthogonal decomposition $W^{2,2}(\Omega) = K \oplus P_1$ with respect to the scalar product $(\ ,\ )_{W^2,2}$ and introduce the factor space $W^{2,2}(\Omega)/P_1$ which is defined as the space of all classes $\tilde{u}$ such that $u, v \in \tilde{u}$ iff $u - v \in P_1$. As usual, $W^{2,2}(\Omega)/P_1$ will be equipped with the norm

$$\|\tilde{u}\|_{W^{2,2}/P_1} = \inf_{u\in\tilde{u}} \|u\|_{W^2,2}.$$

In order that we may treat boundary value problem (1.1)–(1.3) by means of abstract operator methods, it is necessary to introduce on $W^{2,2}(\Omega)/P_1$ a suitable scalar product. To do this, let $0 < \nu < 1$ be a fixed number (ν is in fact the Poisson ratio). We put for $u, v \in W^{2,2}(\Omega)$

$$(u, v)_{W_\nu{}^2,2} = \int_\Omega [u_{xx}v_{xx} + 2(1-\nu)\, u_{xy}v_{xy} + u_{yy}v_{yy} + \nu(u_{xx}v_{yy} + u_{yy}v_{xx})] \,\mathrm{d}x\,\mathrm{d}y$$

and define for $\tilde{u} \in W^{2,2}(\Omega)/P_1$:

$$((\tilde{u}, \tilde{v})) = (u, v)_{W_\nu^{2,2}}, \quad u \in \tilde{u}, \quad v \in \tilde{v} \quad \text{arbitrary},$$
$$|||\tilde{u}||| \quad = ((\tilde{u}, \tilde{u}))^{1/2}.$$

It is easy to see that the inequalities

$$(2.1) \qquad c_1 \|\tilde{u}\|_{W^{2,2}/P_1} \leqq |||\tilde{u}||| \leqq c_2 \|\tilde{u}\|_{W^{2,2}/P_1}$$

in which $c_i = \text{const} > 0$ $(i = 1, 2)$ are true for all $\tilde{u} \in W^{2,2}(\Omega)/P_1$ (cf. [6]). Thus, in the sequel $W^{2,2}(\Omega)/P_1$ will be considered as Hilbert space equipped with the scalar product $((\ ,\ ))$.

Finally, we give a characterization of elements $u \in W^{2,2}(\Omega)$ which belong to K. To this end, let us introduce the functionals

$$f_0(u) = \int_\Omega u \,\mathrm{d}x\,\mathrm{d}y, \quad f_1(u) = \int_\Omega xu \,\mathrm{d}x\,\mathrm{d}y, \quad f_2(u) = \int_\Omega yu \,\mathrm{d}x\,\mathrm{d}y$$

defined on the whole $W^{2,2}(\Omega)$. One then obtains

$$(2.2) \qquad u \in K \quad \text{iff} \quad f_i(u) = 0 \quad \text{for} \quad i = 0, 1, 2.$$

Without particularly referring to it, in all what follows we assume $q \in L^1(\Omega)$.

We now define what is meant by a variational solution of boundary value problem (1.1)–(1.3).

Definition. *The pair w, f with $w \in W^{2,2}(\Omega)$ and $f \in W_0^{2,2}(\Omega)$ is called a variational solution of boundary value problem (1.1)–(1.3) if the following two integral identities are satisfied:*

$$(2.3) \quad \int_\Omega [w_{xx}\varphi_{xx} + 2(1 - \nu)\, w_{xy}\varphi_{xy} + w_{yy}\varphi_{yy} + \nu(w_{xx}\varphi_{yy} + w_{yy}\varphi_{xx})] \,\mathrm{d}x\,\mathrm{d}y =$$
$$= \int_\Omega (w_{xx}\varphi_{yy} - 2w_{xy}\varphi_{xy} + w_{yy}\varphi_{xx}) f \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega q\varphi \,\mathrm{d}x\,\mathrm{d}y \quad \textit{for all} \quad \varphi \in W^{2,2}(\Omega),$$

$$(2.4) \qquad \int_\Omega (f_{xx}\psi_{xx} + 2f_{xy}\psi_{xy} + f_{yy}\psi_{yy}) \,\mathrm{d}x\,\mathrm{d}y =$$
$$= -2 \int_\Omega (w_{xx}w_{yy} - w_{xy}^2)\, \psi \,\mathrm{d}x\,\mathrm{d}y \quad \textit{for all} \quad \psi \in W_0^{2,2}(\Omega).$$

Remarks. – 1. The integral identities (2.3) and (2.4) can be formally obtained by multiplying equations (1.1) by test functions $\varphi \in W^{2,2}(\Omega)$ and $\psi \in W_0^{2,2}(\Omega)$, respectively, and integrating by parts their left hand sides, using the boundary conditions (1.2), (1.3).

2. By our definition above, w is uniquely determined except for a polynomial $p_1 \in P_1$, i.e., all representatives of the class $\tilde{w}$ which contains w (where w, f is the variational solution considered) also satisfy (2.3) and (2.4).

Moreover, if w, f is a variational solution of boundary value problem (1.1)–(1.3) then it holds necessarily that

$$(*) \qquad \int_\Omega q p_1 \, \mathrm{d}x \, \mathrm{d}y = 0 \quad \text{for all} \quad p_1 \in P_1 \, .$$

Theorem 1 below shows that this condition is also sufficient for the existence of a variational solution.

3. Condition (*) may be interpreted as a certain total equilibrium condition.

3. STATEMENT OF MAIN RESULTS

In this section we present our results concerning existence and uniqueness (apart from a polynomial $p_1 \in P_1$) of a variational solution of boundary value problem (1.1)–(1.3).

Theorem 1. *Suppose that condition (*) is satisfied.*

Then:

(i) *There exists at least one pair $\tilde{w}, f$ with $\tilde{w} \in W^{2,2}(\Omega)/P_1$ and $f \in \overset{\circ}{W}{}_0^{2,2}(\Omega)$ such that for each $w \in \tilde{w}$ the pair w, f is a variational solution of boundary value problem* (1.1)–(1.3).

(ii) *In the class $\tilde{w}$ there exists a representative w' such that*

$$\|w'\|_{W^{2,2}} \leqq \text{const} \, \|q\|_{L^1} \, .$$

Moreover, the following estimate holds:

$$\|f\|_{W_0^{2,2}} \leqq \text{const} \, \|q\|_{L^1}^2 \, .$$

(iii) *If $\tilde{w}$ is the class from* (i), *then by the condition*

$$\int_\Omega w p_1 \, \mathrm{d}x \, \mathrm{d}y = 0 \quad \textit{for all} \quad p_1 \in P_1$$

a representative w in $\tilde{w}$ is uniquely determined.

Theorem 2. *Suppose that condition (*) is satisfied.*

If $\|q\|_{L^1}$ is sufficiently small, then $\tilde{w}$ mentioned in Theorem 1, (i) *is uniquely determined.*

Our paper does not include a study of regularity properties of the variational solution obtained in Theorem 1. We only remark that the boundary operators B_1 and B_2 form a normal system which covers Δ^2 on $\partial\Omega$ (this was submitted by O. John to the author). Thus, the linear elliptic theory applies, and under appropriate conditions on q and $\partial\Omega$, regularity results of the variational solution can be expected.

4. OPERATOR FORMULATION

The Proposition below yields our principal methodical tool for the treatment of boundary value problem (1.1)–(1.3) (in variational sense) by abstract operator methods.

Proposition. *Suppose that condition (*) is satisfied. Then the integral identities (2.3) and (2.4) are equivalent to the system of operator equations*

$$\tilde{w} = C_1(\tilde{w}, f) + \tilde{q} \quad \text{in} \quad H\,,^1) \tag{4.1}$$

$$f = C_2(\tilde{w}, \tilde{w}) \qquad \text{in} \quad W_0^{2,2} \tag{4.2}$$

where C_1 is a bilinear operator with domain $H \times W_0^{2,2}$ and range in H, C_2 is a bilinear operator with domain $H \times H$ and range in $W_0^{2,2}$, and $\tilde{q}$ is a fixed element in H, in the following sense:

The pair w, f, where $w \in W^{2,2}$ and $f \in W_0^{2,2}$, is a variational solution of boundary value problem (1.1)–(1.3) if and only if the pair $\tilde{w}, f$, where $\tilde{w} \in H$ denotes the class containing w, is a solution of the system (4.1), (4.2).

Proof. Let $\tilde{w}, \tilde{\varphi} \in H$ and $f \in W_0^{2,2}$ be arbitrarily given. By means of Sobolev's embedding theorem (see [6]) and Schwarz's inequality one obtains for any $w \in \tilde{w}$, $\varphi \in \tilde{\varphi}$ the estimate

$$\left|\int_\Omega w_{xx}\varphi_{yy} f \,\mathrm{d}x\,\mathrm{d}y\right| \leqq (\text{const}\, |||\tilde{w}|||\, \|f\|_{W_0^{2,2}})\, |||\tilde{\varphi}|||\,.$$

Since an estimate of this type is true for the remaining terms in the first integral on the right hand side in (2.3), we get by means of the Riesz representation theorem the existence of a (uniquely determined) element $C_1(\tilde{w}, f) \in H$ such that

$$((C_1(\tilde{w}, f), \tilde{\varphi})) = \int_\Omega (w_{xx}\varphi_{yy} - 2w_{xy}\varphi_{xy} + w_{yy}\varphi_{xx}) f \,\mathrm{d}x\,\mathrm{d}y \tag{4.3}$$

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[1]) Since there is no possibility of confusion, throughout the remainder of the paper, $W^{2,2}(\Omega)/P_1$ will be denoted by H, $W^{2,2}(\Omega)$ and $W_0^{2,2}(\Omega)$ shortly by $W^{2,2}$ and $W_0^{2,2}$, respectively.

for all $\tilde{\varphi} \in H$; here

$$(4.4) \qquad |||C_1(\tilde{w}, f)||| \leqq \text{const}\, |||\tilde{w}|||\, \|f\|_{W_0^{2,2}} \ \forall \tilde{w} \in H\,, \quad f \in W_0^{2,2}\,.$$

Furthermore, by Sobolev's embedding theorem and condition (*),

$$\left|\int_\Omega q\varphi \, \mathrm{d}x\, \mathrm{d}y\right| \leqq (\text{const}\, \|q\|_{L^1})\, |||\tilde{\varphi}|||$$

for any $\varphi \in \tilde{\varphi}$. Thus, there is a unique $\tilde{q} \in H$ such that

$$\int_\Omega q\varphi \, \mathrm{d}x\, \mathrm{d}y = ((\tilde{q}, \tilde{\varphi})) \quad \forall \tilde{\varphi} \in H\,.$$

It is now obvious that (2.3) is equivalent to (4.1) in the sense of the Proposition.

Let be $\tilde{u}, \tilde{v} \in H$. By the same argument as above, we obtain the existence of a unique $C_2(\tilde{u}, \tilde{v}) \in W_0^{2,2}$ such that

$$(4.5) \qquad (C_2(\tilde{u}, \tilde{v}), \psi)_{W_0^{2,2}} = -\int_\Omega (u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx})\, \psi \, \mathrm{d}x\, \mathrm{d}y$$

for all $\psi \in W_0^{2,2}$ and any $u \in \tilde{u}$, $v \in \tilde{v}$; clearly,

$$(4.6) \qquad \|C_2(\tilde{u}, \tilde{v})\|_{W_0^{2,2}} \leqq \text{const}\, |||\tilde{u}|||\, |||\tilde{v}||| \quad \forall \tilde{u}, \tilde{v} \in H\,.$$

Taking into account that

$$(C_2(\tilde{w}, \tilde{w}), \psi)_{W_0^{2,2}} = -2\int_\Omega (w_{xx}w_{yy} - w_{xy}^2)\, \psi \, \mathrm{d}x\, \mathrm{d}y$$

for all $\psi \in W_0^{2,2}$ and $\tilde{w} \in H$ where w is an arbitrary representative of $\tilde{w}$, the second part of the asserted equivalence is proved. Q.E.D.

We now define an operator C of H into itself by

$$(4.7) \qquad C : \tilde{u} \mapsto C(\tilde{u}) \equiv C_1(\tilde{u}, C_2(\tilde{u}, \tilde{u}))\,, \quad \forall \tilde{u} \in H\,.$$

The defining relations (4.3) and (4.5) immediately imply

$$(C_1(\tilde{u}, \psi), \tilde{u})) = -(C_2(\tilde{u}, \tilde{u}), \psi)_{W_0^{2,2}}$$

for all $\tilde{u} \in H$ and all $\psi \in W_0^{2,2}$. Setting $\psi = C_2(\tilde{u}, \tilde{u})$, one gets

$$(4.8) \qquad ((C(\tilde{u}), \tilde{u})) \leqq 0 \quad \forall \tilde{u} \in H\,.$$

Moreover, using (4.4) and (4.6) we obtain for arbitrary $\tilde{u}_1, \tilde{u}_2 \in H$

$$(4.9) \qquad |||C(\tilde{u}_1) - C(\tilde{u}_2)||| \leqq \text{const}\, (|||\tilde{u}_1|||^2 + |||\tilde{u}_2|||^2)\, |||\tilde{u}_1 - \tilde{u}_2|||\,.$$

The following Lemma shows that estimate (4.9) can be improved.

Lemma 1. *Let $\tilde{u}_1, \tilde{u}_2 \in H$ be arbitrarily given.*

Then the estimate

$$|||C(\tilde{u}_1) - C(\tilde{u}_2)||| \leqq \text{const}\,(\|u_1\|^2_{W^{2,2}} + \|u_2\|^2_{W^{2,2}})\,\|u_1 - u_2\|_{W^{1,4}}$$

holds for any $u_1 \in \tilde{u}_1$ and any $u_2 \in \tilde{u}_2$.

Proof. First, we note the divergence form

$$[u, v] = (v_{yy}u_x - v_{xy}u_y)_x + (v_{xx}u_y - v_{xy}u_x)_y$$

which is valid for smooth functions. Integration by parts yields

$$\int_\Omega (u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx})\,\psi\,dx\,dy = \tag{4.10}$$

$$= \int_\Omega [(v_{xy}u_y - v_{yy}u_x)\,\psi_x + (v_{xy}u_x - v_{xx}u_y)\,\psi_y]\,dx\,dy$$

for all $u, v \in \mathscr{E}(\bar{\Omega})$ and all $\psi \in \mathscr{D}(\Omega)$ (here $\mathscr{E}(\bar{\Omega})$ denotes the space of all in Ω infinitely continuously differentiable functions which together with all their derivatives can be continuously continued onto $\bar{\Omega}$). Using Hölder's inequality and Sobolev's embedding theorem, (4.10) implies

$$\left|\int_\Omega (u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx})\,\psi\,dx\,dy\right| \leqq \tag{4.11}$$

$$\leqq (\text{const}\,\|u\|_{W^{1,4}}\,\|\psi\|_{W_0^{2,2}})\,\|v\|_{W^{2,2}} \quad \forall u, v \in \mathscr{E}(\bar{\Omega})\,, \quad \psi \in \mathscr{D}(\Omega)\,.$$

Since $\mathscr{E}(\bar{\Omega})$ is dense in $W^{2,2}$ (see [6]), by passing to limit we see that (4.11) in fact is true for all $u, v \in W^{2,2}$ and all $\psi \in W_0^{2,2}$. Observing the defining relation (4.3), from (4.11) we can conclude that

$$|||C_1(\tilde{w}, \psi)||| \leqq \text{const}\,\|w\|_{W^{1,4}}\,\|\psi\|_{W_0^{2,2}} \tag{4.12}$$

holds for all $\tilde{w} \in H$, $\psi \in W_0^{2,2}$ where w is an arbitrary representative in the class $\tilde{w}$.

Repeating the same reasoning which led us to (4.12), we obtain

$$\|C_2(\tilde{u}, \tilde{v})\|_{W_0^{2,2}} \leqq \text{const}\,\|u\|_{W^{1,4}}\,\|v\|_{W^{2,2}} \tag{4.13}$$

for all $\tilde{u}, \tilde{v} \in H$ and any $u \in \tilde{u}$, $v \in \tilde{v}$.

Observing now that (4.5) immediately implies the symmetry of C_2, the asserted estimate can be easily verified by using (4.12), (4.13) and Sobolev's embedding theorem. Q.E.D.

In proving Theorem 1, the following result based upon Lemma 1 will be essentially used.

Lemma 2. *For any sequence* $\{\tilde{u}_n\} \subset H$ *such that*

$$\tilde{u}_n \to \tilde{u} \quad \textit{weakly in} \quad H$$

there is a subsequence $\{\tilde{u}_\mu\} \subset \{\tilde{u}_n\}$ *such that*

$$C(\tilde{u}_\mu) \to C(\tilde{u}) \quad \textit{strongly in} \quad H\,.$$

Proof. First of all, for each n $(n = 1, 2, \ldots)$ there exists a $u_n \in W^{2,2}$ such that

$$u_n \in \tilde{u}_n\,, \quad \|u_n\|_{W^{2,2}} \leqq \inf_{u \in \tilde{u}_n} \|u\|_{W^{2,2}} + 1\,. \tag{4.14}$$

By virtue of (2.1), the boundedness of $\{\tilde{u}_n\}$ in H implies $\|u_n\|_{W^{2,2}} \leqq$ const, $n = 1, 2, \ldots$, for the sequence $\{u_n\}$ according to (4.14). Thus, from $\{u_n\}$ we can select a subsequence, say $\{u_{n_k}\}$, such that $u_{n_k} \to u^*$ weakly in $W^{2,2}$. Since all (generalized) partial derivatives of order two of u_{n_k} converge separately weakly in $L^2(\Omega)$, one obtains

$$(u_{n_k}, v)_{W_\nu^{2,2}} \to (u^*, v)_{W_\nu^{2,2}} \quad \forall v \in W^{2,2}\,. \tag{4.15}$$

On the other hand,

$$(u_{n_k}, v)_{W_\nu^{2,2}} = ((\tilde{u}_{n_k}, \tilde{v})) \to ((\tilde{u}, \tilde{v})) \tag{4.16}$$

for all $\tilde{v} \in H$ and any $v \in \tilde{v}$. Denoting for arbitrary $v \in W^{2,2}$ by $\tilde{v}$ the class containing v, (4.15) and (4.16) imply $(u^* - u, v)_{W_\nu^{2,2}} = 0$ for any $u \in \tilde{u}$ and all $v \in W^{2,2}$. Thus, $u^* - u = p_1 \in P_1$, i.e., $u^* \in \tilde{u}$.

Finally, Sobolev's embedding theorem yields the existence of a subsequence $\{u_\mu\}$ of $\{u_{n_k}\}$ such that $u_\mu \to u^*$ strongly in $W^{1,4}(\Omega)$.

The assertion now follows by Lemma 1. Q.E.D.

5. PROOF OF THE THEOREMS

Proof of Theorem 1. In order to prove (i) we substitute f in (4.1) according to (4.2) and obtain the operator equation

$$\tilde{u} - C(\tilde{u}) = \tilde{q} \tag{5.1}$$

where C is defined by (4.7). Thus, the identities (2.3), (2.4) are equivalent to (5.1) in the sense described in the Proposition above (see Section 4); in particular, if $\tilde{u} \in H$ is a solution of (5.1), then, by setting $f = C_2(\tilde{u}, \tilde{u})$, each pair u, f where $u \in \tilde{u}$ is a variational solution of (1.1)–(1.3).

By (4.9), C is Lipschitzian uniformly on bounded sets in H. Furthermore, let $\{\tilde{u}_n\} \subset H$ be a bounded sequence in H. By reflexivity of H, we may assume (passing to a subsequence if necessary) that $\tilde{u}_n \to \tilde{u}$ weakly in H. Lemma 2 then implies the following compactness property of C: There exists a subsequence $\{\tilde{u}_\mu\} \subset \{\tilde{u}_n\}$ such that $C(\tilde{u}_\mu) \to C(\tilde{u})$ strongly in H. Finally, (4.8) yields the coerciveness of the operator

$I - C$:

$$(((I - C)\,\tilde{u}, \tilde{u})) \geqq |||\tilde{u}|||^2 \quad \forall \tilde{u} \in H\,. \tag{5.2}$$

From the theory of semi-monotone operators one can now conclude the existence of a solution of (5.1) for arbitrary $\tilde{q} \in H$. For this purpose, we only note the following special result:[1])

Let X be a Hilbert space (with scalar product (,) and norm $\| \ \| = (\ ,\ )^{1/2}$), and let T be a completely continuous[2]) *operator of X into X. Moreover, let be*

$$\lim_{\|u\| \to \infty} \frac{((I + T)\,u, u)}{\|u\|} = \infty\,.$$

Then $I + T$ maps X onto X.

We turn to the proof of (ii). Each $u \in \tilde{u}$ can be represented in a unique manner as $u = u_K + u_{P_1}$ where $u_K \in K$, $u_{P_1} = p_1 \in P_1$; here u_K is uniquely determined by the class $\tilde{u}$. Thus, one obtains

$$\|u_K\|_{W^{2,2}} = \inf_{u \in \tilde{u}} \|u\|_{W^{2,2}} \leqq \text{const}\, |||\tilde{u}|||\,. \tag{5.3}$$

Taking into consideration the defining relation of $\tilde{q}$, one obviously gets $|||\tilde{q}||| \leqq$ $\leqq \text{const}\, \|q\|_{L^1}$. Using this, the first estimate in (ii) follows immediately by (5.2) and (5.3), while the second one, with $f = C_2(\tilde{u}, \tilde{u})$ according to (4.2), is a consequence of (4.6).

To prove (iii), one observes that

$$\int_\Omega u p_1 \,\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{for all} \quad p_1 \in P_1$$

is equivalent to $u \in K$ (cf. (2.2)). Thus, this condition selects the element u_K from the class $\tilde{u}$.

Proof of Theorem 2. From (4.9) one immediately obtains for a certain positive constant α_0 that

$$(((I - C)\,\tilde{u}_1 - (I - C)\,\tilde{u}_2,\ \tilde{u}_1 - \tilde{u}_2)) \geqq \alpha_0 |||\tilde{u}_1 - \tilde{u}_2|||^2$$

for all $\tilde{u}_1, \tilde{u}_2$ lying in a ball with sufficiently small radius, centered at the origin.

We now easily get the assertion by the remark already used that for the solution of (5.1), $|||\tilde{u}||| \leqq \text{const}\, \|q\|_{L^1}$ holds.

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[1]) See the recent paper: *DeFigueiredo, D. G.* and *Gupta, C. P.*: Nonlinear integral equations of Hammerstein type involving unbounded monotone linear mappings. — J. Math. Anal. Appl., 39 (1972), 37—48.

[2]) A mapping T of X into X is said to be completely continuous if T is continuous and maps each bounded set into a compact set. — We remark that this property is slightly weaker than that derived above for the operator C.

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Souhrn

EXISTENČNÍ VĚTA PRO VON KÁRMÁNOVY ROVNICE S VOLNOU HRANICÍ

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Článek se týká von Kármánových rovnic, kterými se řídí průhyb tenké pružné desky, s podmínkou volné hranice. Nejdříve je definováno variační řešení a uvažovaný problém se převádí na ekvivalentní abstraktní operátorovou rovnici, na níž lze aplikovat známé věty z teorie nelineárních operátorů. Hlavním výsledkem je důkaz existence variačního řešení uvažovaného problému.

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