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EXISTENCE THEOREM IN THE LINEAR THEORY OF MULTIPOLAR ELASTICITY

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1. Introduction

The theory of media with microstructure was developed in various papers (see e.g., [1–5]). In [7], I. Hlaváček and J. Nečas established an existence theorem in the classical linear theory of elasticity and in [8], I. Hlaváček and M. Hlaváček proved the existence of the solution in the linear theory of elasticity with couple stresses.

In this paper, we extend the results from [7], [8] to the linear theory of multipolar elasticity for the inhomogeneous anisotropic bodies.

2. Basic Equations

Let Ω be a Lipschitz region in the three-dimensional Euclidean space, occupied by an inhomogeneous and anisotropic multipolar continuum, whose boundary is Γ . The basic equations in the theory of elasticity of multipolar bodies were established by Green and Rivlin in [1], [2]. In the linear theory of multipolar elasticity these equations are:

– the equations of equilibrium

$$(2.1) \quad \sigma_{ji,j} + f_i = 0,$$

$$(2.2) \quad \bar{\sigma}_{ij_1 \dots j_\beta} = \sigma_{kij_1 \dots j_\beta, k} + f_{ij_1 \dots j_\beta} \quad (\beta = 1, 2, \dots, \nu),$$

$$(2.3) \quad \sigma'_{im} = \sigma'_{mi} = \sigma_{im} - \bar{\sigma}_{im},$$

– the geometrical equations

$$(2.4) \quad \begin{aligned} 2\varepsilon_{ij} &= \mu_{i,j} + \mu_{j,i}, \\ d_{ij} &= \mu_{j,i} + \mu_{ij}, \\ d_{ij_1j_2\dots j_\beta} &= u_{ij_1j_2\dots j_\beta} \quad (\beta = 2, 3, \dots, \nu), \\ e_{ij_1j_2\dots j_\beta k} &= \mu_{ij_1j_2\dots j_\beta, k} \quad (\beta = 1, 2, 3, \dots, \nu), \end{aligned}$$

– the constitutive equations

$$(2.5) \quad \begin{aligned} \sigma'_{ij} &= \frac{\partial \mathcal{A}}{\partial \varepsilon_{ij}}, \quad \sigma_{kij_1\dots j_\beta} = \frac{\partial \mathcal{A}}{\partial e_{ij_1\dots j_\beta k}} \quad (\beta = 1, 2, \dots, \nu) \\ \bar{\sigma}_{ij} &= \frac{\partial \mathcal{A}}{\partial d_{ij}}, \quad \bar{\sigma}_{ij_1j_2\dots j_\beta} = \frac{\partial \mathcal{A}}{\partial d_{ij_1j_2\dots j_\beta}} \quad (\beta = 2, 3, \dots, \nu). \end{aligned}$$

We take the conditions on the boundary Γ in the form:

$$(2.6) \quad \begin{aligned} u_i &= \hat{u}_i, \quad u_{ij_1\dots j_\beta} = \hat{u}_{ij_1\dots j_\beta} \quad (\beta = 1, 2, \dots, \nu) \quad \text{on } \Gamma_U \\ t_i &\equiv \sigma_{ji}n_j = \hat{t}_i, \quad t_{ij_1\dots j_\beta} \equiv \sigma_{kij_1\dots j_\beta}n_k = \hat{t}_{ij_1\dots j_\beta} \quad (\beta = 1, 2, \dots, \nu) \quad \text{on } \Gamma_T \end{aligned}$$

where Γ_U, Γ_T represent a disjoint decomposition of $\Gamma, \Gamma = \Gamma_U \cup \Gamma_T \cup N$ (N has the surface measure zero).

In the relations (2.4) $\varepsilon_{ij}, d_{ij_1\dots j_\beta}, e_{ij_1\dots j_\beta k}$ ($\beta = 1, 2, \dots, \nu$) are the classical strain tensor and the multipolar strain tensors, \mathcal{A} is the elastic energy per unit volume and the other notation in the above relations follows that from [2].

For the linear theory of elasticity, the internal energy has the following form:

$$(2.7) \quad \begin{aligned} \mathcal{A}(\varepsilon_{ij}, d_{ij_1\dots j_\beta}, e_{ij_1\dots j_\beta k}) &= \frac{1}{2}A_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2}B_{ijklm} d_{ij} d_{klm} + \\ &+ \frac{1}{2} \sum_{\alpha=2}^{\nu} \sum_{\gamma=2}^{\nu} C_{ij_1j_2\dots j_\alpha kh_1h_2\dots h_\gamma} d_{ij_1j_2\dots j_\alpha} d_{kh_1h_2\dots h_\gamma} + \\ &+ \frac{1}{2} \sum_{\alpha=1}^{\nu} \sum_{\gamma=1}^{\nu} D_{ij_1\dots j_\alpha jkh_1\dots h_\gamma h} e_{ij_1\dots j_\alpha j} e_{kh_1\dots h_\gamma h} + E_{ijklm} \varepsilon_{ij} d_{klm} + \\ &+ \sum_{\gamma=2}^{\nu} F_{ijkh_1h_2\dots h_\gamma} \varepsilon_{ij} d_{kh_1h_2\dots h_\gamma} + \sum_{\gamma=1}^{\nu} G_{ijkh_1\dots h_\gamma h} \varepsilon_{ij} e_{kh_1\dots h_\gamma h} + \\ &+ \sum_{\gamma=2}^{\nu} H_{ijkh_1h_2\dots h_\gamma} d_{ij} d_{kh_1h_2\dots h_\gamma} + \sum_{\gamma=1}^{\nu} I_{ijkh_1\dots h_\gamma h} d_{ij} e_{kh_1\dots h_\gamma h} + \\ &+ \sum_{\alpha=2}^{\nu} \sum_{\gamma=1}^{\nu} J_{ij_1j_2\dots j_\alpha kh_1\dots h_\gamma h} d_{ij_1j_2\dots j_\alpha} e_{kh_1\dots h_\gamma h}, \end{aligned}$$

where

$$(2.8) \quad \begin{aligned} A_{ijkm} &= A_{kmij} = A_{jikm}, & B_{ijkm} &= B_{kmij}, & E_{ijkm} &= E_{jikm}, \\ C_{ij_1j_2\dots j_\alpha kh_1h_2\dots h_\gamma} &= C_{kh_1h_2\dots h_\gamma ij_1j_2\dots j_\alpha}, & F_{ijkh_1h_2\dots h_\gamma} &= F_{jikh_1h_2\dots h_\gamma} \quad (\alpha, \gamma = 2, \dots, \nu), \\ D_{ij_1\dots j_\alpha jkh_1\dots h_\gamma h} &= D_{kh_1\dots h_\gamma hij_1\dots j_\alpha j}, & G_{ijkh_1\dots h_\gamma h} &= G_{jikh_1\dots h_\gamma h} \quad (\alpha, \gamma = 1, 2, \dots, \nu) \end{aligned}$$

and $A_{ijkm}, B_{ijkm}, C_{ij_1j_2\dots j_\alpha kh_1h_2\dots h_\gamma}, D_{ij_1\dots j_\alpha jkh_1\dots h_\gamma h}, E_{ijkm}, F_{ijkh_1h_2\dots h_\gamma}, G_{ijkh_1\dots h_\gamma h}, H_{ijkh_1h_2\dots h_\gamma}, I_{ijkh_1\dots h_\gamma h}, J_{ij_1j_2\dots j_\alpha kh_1\dots h_\gamma h}$ are bounded and measurable functions in $\bar{\Omega} = \Omega \cup \Gamma$.

We suppose that the form (2.7) is uniformly positive definite in the tensors ε_{ij} and $e_{ij_1\dots j_\beta}$, i.e.

$$(2.9) \quad \mathcal{A}(\varepsilon_{ij}, d_{ij_1\dots j_\beta}, e_{ij_1\dots j_\beta}) \geq c \sum_{i,j_1,\dots,j_\nu,j=1}^3 (\varepsilon_{ij}^2 + \sum_{\beta=1}^\nu e_{ij_1\dots j_\beta}^2) \quad (\beta = 1, 2, \dots, \nu)$$

for each point $x \in \bar{\Omega}$ and c is a positive constant.

3. Existence and uniqueness of the weak solution

In this section we shall apply the method given in [7], [8] to prove the existence and uniqueness of a weak solution of the mixed boundary-value problem in the linear elasticity of multipolar bodies. $L_2(\Omega)$ denotes the space of real functions square-integrable in Ω in the Lebesgue sense and $W_2^{(k)}(\Omega)$ denotes the subspace of $L_2(\Omega)$ which contains the functions whose derivatives up to the order k in the sense of distributions are in $L_2(\Omega)$.

Next, W is the Cartesian product $\prod_{s=1}^m W_2^{(\kappa_s)}(\Omega)$, $s = 1, 2, \dots, m$, where m, κ_s are positive integers.

In our case we have

$$m = \frac{1}{2}(3(3^{\nu+1} - 1)), \quad s = 1, 2, \dots, m, \quad \kappa_1 = \kappa_2 = \dots = \kappa_m = 1,$$

so that

$$W = \prod_{s=1}^m W_2^{(1)}(\Omega).$$

If we denote

$$(3.1) \quad u = \{u_i, u_{ij_1\dots j_\beta}\} \quad (\beta = 1, 2, \dots, \nu)$$

then

$$(3.2) \quad W = \{u \mid u_i, u_{ij_1\dots j_\beta} \in W_2^{(1)}(\Omega), \beta = 1, 2, \dots, \nu\}$$

with the norm

$$(3.3) \quad \|u\|_W^2 = \sum_{i,j_1,j_2,\dots,j_\nu=1}^3 (\|u_i\|_{W^{(1)}_2(\Omega)}^2 + \sum_{\beta=1}^\nu \|u_{ij_1j_2\dots j_\beta}\|_{W^{(1)}_2(\Omega)}^2),$$

where

$$\|v\|_{W^{(k)}_2(\Omega)}^2 = (v, v)_k = \sum_{|\alpha| \leq k} \int_{\Omega} D^\alpha v D^\alpha v \, dx, \quad D^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}.$$

Let $\mathcal{D}(\Omega)$ be the space of real functions with compact support in Ω which are infinitely differentiable. Let $\dot{W}_2^{(k)}(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $W_2^{(k)}(\Omega)$ and $\dot{W} = \prod_{s=1}^m \dot{W}_2^{(s_s)}(\Omega)$.

If V denotes a closed subspace of W such that $\dot{W} \subset V \subset W$, then we obtain

$$(3.4) \quad V = \{u \mid u \in W, u_i = u_{ij_1\dots j_\beta} = 0 \text{ on } \Gamma_U, \beta = 1, 2, \dots, \nu\}.$$

According to [6], we define a bilinear form $A(v, u)$ on $W \times W$ by

$$(3.5) \quad A(v, u) = \int_{\Omega} [A_{ijkm} \varepsilon_{ij}(v) \varepsilon_{km}(u) + B_{ijkm} d_{ij}(v) d_{km}(u) + \sum_{\alpha=2}^{\nu} \sum_{\beta=2}^{\nu} C_{ij_1j_2\dots j_\alpha kh_1h_2\dots h_\beta} d_{ij_1j_2\dots j_\alpha}(v) d_{kh_1h_2\dots h_\beta}(u) + \sum_{\alpha=1}^{\nu} \sum_{\beta=1}^{\nu} D_{ij_1\dots j_\alpha j_1k h_1\dots h_\beta} e_{ij_1\dots j_\alpha}(v) e_{kh_1\dots h_\beta}(u) + E_{ijkm} (\varepsilon_{ij}(v) d_{km}(u) + \varepsilon_{ij}(u) d_{km}(v)) + \sum_{\beta=2}^{\nu} F_{ijkh_1h_2\dots h_\beta} (\varepsilon_{ij}(v) d_{kh_1h_2\dots h_\beta}(u) + \varepsilon_{ij}(u) d_{kh_1h_2\dots h_\beta}(v)) + \sum_{\beta=1}^{\nu} G_{ijkh_1\dots h_\beta} (\varepsilon_{ij}(v) e_{kh_1\dots h_\beta}(u) + \varepsilon_{ij}(u) e_{kh_1\dots h_\beta}(v)) + \sum_{\beta=2}^{\nu} H_{ijkh_1h_2\dots h_\beta} (d_{ij}(v) d_{kh_1h_2\dots h_\beta}(u) + d_{ij}(u) d_{kh_1h_2\dots h_\beta}(v)) + \sum_{\beta=1}^{\nu} I_{ijkh_1\dots h_\beta} (d_{ij}(v) e_{kh_1\dots h_\beta}(u) + d_{ij}(u) e_{kh_1\dots h_\beta}(v)) + \sum_{\alpha=2}^{\nu} \sum_{\beta=1}^{\nu} J_{ij_1j_2\dots j_\alpha kh_1\dots h_\beta} (d_{ij_1j_2\dots j_\alpha}(v) e_{kh_1\dots h_\beta}(u) + d_{ij_1j_2\dots j_\alpha}(u) e_{kh_1\dots h_\beta}(v))] \, dx$$

and the functionals

$$(3.6) \quad f(v) = \int_{\Omega} (f_i v_i + \sum_{\beta=1}^{\nu} f_{ij_1\dots j_\beta} v_{ij_1\dots j_\beta}) \, dx, \\ g(v) = \int_{\Gamma_T} (\hat{t}_i v_i + \sum_{\beta=1}^{\nu} \hat{t}_{ij_1\dots j_\beta} v_{ij_1\dots j_\beta}) \, d\Gamma,$$

for each $v \in W$.

By virtue of (2.7), (2.8), we can write

$$(3.7) \quad \int_{\Omega} 2\mathcal{A}(\varepsilon_{ij}, d_{ij_1\dots j_\beta}, e_{ij_1\dots j_\beta}) dx = A(u, u), \quad A(u, v) = A(v, u).$$

From (2.9) and (3.7)₁ we obtain

$$(3.8) \quad A(v, v) \geq 2c \sum_{i, j_1, \dots, j_v, j=1}^3 \int_{\Omega} \left(\varepsilon_{ij}^2 + \sum_{\beta=1}^v e_{ij_1\dots j_\beta}^2 \right) dx.$$

Let $\hat{u} = \{\hat{u}_i, \hat{u}_{ij_1\dots j_\beta}\} \in W$, ($\beta = 1, 2, \dots, v$) and $\hat{t}_i, \hat{t}_{ij_1\dots j_\beta} \in L_2(\Gamma_T)$, ($\beta = 1, 2, \dots, v$) so that the boundary conditions (2.6) are met in the sense of traces and in the sense of $L_2(\Gamma_T)$, respectively. We say that $u \in W$ is a weak solution of the mixed boundary-value problem, if

$$(3.9) \quad u - \hat{u} \in V$$

$$(3.10) \quad A(u, v) = f(v) + g(v)$$

for each $v \in V$.

The operators $N_l v = \sum_{r=1}^m \sum_{|\alpha| \leq \kappa_r} n_{lr\alpha} D^\alpha v_r$ ($l = 1, 2, \dots, h$), [6], $h = 3m - 3$, mapping

W into $L_2(\Omega)$ are taken as follows:

$$(3.11) \quad \begin{aligned} N_1 v &= v_{1,1}, & N_2 v &= \frac{1}{2}(v_{2,1} + v_{1,2}), & N_3 v &= \frac{1}{2}(v_{3,1} + v_{1,3}), \\ N_4 v, \dots, & & N_m v &= v_{ij_1 j_2 \dots j_\beta, 1}; \\ N_{m+1} v &= v_{2,2}, & N_{m+2} v &= \frac{1}{2}(v_{3,2} + v_{2,3}), \\ N_{m+3} v, \dots, & & N_{2m-1} v &= v_{ij_1 j_2 \dots j_\beta, 2}; \\ N_{2m} v &= v_{3,3}, & N_{2m+1} v, \dots, N_{3m-3} v &= v_{ij_1 j_2 \dots j_\beta, 3}, \end{aligned}$$

where $\beta = 1, 2, 3, \dots, v$, and $i, j_1, j_2, \dots, j_v = 1, 2, 3$. We have

$$(3.12) \quad \sum_{l=1}^h \|N_l v\|_{L_2(\Omega)}^2 = \sum_{i, j_1, \dots, j_v, j=1}^3 \int_{\Omega} \left(\varepsilon_{ij}^2 + \sum_{\beta=1}^v e_{ij_1\dots j_\beta}^2 \right) dx,$$

so that (3.8) implies

$$(3.13) \quad A(v, v) \geq 2c \sum_{l=1}^h \|N_l v\|_{L_2(\Omega)}^2.$$

From (3.11) it follows that the matrix $N_{ls} \xi = \sum_{|\alpha| \leq \kappa_s} n_{ls\alpha} \xi^\alpha$ has the rank m for each $\xi \neq 0$, $\xi \in C_3$.

By virtue of Theorem 3.1 from [6], we can conclude that the system of operators $N_l v$ ($l = 1, 2, 3, \dots, h$) is coercive on W .

For \mathcal{P} , we obtain

$$(3.14) \quad \mathcal{P} = \{v \mid v \in V, v_i = a_i + b_{ij}x_j, v_{ij} = b_{ij}, v_{ij_1j_2\dots j_\beta} = 0, \beta = 2, 3, \dots, v\}$$

where $a_i, b_{ij} = -b_{ji}$ are constants.

The definition (3.4) of V implies $a_i = b_{ij} = 0$, hence $\mathcal{P} = \{0\}$.

All assumptions of Theorems 2.1 and 2.2 from [6] are satisfied. According to these theorems, we have

$$(3.15) \quad A(v, v) \geq c_1 \|v\|_V^2,$$

for each $v \in V$, and the mixed boundary-value problem has one and only one weak solution $u \in W$, continuously dependent on the given data.

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Souhrn

EXISTENČNÍ VĚTY V LINEÁRNÍ TEORII MULTIPOLÁRNÍ PRUŽNOSTI

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Tento článek je věnován kombinované úloze lineární pružnosti pro multipolární tělesa. Užívá se některých výsledků z teorie lineárních eliptických parciálních diferenciálních rovnic. Pro odpovídající bilineární formu vnitřní energie je odvozena V -elipticita a je dokázána existence a unicita řešení, jakož i spojitá závislost řešení na okrajových podmínkách.

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