Ivan Hlaváček

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ON A CONJUGATE SEMI-VARIATIONAL METHOD FOR PARABOLIC EQUATIONS

IVAN HLAVÁČEK

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INTRODUCTION

In the physical literature it has been long recognized that problems of heat conduction can be formulated and solved either in terms of temperature or of heat flux field [1], [2]. The first approach leads to the usual formulation of the well-known parabolic equation for heat conduction. Efficient variational methods have been established and analysed to solve the mixed problems with parabolic equations, e.g. Crank-Nicholson-Galerkin procedure [3], semi-variational method [4] a.o.

The second approach, using the heat flux field (or entropy displacements, respectively), is less known, though it may bring some new results both in theory and practice.

The present paper is devoted to the study of the second approach, when applied to a simple mathematical model of one parabolic equation of the second order. Our formulation is similar to that of a conjugate problem as defined by Aubin and Burchard in [5] and the method can be extended to parabolic problems of more complex type, such as parabolic systems and equations of higher order.

In Section 1, the conjugate variational formulation is shown to be a particular case of a general parabolic equation with two positive operators and therefore the results of Section 3 of [4]-II apply, when completed by the proof of a new auxiliary inequality. In Section 2 we define the first and second semi-variational approximation to the solution of the conjugate problem. The a priori error estimates show that the first and second semi-variational approximations are second and fourth order correct in the time increment, respectively.

Using the terminology of the finite element method, the conjugate method belongs to the "equilibrium models", whereas the usual method presents a "displacement model". In contradiction to the equilibrium models for elliptic problems, however, no difficulties occur here during the construction of the basis functions. Almost the same shape and basis functions can be employed to the "equilibrium type" elements as to the "displacement type" ones.

1. CONJUGATE PARABOLIC PROBLEM

There is a close formal analogy between the conjugate elliptic problems [5] and parabolic problems, if only the time-derivative operator is treated as a time-independent parameter. We are going to show this analogy on the following example.

Let us consider the mixed problem for the equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0 u = f\,, \tag{1.1}$$

$$x = (x_1, x_2, \ldots, x_n) \in \Omega\,, \quad 0 < t \leqq T < \infty\,,$$

where Ω is a bounded region with a Lipschitz boundary Γ, repeated Latin index implies summation over the range $1, 2, \ldots, n$.

The initial condition takes the form

$$u(\cdot, 0) = \varphi_0\,. \tag{1.2}$$

The boundary Γ consists of disjoint parts

$$\Gamma = \Gamma_u \cup \Gamma_h \cup \Gamma_v \cup \Gamma_0$$

where mes Γ_0 is zero and each of Γ_u, Γ_h, Γ_v is either open in Γ or empty.

The boundary conditions are

$$u = g \qquad \text{on} \quad \Gamma_u \times (0, T\rangle\,, \tag{1.3}$$

$$a_{ij}\frac{\partial u}{\partial x_j}\nu_i = P \qquad \text{on} \quad \Gamma_h \times (0, T\rangle\,, \tag{1.4}$$

$$\alpha u + a_{ij}\frac{\partial u}{\partial x_j}\nu_i = P \quad \text{on} \quad \Gamma_v \times (0, T\rangle\,. \tag{1.5}$$

Here $\varphi_0(x)$, $f(x, t)$, $g(x, t)$, $P(x, t)$ are given functions, ν_i denote the components of the unit outward normal to Γ.

Let us denote

$$a_{ij}\frac{\partial u}{\partial x_j} = h_i\,, \quad \frac{\partial h_i}{\partial x_i} = \operatorname{div} \mathbf{h}\,, \quad h_i\nu_i = h_\nu\,, \tag{1.6}$$

so that

$$\frac{\partial u}{\partial x_k} = b_{ki}h_i \tag{1.7}$$

where $b_{ij}(x)$ represent the matrix inverse to $\boldsymbol{a} = (a_{ij}(x))$.

Differentiating (1.1) with respect to x_k, we obtain

$$\left(a_0 + \frac{\partial}{\partial t}\right)\frac{\partial u}{\partial x_k} - \frac{\partial^2}{\partial x_i\,\partial x_k}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)] = \frac{\partial f}{\partial x_k}\,,$$

which can be rewritten, making use of (1.6), (1.7), as follows

$$(1.8) \qquad \left(a_0 + \frac{\partial}{\partial t}\right) b_{ki} h_i - \frac{\partial}{\partial x_k} \operatorname{div} \boldsymbol{h} = \frac{\partial f}{\partial x_k}, \quad k = 1, 2, \ldots, n.$$

The initial condition (1.2) in terms of the vector-function $\boldsymbol{h}$ is

$$(1.9) \qquad h_i(\cdot, 0) = a_{ij} \frac{\partial \varphi_0}{\partial x_j}$$

or more concisely

$$\boldsymbol{h}(\cdot, 0) = \boldsymbol{a} \operatorname{grad} \varphi_0$$

and the boundary conditions (1.3)–(1.5) are equivalent to

$$(1.10) \qquad f + \operatorname{div} \boldsymbol{h} = \left(a_0 + \frac{\partial}{\partial t}\right) g \qquad \text{on} \quad \Gamma_u \times (0, T\rangle,$$

$$(1.11) \qquad h_v = P \qquad \text{on} \quad \Gamma_h \times (0, T\rangle,$$

$$(1.12) \qquad \alpha(f + \operatorname{div} \boldsymbol{h}) + \left(a_0 + \frac{\partial}{\partial t}\right) h_v = \left(a_0 + \frac{\partial}{\partial t}\right) P \quad \text{on} \quad \Gamma_v \times (0, T\rangle.$$

We are led to the problem (1.8)–(1.12), which we call *conjugate* to the original mixed problem (1.1)–(1.5) (cf. [5] and [6] – eqs. (3.13)–(3.15)).

Suppose that the coefficients $a_{ij}(x)$, $\alpha(x)$ are bounded measurable functions on $\bar{\Omega}$ and Γ_v, respectively, $a_0 =$ konst., and the matrix $\boldsymbol{a} = a_{ij}(x)$ is symmetric and positive definite with its spectrum bounded above and below by positive numbers, which are independent of $x \in \bar{\Omega}$,

$$a_0 \geqq 0, \quad \alpha_1 \geqq \alpha(x) \geqq \alpha_0 > 0, \quad x \in \Gamma_v.$$

Definition 1. *Let us introduce the following linear spaces, bilinear forms and norms*:

$$(1.13) \qquad H_B = \{\chi \in [L_2(\Omega)]^n, \ \chi_v \in L_2(\Gamma_v) \text{ on } \Gamma_v\},$$

$$\bar{H}_A = \{\chi \in [L_2(\Omega)]^n, \ \operatorname{div} \chi \in L_2(\Omega), \ \chi_v \sqrt{(a_0)} \in L_2(\Gamma_v) \text{ on } \Gamma_v\},$$

$$H_A = \{\chi \in \bar{H}_A, \ \chi_v = 0 \text{ on } \Gamma_h\},$$

$$\mathscr{V} = H_A \cap H_B,$$

$$(\varphi, \psi) = \int_\Omega \varphi \psi \, dx; \quad \varphi, \psi \in L_2(\Omega),$$

$$(\varphi, \psi)_{\Gamma_s} = \int_{\Gamma_s} \varphi \psi \, d\Gamma; \quad \varphi, \psi \in L_2(\Gamma_s), \quad \Gamma_s = \Gamma_u \quad \text{or} \quad \Gamma_s = \Gamma_v.$$

$$(1.14) \qquad B(h, \chi) = (b_{ki} h_i, \chi_k) + (\alpha^{-1} h_\nu, \chi_\nu)_{\Gamma_v}\,; \quad h, \chi \in H_B\,,$$

$$A(h, \chi) = (\operatorname{div} h, \operatorname{div} \chi) + B(a_0 h, \chi)\,; \quad h, \chi \in \bar{H}_A\,,$$

$$B(\chi, \chi) = \|\chi\|_B^2\,, \quad A(\chi, \chi) = \|\chi\|_A^2\,, \quad \|\chi\|^2 = \|\chi\|_A^2 + \|\chi\|_B^2\,.$$

Let $L_2(\langle 0, T\rangle, H)$ *denote the space of measurable mappings* $u(t)$ *of* $\langle 0, T\rangle$ *into a normed space* H *such that*

$$\int_0^T \|u(t)\|_H^2 \,\mathrm{d}t < \infty\,.$$

Suppose that a vector-function $g_P(x, t)$ *exists such that*

$$g_P(\cdot, t)\,, \quad \frac{\partial}{\partial t} g_P(\cdot, t) \in [W_2^{(1)}(\Omega)]^n\,, \quad t \in \langle 0, T\rangle\,,$$

$$g_{P\nu} = P \quad on \quad \Gamma_h \times \langle 0, T\rangle\,.$$

Moreover, let $\varphi_0(x)$ *be such that*

$$\operatorname{grad} \varphi_0 \in H_B\,,$$

$$f(\cdot, t) \in L_2(\Omega)\,, \quad a_0\, g(\cdot, t)\,, \quad \frac{\partial}{\partial t} g(\cdot, t) \in W_2^{(1)}(\Omega)\,,$$

$$a_0\, P(\cdot, t) \in L_2(\Gamma_v)\,, \quad \frac{\partial}{\partial t} P(\cdot, t) \in L_2(\Gamma_v)\,, \quad t \in \langle 0, T\rangle\,.$$

Remark 1.1. It is easy to see that the norm $\|\chi\|_B$ is equivalent with

$$[(\chi_i, \chi_i) + (\chi_\nu, \chi_\nu)_{\Gamma_v}]^{1/2}\,.$$

Moreover,

$$\|h\|_A < \infty \quad \text{for} \quad h \in \bar{H}_A \quad \text{and} \quad \|h\|_B < \infty \quad \text{for} \quad h \in H_B\,,$$

$$[W_2^{(1)}(\Omega)]^n \subset \bar{H}_A \cap H_B$$

and

$$B(a_0 h, \chi) = B(h \sqrt{a_0}, \chi \sqrt{a_0})\,.$$

As A and B are symmetric, positive bilinear forms on $\bar{H}_A \times \bar{H}_A$ and $H_B \times H_B$, respectively, the Schwartz inequality implies

$$|A(h, \chi)| \leqq \|h\|_A \|\chi\|_A\,,$$

$$|B(h, \chi)| \leqq \|h\|_B \|\chi\|_B\,.$$

Definition 2. *We say that $h(x, t)$ is a weak solution of the conjugate problem* (1.8)–(1.12), *if*

$$h - g_P \in L_2(\langle 0, T\rangle, \mathscr{V}) \tag{1.15}$$

$$\frac{\partial}{\partial t}(h - g_P) \in L_2(\langle 0, T\rangle, H_B),$$

$$B\left(\frac{\partial h}{\partial t}, \chi\right) + A(h, \chi) = -(f, \operatorname{div} \chi) + \left(a_0 g + \frac{\partial g}{\partial t}, \chi_v\right)_{\Gamma_u} + \tag{1.16}$$

$$+ \left(\alpha^{-1} a_0 P + \alpha^{-1} \frac{\partial P}{\partial t}, \chi_v\right)_{\Gamma_v}, \quad 0 < t \leqq T, \quad \chi \in \mathscr{V},$$

$$B(h(\cdot, 0) - a \operatorname{grad} \varphi_0, \chi) = 0, \quad \chi \in \mathscr{V}. \tag{1.17}$$

Remark 1.2. We can derive (1.16) formally, multiplying equations (1.8) by the test function χ, integrating by parts and using both the definition of $\mathscr{V}$ and the boundary conditions (1.10), (1.12).

Setting

$$h = g_P + k,$$

we reformulate the definition by means of k, as follows in

Definition 3. *We say that $h = g_P + k$ is a weak solution of the conjugate problem* (1.8)–(1.12), *if*

$$k \in L_2(\langle 0, T\rangle, \mathscr{V}), \tag{1.18}$$

$$\frac{\partial k}{\partial t} \in L_2(\langle 0, T\rangle, H_B),$$

$$B\left(\frac{\partial k}{\partial t}, \chi\right) + A(k, \chi) = -(f, \operatorname{div} \chi) - A(g_P, \chi) - \tag{1.19}$$

$$- B\left(\frac{\partial g_P}{\partial t}, \chi\right) + \left(\alpha^{-1} a_0 P + \alpha^{-1} \frac{\partial P}{\partial t}, \chi_v\right)_{\Gamma_v} + \left(a_0 g + \frac{\partial g}{\partial t}, \chi_v\right) \Gamma_u,$$

$$0 < t \leqq T, \quad \chi \in \mathscr{V}.$$

$$B(k(\cdot, 0) - \psi_0, \chi) = 0, \quad \chi \in \mathscr{V}, \tag{1.20}$$

where

$$\psi_0 = a \operatorname{grad} \varphi_0 - g_P(\cdot, 0).$$

Lemma 1.1. *The right-hand side of* (1.19) *defines a linear functional* $\langle \mathcal{J}(t), \chi \rangle$, *which is continuous on* $\mathcal{V}$.

Proof. Using Remark 1.1, the assertion can be easily verified for all the terms except the last one. Integrating by parts, we obtain

$$(G, \chi_\nu)_{\Gamma_u} = \int_{\Gamma_u} G\chi_\nu \, d\Gamma = - \int_{\Gamma_v} G\chi_\nu \, d\Gamma + \int_\Omega \left(\frac{\partial G}{\partial x_i} \chi_i + G \operatorname{div} \chi \right) dx$$

for any $G \in W_2^{(1)}(\Omega)$, $\chi \in \mathcal{V}$. Consequently

$$|(G, \chi_\nu)_{\Gamma_u}| \leqq C \|G\|_{W_2^{(1)}(\Omega)} \|\chi\| ,$$

and because we have

$$a_0 g + \frac{\partial g}{\partial t} \in W_2^{(1)}(\Omega) ,$$

the last term is continuous on $\mathcal{V}$, as well.

Remark 1.3. If the time derivatives in (1.19) are taken in the sense of distributions (see e.g. [7], chpt. IV., Th. 7.1), there exists precisely one weak solution of the conjugate problem.

Henceforth we shall assume the existence of a solution $k(x, t)$ of the problem (1.18)–(1.20). The uniqueness then follows from Remark 1.3.

Moreover, we suppose that

$$(1.21) \qquad \lim_{t \to 0+} \|k(\cdot, t) - k(\cdot, 0)\|_A = 0 ,$$

$$\lim_{t \to 0+} \left\| \frac{\partial}{\partial t} k(\cdot, t) - \frac{\partial}{\partial t} k(\cdot, 0+) \right\|_B = 0 .$$

2. SEMI-VARIATIONAL APPROXIMATIONS

Comparing Definition 3 with the problem (3.8)–(3.10) of the Section 3 in [4]-II, we can see that the problem (1.18)–(1.20) is of the same type as the latter, if we set

$$H = [L_2(\Omega)]^n , \quad \mathcal{V}_0 = H_A , \quad \mathcal{V}_1 = H_B ,$$
$$\|\chi\|_0^2 = [\chi, \chi]_A = \|\chi\|_A^2 , \quad \|\chi\|_1^2 = [\chi, \chi]_B = \|\chi\|_B^2 .$$

The only difference is, that the supposition (3.4) of [4]-II, i.e.

$$H_A \subset H_B , \quad \|\chi\|_B < C\|\chi\|_A$$

does not hold and the corresponding operator A is only positively semi-definite in H.

Nevertheless, the conclusions of Section 3 of [4]-II hold for the problem (1.18)–(1.20), as well. Thus we may construct the sequence of semi-variational approximations with an increasing accuracy in the time-increment. In the following, we present the first and second approximations and the error estimates in detail.

First approximation (Crank-Nicholson-Galerkin type)

Let a finite-dimensional subspace $\mathcal{M}$ of $\mathcal{V}$ be spanned by elements $\chi^1, \chi^2, \ldots, \chi^N$, which are linearly independent in H_B. Let the interval $\langle 0, T \rangle$ be divided into M subintervals, the length of which is $\tau = T/M$.

Denote

$$K_m = k^{(1)}(., m\tau) = \sum_{i=1}^{N} w_i^m \chi^i, \quad m = 0, 1, \ldots, M,$$

$$\bar{f}_m = \bar{f}(m\tau), \quad \tfrac{1}{2}(\bar{f}_m + \bar{f}_{m+1}) = \bar{f}_{m+1/2},$$

$$\tfrac{1}{2}(K_m + K_{m+1}) = K_{m+1/2}, \quad K_{m+1} - K_m = \delta K_m .$$

The first semi-variational approximation $k^{(1)}$ of the conjugate problem will be defined by the system

(2.1)
$$\frac{1}{\tau} B(\delta K_m, \chi) + A(K_{m+1/2}, \chi) = \langle \bar{f}_{m+1/2}, \chi \rangle,$$
$$m = 0, 1, \ldots, M - 1, \quad \chi \in \mathcal{M}$$

and the initial condition

(2.2)
$$B(K_0 - \psi_0, \chi) = 0, \quad \chi \in \mathcal{M} .$$

The matrix form of (2.1), (2.2) is (cf. (3.5) of [4]-II)

$$\left(\mathcal{B} + \frac{\tau}{2} \mathcal{A}\right) \mathbf{a}^m = \mathcal{B}\mathbf{w}^m + \frac{\tau}{4} \left[\mathbf{F}(m\tau) + \mathbf{F}(m\tau + \tau)\right],$$

$$\mathcal{B}\mathbf{w}^0 = \omega_0, \quad \mathbf{w}^{m+1} = 2\mathbf{a}^m - \mathbf{w}^m,$$

where

$$\mathcal{A}_{ij} = A(\chi^i, \chi^j), \quad \mathcal{B}_{ij} = B(\chi^i, \chi^j),$$
$$\omega_{0j} = B(\psi_0, \chi^j), \quad F_j(m\tau) = \langle \bar{f}_m, \chi^j \rangle .$$

As $\mathcal{B}$ and $\mathcal{A}$ are positive definite and positive semi-definite matrices, respectively, the system has a unique solution for any m and any positive τ.

Remark 2.1. There is another alternative of the right-hand side in (2.1); namely we can set

(2.1′)
$$\langle f^*_{m+1/2}, \chi \rangle + \frac{1}{\tau} \langle \delta \hat{g}_m, \chi \rangle$$

instead of $\langle \bar{f}_{m+1/2}, \chi \rangle$, where

$$\langle f^*, \chi \rangle = -(f, \operatorname{div} \chi) - A(g_P, \chi) + (\alpha^{-1} a_0 P, \chi_v)_{\Gamma_v} + (a_0 g, \chi_v)_{\Gamma_u},$$
$$\langle \hat{g}, \chi \rangle = -B(g_P, \chi) + (\alpha^{-1} P, \chi_v)_{\Gamma_v} + (g, \chi_v)_{\Gamma_u}.$$

This alternative can be derived on the base of a convolution variational principle (secondary β-differential in [6]) and a projection, likewise the first approximation in Section 1 of [4]-I.

Theorem 1. *Suppose that the solution $k(x, t)$ of the problem* (1.18)–(1.20) *satisfies* (1.21) *and possesses continuous and bounded derivatives $\partial^3 k/\partial t^3$ on $(\Omega \cup \Gamma_v) \times \times (0, T)$.*

Denote $z_m = k_m - K_m$, where K_m is a solution of (2.1), (2.2), *δ_{jk} the Kronecker's delta.*

Let $\tilde{k}$ be any function of the form

(2.3) $$\tilde{k}(x, t) = \sum_{i=1}^{N} \alpha_i(t)\, \chi^i(x).$$

Then there exist positive constants C and τ_0, independent of τ, such that

(2.4) $$\|z_m\|_B^2 + \sum_{p=0}^{m-1} \tau \|z_{p+1/2}\|_A^2 \leqq$$
$$\leqq C\left\{\sum_{p=0}^{m-1} \tau \left[\|(k - \tilde{k})_{p+1/2}\|^2 + (1 - \delta_{1m}) \left\|\frac{1}{\tau}\, \delta(k - \tilde{k})_{p-1/2}\right\|_B^2\right] + \right.$$
$$\left. + \|(k - \tilde{k})_0\|_B^2 + \|(k - \tilde{k})_{1/2}\|_B^2 + \|(k - \tilde{k})_{m-1/2}\|_B^2 + \tau^4\right\}$$

holds for $m = 1, 2, \ldots, M$ and $\tau \leqq \tau_0$.

Proof is merely a slight modification of the proof of Theorem 4.1 of [3] (cf. also Theorem 3.1 and Remark 3.2 of [4]-II).

Remark 2.2. In case of the alternative (2.1′) Theorem 1 holds under the following additional assumptions: $\partial^3 g/\partial t^3$ is continuous and bounded on $\Gamma_u \times (0, T)$, $\partial^3 g_P/\partial t^3$ on $(\Omega \cup \Gamma_v) \times (0, T)$ and $\partial^3 P/\partial t^3$ on $\Gamma_v \times (0, T)$.

Remark 2.3. Let $\bar{f} = 0$ in (2.1) (or $f^* = 0$, $\delta \hat{g}_m = 0$ in (2.1′)). Then

(2.5) $$\|K_{m+1}\|_B \leqq \|K_m\|_B \leqq \|\psi_0\|_B, \quad m = 0, 1, \ldots, M - 1.$$

The assertion follows directly from (2.1) by inserting $\chi = K_m + K_{m+1}$ and from (2.2).

Remark 2.4. The form of the left-hand side of (2.4) leads to a suggestion that rather $k(\cdot, p\tau + \frac{1}{2}\tau)$ should be approximated by $K_{p+1/2}$ than $k(\cdot, p\tau)$ by K_p (cf. also [3]).

Second approximation

The second semi-variational approximation $k^{(2)}$ of the conjugate problem will be defined by the system

$$(2.6)\quad \frac{4}{\tau} B(K_m - 2K_{m+1/2} + K_{m+1}, \chi) + A(K_{m+1} - K_m, \chi) = \langle \tilde{f}_{m+1} - \tilde{f}_m, \chi \rangle ,$$

$$(2.7)\quad \frac{1}{\tau} B(K_{m+1} - K_m, \chi) + \tfrac{1}{6}A(K_m + 4K_{m+1/2} + K_{m+1}, \chi) =$$
$$= \tfrac{1}{6}\langle \tilde{f}_m + 4\tilde{f}_{m+1/2} + \tilde{f}_{m+1}, \chi \rangle , \quad m = 0, 1, \ldots, M-1 , \quad \chi \in \mathcal{M} ,$$

and the initial condition (2.2),

where

$$K_{m+1/2} = k^{(2)}(\cdot, m\tau + \tfrac{1}{2}\tau) , \quad \tilde{f}_{m+1/2} = \tilde{f}(m\tau + \tfrac{1}{2}\tau) .$$

The matrix form of (2.6), (2.7) is

$$(2.8)\quad \mathcal{B}\mathbf{c}^m - \left(\frac{\tau}{12}\mathcal{A} + \tfrac{1}{2}\mathcal{B}\right)\mathbf{b}^m = \mathcal{B}\mathbf{w}^m + \frac{\tau}{12}\left[\mathbf{F}(m\tau) - \mathbf{F}(m\tau + \tau)\right] ,$$

$$\mathcal{A}\mathbf{c}^m + \frac{1}{\tau}\mathcal{B}\mathbf{b}^m = \tfrac{1}{6}\left[\mathbf{F}(m\tau) + 4\mathbf{F}(m\tau + \tfrac{1}{2}\tau) + \mathbf{F}(m\tau + \tau)\right] ,$$

where

$$\mathbf{c}^m = \tfrac{1}{6}(\mathbf{w}^m + 4\mathbf{w}^{m+1/2} + \mathbf{w}^m) , \quad \mathbf{b}^m = \mathbf{w}^{m+1} - \mathbf{w}^m ,$$

$$K_m = k^{(2)}(\cdot, m\tau) = \sum_{i=1}^{N} w_i^m \chi^i .$$

It is easy to show that the system (2.8) has a unique solution for any m and positive τ.

Theorem 2. *Suppose that the solution $k(x, t)$ of the problem (1.18)–(1.20) satisfies (1.21) and possesses continuous and bounded derivatives $\partial^5 k/\partial t^5$ on $(\Omega \cup \Gamma_v) \times (0, T)$.*

Denote $z_m = k_m - K_m$, $s_m^\wedge = \frac{1}{6}(s_m + 4s_{m+1/2} + s_{m+1})$, where K_m is a solution of (2.6), (2.7) with the initial condition (2.2). Let $\tilde{k}$ be any function of the form (2.3).

Then there exist positive constants C and τ_0, independent of τ, such that

$$\|z_m\|_B^2 + \tau \sum_{p=0}^{m-1} (\|\delta z_p\|_A^2 + \|z_p^\wedge\|_A^2) \leqq \tag{2.9}$$

$$\leqq C \left\{ \sum_{p=0}^{m-1} \tau \left[\|(k - \tilde{k})_p^\wedge\|^2 + \left\| \frac{1}{\tau} \delta(k - \tilde{k})_p \right\|_B^2 + \|\delta(k - \tilde{k})_p\|_A^2 \right] + \right.$$

$$\left. + \sum_{p=0}^{m-2} \tau \left\| \frac{1}{\tau} \delta(k - \tilde{k})_{p+1/2} \right\|_B^2 + \|(k - \tilde{k})_0\|_B^2 + \|(k - \tilde{k})_0^\wedge\|_B^2 + \|(k - \tilde{k})_{m-1}^\wedge\|_B^2 + \tau^8 \right\}$$

holds for $m = 2, 3, \ldots, M$ and $\tau \leqq \tau_0$.

Proof is similar to that of Theorem I.2.1 of [4]-I with the only change that instead of the inequality of the type

$$\|z_m^\wedge\|_B \leqq C\|z_m^\wedge\|_A ,$$

which was used to deduce (2.20) of [4]-I, we employ the inequality

$$\|z_m^\wedge\|_B^2 \leqq C\{\|\delta z_m\|_A^2 + \|z_m^\wedge\|_A^2 + \|z_m\|_B^2 + \|z_{m+1}\|_B^2 + \\ + \|(k - \tilde{k})_m^\wedge\|^2 + \|\tau\zeta_m\|_B^2\} , \tag{2.10}$$

where

$$\zeta_m = \frac{4}{\tau} \Delta^2 k_m - \delta(\partial k/\partial t)_m ,$$

$$\Delta^2 k_m = k_m - 2k_{m+1/2} + k_{m+1} .$$

In order to prove (2.10), we derive, as in the proof of Theorem I.2.1 of [4]-I,

$$\frac{4}{\tau} B(\Delta^2 k_m, \chi) + A(\delta k_m, \chi) = B(\zeta_m, \chi) + \langle \delta \bar{f}_m, \chi \rangle , \tag{2.11}$$

$$m = 0, 1, \ldots, M - 1 , \quad \chi \in \mathcal{M} .$$

If we subtract (2.6) from (2.11) and insert

$$\chi = (\tilde{k} - K)_m^\wedge = z_m^\wedge + (\tilde{k} - k)_m^\wedge ,$$

$$\Delta^2 z_m = -3z_m^\wedge + \tfrac{3}{2}(z_m + z_{m+1}) ,$$

we obtain

$$4B(-3z_m^\wedge + \tfrac{3}{2}(z_m + z_{m+1}) , \quad z_m^\wedge + (\tilde{k} - k)_m^\wedge) + \\ + \tau A(\delta z_m, z_m^\wedge + (\tilde{k} - k)_m^\wedge) = \tau B(\zeta_m, z_m^\wedge + (\tilde{k} - k)_m^\wedge) .$$

From there it follows, with the use of Remark 1.1, that

$$12\|z_m^\wedge\|_B^2 \leqq 12\|z_m^\wedge\|_B \|(\tilde{k} - k)_m^\wedge\|_B + 6(\|z_m\|_B + \|z_{m+1}\|_B)(\|z_m^\wedge\|_B + \|(\tilde{k} - k)_m^\wedge\|_B) + \\ + \tau\|\delta z_m\|_A (\|z_m^\wedge\|_A + \|(\tilde{k} - k)_m^\wedge\|_A) + \|\tau\zeta_m\|_B (\|z_m^\wedge\|_B + \|(\tilde{k} - k)_m^\wedge\|_B) \leqq \\ \leqq 3\varepsilon\|z_m^\wedge\|_B^2 + C\{\|z_m\|_B^2 + \|z_{m+1}\|_B^2 + \|\delta z_m\|_A^2 + \|z_m^\wedge\|_A^2 + \|(\tilde{k} - k)_m^\wedge\| + \|\tau\zeta_m\|_B^2\} .$$

Choosing ε small enough, we come to (2.10).

Remark 2.5. Let $\bar{f} = 0$ in (2.6), (2.7). Then

$$\|K_{m+1}\|_B \leqq \|K_m\|_B \leqq \|\psi_0\|_B,$$
$$\|K_m^{\wedge}\|_B \leqq \|K_m\|_B, \quad \|K_{m+1/2}\|_B \leqq 2\|K_m\|_B$$

holds for $m = 0, 1, \ldots, M - 1$.

The proof (cf. Th. II.3.6 of [4]-II) is analogous to that of Th. I.2.3 in [4]-I. A stronger relation, like that of Remark II.1.2 of [4]-II, could be derived, as well.

Remark 2.6. The form of the left-hand side of (2.9) leads to a suggestion that rather $k(\cdot, p\tau + \frac{1}{2}\tau)$ should be approximated by $K_p^{\wedge}$ than $k(\cdot, p\tau)$ by K_p.

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Souhrn

O KONJUGOVANÉ POLOVARIAČNÍ METODĚ PRO PARABOLICKÉ ROVNICE

IVAN HLAVÁČEK

Smíšenou úlohu pro parabolickou rovnici 2. řádu lze formulovat podobně jako u eliptických problémů (srv. [5]) též konjugovaným způsobem, tj. prostřednictvím vektorové funkce „ko-gradientu". Ukazuje se, že konjugovaná úloha je zvláštním případem obecné parabolické rovnice se dvěma positivními operátory, pro níž byla v práci [4]-II navržena polovariační metoda řešení. Zde je uvedena 1. a 2. polovariační aproximace řešení konjugované úlohy spolu s apriorními odhady chyb.

Author's address: Ing. *Ivan Hlaváček* CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.