Aplikace matematiky

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Aplikace matematiky, Vol. 19 (1974), No. 1, 7–16

Persistent URL: http://dml.cz/dmlcz/103509

ON A BOUNDARY VALUE PROBLEM IN NONLINEAR THEORY OF THIN ELASTIC PLATES

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(Received February 27, 1973)

1. Introduction

The purpose of the present paper is to solve a boundary value problem for a system of nonlinear partial differential equations governing the equilibrium state of a thin elastic plate, subject to a perpendicular load and to prescribed displacements u_0, v_0 along the boundary. Using the variational character of the boundary value problem considered, a solution will be obtained as a critical point of the associated potential when proving the existence of its absolute minimum.

Let Ω be a bounded domain in the x, y-plane (representing the shape of the plate) with boundary Γ. We then consider in Ω the system

$$\frac{D}{h}\Delta^2 w = \sigma_{11}w_{xx} + \sigma_{22}w_{yy} + 2\sigma_{12}w_{xy} + f\,, \tag{1.1}$$

$$\Delta u + \frac{1+\mu}{1-\mu}\Theta_x = -\frac{2}{1-\mu}\left(w_x w_{xx} + \mu w_y w_{xy}\right) - w_y w_{xy} - w_x w_{yy}\,,$$

$$\Delta v + \frac{1+\mu}{1-\mu}\Theta_y = -\frac{2}{1-\mu}\left(w_x w_{yy} + \mu w_x w_{xy}\right) - w_x w_{xy} - w_y w_{xx}\,,$$

where

$$\Theta = u_x + v_y\,,$$

$$\sigma_{11} = \frac{E}{1-\mu^2}\left[u_x + \tfrac{1}{2}w_x^2 + \mu\left(v_y + \tfrac{1}{2}w_y^2\right)\right],$$

$$\sigma_{22} = \frac{E}{1-\mu^2}\left[v_y + \tfrac{1}{2}w_y^2 + \mu\left(u_x + \tfrac{1}{2}w_x^2\right)\right],$$

$$\sigma_{12} = \sigma_{21} = \frac{E}{2(1+\mu)}\left(u_y + v_x + w_x w_y\right).$$

*) The paper was written while the second author was staying at the Department of Mathematics, Charles University, Prague.

Here $f = f(x, y)$ denotes the given perpendicular load, while the constants have the following meaning:

h – plate thickness ,
E – compression modulus of elasticity ,
μ – Poisson number ,
D – plate stiffness .

To formulate boundary conditions for the system (1.1), in the whole paper we assume that Γ is decomposed into two parts Γ_1 and Γ_2 such that

$$(*) \qquad \Gamma = \Gamma_1 \cup \Gamma_2 \quad \text{where} \quad \operatorname{mes}(\Gamma_1) > 0 .$$

Then the boundary conditions imposed on w are

$$(1.2) \quad w = \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma_1 ,$$

$$w = 0 , \quad \mu \, \Delta w + (1 - \mu)(n_x^2 w_{xx} + 2n_x n_y w_{xy} + n_y^2 w_{yy}) = g_0 \quad \text{on} \quad \Gamma_2$$

where $n = (n_x, n_y)$ is the outer normal with respect to Ω. From the mechanical point of view, these conditions mean that the plate is clamped along Γ_1 and is simply supported along Γ_2 with prescribed bending moment.

We complete the boundary conditions for (1.1) by

$$(1.3) \qquad u = u_0 , \quad v = v_0 \quad \text{on} \quad \Gamma .$$

The system (1.1) together with boundary conditions (1.2), (1.3) constitute the problem investigated below.

In [6], Nečas, Poracká, Kodnar have proved by the variational approach the existence of a solution of (1.1) under sufficiently small traction conditions on Γ. Vorovich ([8], [9], [10]) has proved the existence of a solution for the corresponding shell problem (under somewhat simpler boundary conditions on w). After reducing the problem to a single equation, in the papers [8], [9] the solution is also obtained as the minimizing point of the associated energy functional; however, in the plate case, this functional differs somewhat from that used in [6].

Various existence theorems for the v. Kármán equations of a thin elastic plate have been established in [1], [2], [4] and [7]. In particular, using a-priori-estimates, Knightly [1] was able to prove an existence theorem for a clamped plate subject to combined normal and edge loading.

In section 2 we introduce the terminology used which is necessary for putting (1.1)–(1.3) into the framework of elliptic boundary value problems. The following section contains the precise definition of the notion of solution of (1.1)–(1.3) as well as the statement of our main result. Section 4 presents its proof whose crucial point consists in deriving the coerciveness of the associated functional.

2. Terminology

Let Ω be a bounded domain in the x, y-plane whose boundary Γ is Lipschitzian (see [5] for details). $L^p(\Omega)$ will denote the space of real functions which are integrable on Ω with power $1 \leqq p < \infty$ (with respect to the Lebesgue measure $\mathrm{d}x\,\mathrm{d}y$).

Using the usual notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\,\partial y^{\alpha_2}}, \quad |\alpha| = \alpha_1 + \alpha_2 ,$$

we define for an integer $m \geqq 1$

$$W^{m,2}(\Omega) = \left\{ u \;\middle|\; u \in L^2(\Omega),\ D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leqq m \right\}$$

(the derivatives are to be understood in the sense of distributions). The scalar product

$$(u, v)_{W^{m,2}} = \sum_{|\alpha| \leqq m} \int_\Omega D^\alpha u D^\alpha v \,\mathrm{d}x\,\mathrm{d}y$$

turns $W^{m,2}(\Omega)$ into a Hilbert space.

For the treatment of boundary value problem (1.1)–(1.3) we introduce the space

$$V = \left\{ u \;\middle|\; u \in W^{2,2}(\Omega),\ u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_1,\ u = 0 \text{ on } \Gamma_2 \right\}$$

which is a closed subspace of $W^{2,2}(\Omega)$ with respect to the norm $\|\ \|_{W^{2,2}} = (\ ,\ )^{1/2}_{W^{2,2}}$. Taking into account condition (*), it is readily seen that

$$\|u\|_{W^{2,2}} \leqq c \left\{ \iint_\Omega [u_{xx}^2 + 2(1-\mu)\,u_{xy}^2 + u_{yy}^2 + 2\mu u_{xx}u_{yy}] \,\mathrm{d}x\,\mathrm{d}y \right\}^{1/2} \tag{2.1}$$

for all $u \in V$ where $c = \text{const} > 0$ and $0 < \mu < 1$ (in our considerations in sections 3 and 4, μ is in fact the Poisson number which satisfies $0 < \mu < \frac{1}{2}$).

Furthermore, let $\mathscr{D}(\Omega)$ be the space of all real infinitely continuously differentiable functions with support in Ω, and let $W_0^{1,2}(\Omega)$ be the closure of $\mathscr{D}(\Omega)$ with respect to the norm $\|\ \|_{W^{1,2}} = (\ ,\ )^{1/2}_{W^{1,2}}$.

We denote by $C_0(\bar{\Omega})$ the space of all real functions which are continuous on $\bar{\Omega}$ and vanish on Γ, and by $C_c(\Omega)$ the space of all real functions which are continuous on Ω and vanish outside some compact subset of Ω, both spaces being furnished with the usual maximum-norm. By Sobolev's embedding theorem (cf. [5]) we have $V \subset C_0(\bar{\Omega})$.

Since $C_c(\Omega)$ is dense in $C_0(\bar{\Omega})$, each element in $(C_0(\bar{\Omega}))'$ can be identified by transposition with an element in $(C_c(\Omega))'$. Thus, in our discussion below, the Dirac measure $\delta = \delta_{(x_0,y_0)}((x_0, y_0) \in \Omega)$ is included as the perpendicular load in (1.1).

Moreover, by a standard argument, $L^1(\Omega) \subset (C_0(\bar{\Omega}))'$ where, denoting by $\langle f, \varphi \rangle$ the value of $f \in (C_0(\bar{\Omega}))'$ in $\varphi \in C_0(\bar{\Omega})$, it holds that

$$\langle f, \varphi \rangle = \int_\Omega f(x, y)\, \varphi(x, y)\, \mathrm{d}x\, \mathrm{d}y .$$

3. Statement of the theorem

Before passing to the definition of the notion of a variational solution of (1.1)–(1.3), we specify the assumptions on the boundary data in (1.2), (1.3). Since Γ is assumed to be Lipschitzian, one imposes

$$g_0 \in L^p(\Gamma_2) \quad \text{for} \quad 1 < p < \infty , \tag{3.1}$$

$$u_0 \in W^{1/2,2}(\Gamma), \quad v_0 \in W^{1/2,2}(\Gamma) . \text{ [1]} \tag{3.2}$$

The condition (3.2) implies the existence of elements u^*, $v^* \in W^{1,2}(\Omega)$ such that

$$u^* = u_0 , \quad v^* = v_0 \quad \text{on} \quad \Gamma \tag{3.3}$$

in the trace sense.

We then state

Definition. *The triple* $(w, u, v) \in V \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ *is called a variational solution of the boundary value problem* (1.1)–(1.3) *if*

(*i*) $u - u^* \in W_0^{1,2}(\Omega)$, $v - v^* \in W_0^{1,2}(\Omega)$,

(*ii*) *the identity*

$$\begin{aligned} &\frac{D}{h}\int_\Omega [w_{xx}\varphi_{xx} + 2(1-\mu)\, w_{xy}\varphi_{xy} + w_{yy}\varphi_{yy} + \mu(w_{xx}\varphi_{yy} + w_{yy}\varphi_{xx})]\, \mathrm{d}x\, \mathrm{d}y + \\ &+ \int_\Omega (\sigma_{11}w_x\varphi_x + \sigma_{12}w_x\varphi_y + \sigma_{21}w_y\varphi_x + \sigma_{22}w_y\varphi_y)\, \mathrm{d}x\, \mathrm{d}y + \\ &+ \int_\Omega (\sigma_{11}\psi_x + \sigma_{12}\psi_y + \sigma_{21}\zeta_x + \sigma_{22}\zeta_y)\, \mathrm{d}x\, \mathrm{d}y = \\ &= \int_{\Gamma_2} g_0\varphi_n\, \mathrm{d}s + \langle f, \varphi \rangle \text{ [2]} \end{aligned} \tag{3.4}$$

is satisfied for all $(\varphi, \psi, \zeta) \in V \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$.

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[1]) For details concerning the definition and investigation of the spaces $L^p(\Gamma)$, $W^{1/2,2}(\Gamma)$, we refer the reader to [5].

[2]) Hence forth we shall denote $\varphi_n = \partial\varphi/\partial n$.

Remarks. — 1. The integral identity (3.4) can be obtained in a formal way by multiplying equations (1.1) by test functions $\varphi \in V$, ψ, $\zeta \in W_0^{1,2}(\Omega)$, respectively, and integrating by parts, using boundary conditions (1.2).

2. Besides their belonging to $W_0^{1,2}(\Omega)$ and satisfying boundary conditions (3.3), the functions u^*, v^* are not subjected to additional conditions (cf. [8], [9]).

If the pair u^*, v^* is the solution of the two-dimensional equilibrium problem of linear elasticity, identity (3.4) as well as the associated potential get simplified. Proceeding in this way, it seems that instead of (1.3) more general boundary conditions can be handled. However, our approach does not depend on this argument.

We now formulate the main result.

Theorem. *For arbitrary boundary data satisfying* (3.1), (3.2), *and arbitrary* $f \in (C_0(\bar{\Omega}))'$, *boundary value problem* (1.1)–(1.3) *possesses at least one variational solution*; *in the case of zero boundary data on* u, v, *the variational solution coincides with the absolute minimum of the corresponding potential.*

4. Proof of the theorem

Introducing the notations

$$u = \bar{u} + u^*, \quad v = \bar{v} + v^* \tag{4.1}$$

where $\bar{u}$, $\bar{v} \in W_0^{1,2}(\Omega)$ and u^*, v^* is the fixed pair satisfying (3.3) and setting

$$\bar{\sigma}_{11} = \frac{E}{1-\mu^2}\left[\bar{u}_x + \tfrac{1}{2}w_x^2 + \mu(\bar{v}_y + \tfrac{1}{2}w_y^2)\right],$$

$$\bar{\sigma}_{22} = \frac{E}{1-\mu^2}\left[\bar{v}_y + \tfrac{1}{2}w_y^2 + \mu(\bar{u}_x + \tfrac{1}{2}w_x^2)\right],$$

$$\bar{\sigma}_{12} = \bar{\sigma}_{21} = \frac{E}{2(1+\mu)}(\bar{u}_y + \bar{v}_x + w_x w_y),$$

and

$$\sigma_{11}^* = \frac{E}{1-\mu^2}(u_x^* + \mu v_y^*), \quad \sigma_{22}^* = \frac{E}{1-\mu^2}(v_y^* + \mu u_x^*),$$

$$\sigma_{12}^* = \sigma_{21}^* = \frac{E}{2(1+\mu)}(u_y^* + v_x^*),$$

identity (3.4) may be written in the form

$$(4.2)\quad \frac{D}{h}\int_\Omega [w_{xx}\varphi_{xx} + 2(1-\mu)\, w_{xy}\varphi_{xy} + w_{yy}\varphi_{yy} + \mu(w_{xx}\varphi_{yy} + w_{yy}\varphi_{xx})]\,\mathrm{d}x\,\mathrm{d}y +$$
$$+ \int_\Omega (\bar{\sigma}_{11} w_x\varphi_x + \bar{\sigma}_{12} w_x\varphi_y + \bar{\sigma}_{21} w_y\varphi_x + \bar{\sigma}_{22} w_y\varphi_y)\,\mathrm{d}x\,\mathrm{d}y +$$
$$+ \int_\Omega (\bar{\sigma}_{11}\psi_x + \bar{\sigma}_{12}\psi_y + \bar{\sigma}_{21}\zeta_x + \bar{\sigma}_{22}\zeta_y)\,\mathrm{d}x\,\mathrm{d}y +$$
$$+ \int_\Omega [\sigma^*_{11}(\psi_x + w_x\varphi_x) + \sigma^*_{12}(\psi_y + \zeta_x + w_x\varphi_y + w_y\varphi_x) + \sigma^*_{22}(\zeta_y + w_y\varphi_y)]\,\mathrm{d}x\,\mathrm{d}y =$$
$$= \int_{\Gamma_2} g_0\varphi_n\,\mathrm{d}s + \langle f, \varphi\rangle\,.$$

Thus, if $(w, \bar{u}, \bar{v}) \in V \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ satisfies identity (4.2) for all $(\varphi, \psi, \zeta) \in \in V \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, the triple (w, u, v) with u, v according to (4.1) presents a variational solution of (1.1)–(1.3).

1° *The associated potential.* We set

$$\boldsymbol{H} = V \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$$

and denote the elements of $\boldsymbol{H}$ by $\boldsymbol{h} = (w, \bar{u}, \bar{v}), \ldots$. $\boldsymbol{H}$ is a Hilbert space with respect to the scalar product

$$((\boldsymbol{h}_1, \boldsymbol{h}_2)) = (w_1, w_2)_{W^{2,2}} + (\bar{u}_1, \bar{u}_2)_{W^{1,2}} + (\bar{v}_1, \bar{v}_2)_{W^{1,2}}\,.$$

It is obvious that the left hand side of (4.2) defines a (nonlinear) operator T of $\boldsymbol{H}$ into itself. A simple calculation shows that T is weakly differentiable where $T'(\boldsymbol{h})$, for any $\boldsymbol{h} \in \boldsymbol{H}$, is symmetric and satisfies the continuity property guaranteeing the applicability of Theorem 2 in [3].[1]) Thus, T is the gradient of the functional

$$F(\boldsymbol{h}) = \int_0^1 ((T(s\boldsymbol{h}), \boldsymbol{h}))\,\mathrm{d}s$$

and the potential associated to boundary value problem (1.1)–(1.3) is then given by

$$\Phi(\boldsymbol{h}) = F(\boldsymbol{h}) - \int_{\Gamma_2} g_0 w_n\,\mathrm{d}s - \langle f, w\rangle\,.$$

Clearly, under zero boundary conditions on u, v, each critical point of Φ (with respect to $\boldsymbol{H}$) is a variational solution of (1.1)–(1.3).

[1]) For the corresponding result slightly stronger differentiability and continuity properties are required in: *Vaijnberg, M. M.:* Variational methods for the investigation of nonlinear operators (Russian). — Moscow, GITTL 1956.

It is easy to see that F may be written in the form $F \lesseqgtr F_1 + F_2$ in which

$$(4.3)\quad F_1(\boldsymbol{h}) =$$

$$= \frac{D}{2h}\int_\Omega [w_{xx}^2 + 2(1-\mu)\, w_{xy}^2 + w_{yy}^2 + 2\mu w_{xx}w_{yy}]\, dx\, dy +$$

$$+ a_1 \int_\Omega [\mu(\bar{u}_x + \bar{v}_y)^2 + (1-\mu)(\bar{u}_x^2 + \bar{v}_y^2) + \tfrac{1}{2}(1-\mu)(\bar{u}_y + \bar{v}_x)^2]\, dx\, dy +$$

$$+ a_1 \int_\Omega (\bar{u}_x w_x^2 + \bar{v}_y w_y^2 + \mu\bar{u}_x w_y^2 + \mu\bar{v}_y w_x^2)\, dx\, dy +$$

$$+ \frac{a_1}{4}\int_\Omega (w_x^2 + w_y^2)^2\, dx\, dy + a_2 \int_\Omega (\bar{u}_y + \bar{v}_x)\, w_x w_y\, dx\, dy\,,$$

or equivalently, rearranging the formula,

$$(4.4)\quad F_1(\boldsymbol{h}) =$$

$$= \frac{D}{2h}\int_\Omega [w_{xx}^2 + 2(1-\mu)\, w_{xy}^2 + w_{yy}^2 + 2\mu w_{xx}w_{yy}]\, dx\, dy +$$

$$+ a_1\mu \int_\Omega (\bar{u}_x + \bar{v}_y + \tfrac{1}{2}w_x^2 + \tfrac{1}{2}w_y^2)^2\, dx\, dy +$$

$$+ \frac{a_2}{2}\int_\Omega (\bar{u}_y + \bar{v}_x + w_x w_y)^2\, dx\, dy +$$

$$+ a_2 \int_\Omega [(\bar{u}_x + \tfrac{1}{2}w_x^2)^2 + (\bar{v}_y + \tfrac{1}{2}w_y^2)^2]\, dx\, dy\,,$$

while F_2 is given by

$$(4.5)\quad F_2(\boldsymbol{h}) =$$

$$= 2a_1 \int_\Omega [(u_x^* + \mu v_y^*)(\bar{u}_x + \tfrac{1}{2}w_x^2) + (v_y^* + \mu u_x^*)(\bar{v}_y + \tfrac{1}{2}w_y^2)]\, dx\, dy +$$

$$+ a_2 \int_\Omega (u_y^* + v_x^*)(\bar{u}_y + \bar{v}_x + w_x w_y)\, dx\, dy\;;$$

here we have set

$$a_1 \equiv \frac{E}{2(1-\mu^2)}\,, \quad a_2 \equiv \frac{E}{2(1+\mu)}\,.$$

2° *Estimation of Φ from below.* We now derive two inequalities implying the coerciveness of Φ.

Let us turn to (4.3). First of all, by Schwarz's inequality and Sobolev's embedding theorem, we have[1])

$$\left|\int_\Omega (u_y + v_x)\, w_x w_y \,\mathrm{d}x\,\mathrm{d}y\right| \leqq \text{const}\,(\|u\|_{W^{1,2}} + \|v\|_{W^{1,2}})\,\|w\|^2_{W^{2,2}}$$

for any $(w, u, v) \in V \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. Proceeding similarly with the third integral in (4.3), integrating by parts the second one (which presents a special case of Korn's inequality, see [5]), and using (2.1), one obtains

$$(4.6)\quad F_1(\boldsymbol{h}) \geqq c_1(\|w\|^2_{W^{2,2}} + \|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}}) + c_1\int_\Omega (w_x^2 + w_y^2)^2\,\mathrm{d}x\,\mathrm{d}y - \\ - c_2(\|u\|_{W^{1,2}} + \|v\|_{W^{1,2}})\,\|w\|^2_{W^{2,2}}$$

for all $\boldsymbol{h} \in \boldsymbol{H}$ where $c_i = \text{const} > 0$, $i = 1, 2$.

With respect to this, F_2 will be estimated as follows:

$$(4.7)\quad |F_2(\boldsymbol{h})| \leqq \frac{\varepsilon_1}{2}\left\{\|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}} + \int_\Omega (w_x^2 + w_y^2)^2\,\mathrm{d}x\,\mathrm{d}y\right\} + \\ + \frac{1}{2\varepsilon_1}\cdot\text{const}\,(\|u^*\|^2_{W^{1,2}} + \|v^*\|^2_{W^{1,2}})\ \forall\, \boldsymbol{h} \in \boldsymbol{H}\,;$$

here ε_1 denotes an arbitrary positive constant. Using the continuity of the trace operator on $W^{2,2}(\Omega)$ (see [5]) and Sobolev's embedding theorem, we get

$$(4.8)\quad \left|\int_{\Gamma_2} g_0 w_n\,\mathrm{d}s + \langle f, w\rangle\right| \leqq \\ \leqq \frac{\varepsilon_2}{2}\,\|w\|^2_{W^{2,2}} + \frac{1}{2\varepsilon_2}\cdot\text{const}\,(\|g_0\|^2_{L^p(\Gamma_2)} + \|f\|^2_{(C_0(\bar\Omega))'})$$

which is valid for all $w \in V$ and any $\varepsilon_2 > 0$.

Setting $\varepsilon_1 = \varepsilon_2 = c_1$ in (4.7), (4.8), respectively, it follows by (4.6) that

$$(4.9)\quad \Phi(\boldsymbol{h}) \geqq \frac{c_1}{2}(\|w\|^2_{W^{2,2}} + \|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}}) - \\ - c_2(\|u\|_{W^{1,2}} + \|v\|_{W^{1,2}})\,\|w\|^2_{W^{2,2}} - c_3$$

for all $(w, u, v) = \boldsymbol{h} \in \boldsymbol{H}$ where

$$c_3 = \text{const}\,(\|u^*\|^2_{W^{1,2}} + \|v^*\|^2_{W^{1,2}} + \|g_0\|^2_{L^p(\Gamma_2)} + \|f\|^2_{(C_0(\bar\Omega))'})$$

which yields the first estimate we are interested in.

[1]) Since there is no danger of confusion, throughout the remainder of the paper we drop the bar in $\bar{u}$, $\bar{v}$.

In order to obtain the second one, we start with (4.4). Obviously, F_2 can be estimated in the following way:

$$|F_2(\boldsymbol{h})| \leqq a_2 \int_\Omega [(u_x + \tfrac{1}{2}w_x^2)^2 + (v_y + \tfrac{1}{2}w_y^2)^2] \,\mathrm{d}x\,\mathrm{d}y + \frac{a_2}{2} \int_\Omega (u_y + v_x + \\ + w_x w_y)^2 \,\mathrm{d}x\,\mathrm{d}y + \mathrm{const}\,(\|u^*\|^2_{W^{1,2}} + \|v^*\|^2_{W^{1,2}})\ \forall\, \boldsymbol{h} \in \boldsymbol{H}\,.$$

Arguing as above, we finally get

$$\Phi(\boldsymbol{h}) \geqq c_1' \|w\|^2_{W^{2,2}} - c_3' \tag{4.10}$$

for all $(w, u, v) = \boldsymbol{h} \in \boldsymbol{H}$ where $c_i' = \mathrm{const} > 0$, $i = 1, 3$, and c_3' has the same structure as c_3 above.

3° *Proof of the theorem completed.* Set

$$|||\boldsymbol{h}||| = (\|w\|^2_{W^{2,2}} + \|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}})^{1/2} = r\,.$$

We suppose $r > 2c_1^{-1}$. Using (4.9) if $\|w\|^2_{W^{2,2}} \leqq (c_2 \,.\, 2^{1/2})^{-1}\,(\tfrac{1}{2}c_1 r - 1)$, and using (4.10) in the other case, one easily gets the coerciveness

$$\Phi(\boldsymbol{h}) \geqq k_1 |||\boldsymbol{h}||| - k_2 \quad \text{for all} \quad |||\boldsymbol{h}||| > 2c_1^{-1}$$

where $k_i = \mathrm{const} > 0$, $i = 1, 2$.

The weak lower semicontinuity of Φ follows immediately by Korn's inequality and Sobolev's embedding theorem.

Thus, there exists $(\tilde{w}, \tilde{u}, \tilde{v}) = \tilde{\boldsymbol{h}} \in \boldsymbol{H}$ in which Φ attains its absolute minimum on $\boldsymbol{H}$; the triple $(\tilde{w}, \tilde{u} + u^*, \tilde{v} + v^*)$ then presents a variational solution of (1.1)–(1.3).

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Souhrn

O JEDNOM OKRAJOVÉM PROBLÉMU Z NELINEÁRNÍ TEORIE TENKÝCH PRUŽNÝCH DESEK

JINDŘICH NEČAS, JOACHIM NAUMANN

V této práci jsou řešeny okrajové úlohy pro systém nelineárních parciálních diferenciálních rovnic pro posunutí, popisujících průhyb tenkých desek. Užívá se abstraktního variačního počtu.

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