Vladimír Matas

On non-existence of periodic solutions of an important differential equation

Aplikace matematiky, Vol. 18 (1973), No. 4, 213–226

Persistent URL: http://dml.cz/dmlcz/103474

ON NON-EXISTENCE OF PERIODIC SOLUTIONS OF AN IMPORTANT DIFFERENTIAL EQUATION

VLADIMÍR MATAS

(Received September 23, 1971)

1. INTRODUCTION

The equations of variation with respect to the straight-line equilibrium points L_1, L_2, L_3 of the elliptic three-dimensional restricted problem of three bodies are equivalent to the system of differential equations (see e.g. [1], p. 261)

$$\frac{d^2\xi}{dv^2} - 2\frac{d\eta}{dv} = \frac{1}{1 + e\cos v}(1 + 2A_i)\,\xi\,,$$

$$\frac{d^2\eta}{dv^2} + 2\frac{d\xi}{dv} = \frac{1}{1 + e\cos v}(1 - A_i)\,\eta\,,$$

$$\frac{d^2\zeta}{dv^2} = \frac{1}{1 + e\cos v}(-e\cos v - A_i)\,\zeta\,,$$

$$i = 1, 2, 3\,,$$

where e and A_i are constants;

$$0 < e < 1\,,$$

$$A_i > 1\,, \quad i = 1, 2, 3\,.$$

The question of existence or non-existence of nontrivial periodic solutions of the above system is very important because of its close connection with the problem of existence of periodic solutions – see e.g. [2], p. 250 – of a disturbed restricted three-body problem. In the present paper our attention will be paid to a proof of non-existence of nontrivial periodic solutions of the last differential equation of the system given above.

2. THE PROOF OF NON-EXISTENCE OF PERIODIC SOLUTIONS

Consider the differential equation

$$\frac{d^2\zeta}{dv^2} = - \frac{A + e\cos v}{1 + e\cos v}\,\zeta \tag{1}$$

and assume

$$A > 1 \tag{2}$$

and

$$0 < e < 1\,. \tag{3}$$

It is sufficient to prove that equation (1) has no nontrivial periodic solution with the period $2\pi q$, q a positive integer.

Since the expression

$$\frac{A + e\cos v}{1 + e\cos v} \tag{4}$$

has finite and continuous derivatives of all orders with respect to v for all real v, equation (1) evidently yields that every its solution ζ has

$$\text{finite and continuous } \frac{d^k\zeta}{dv^k}\,, \quad k = 0, 1, 2, \ldots\,, \quad v \in (-\infty, +\infty)\,. \tag{5}$$

Let q be an arbitrary (fixed) positive integer. Assume that a nontrivial $2\pi q$-periodic solution $\zeta(v)$ of equation (1) exists. A consequence of property (5) is that this solution and its first and second derivatives may be written in a form of Fourier series (of the functions $\zeta(v)$ and $\zeta'(v)$, $\zeta''(v)$) convergent uniformly and absolutely on the interval $(-\infty, +\infty)$ to $\zeta(v)$ and $\zeta'(v)$, $\zeta''(v)$ respectively – see e.g. [3], p. 44 – and we have

$$\zeta(v) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} \left(a_k \cos\frac{k}{q}v + b_k \sin\frac{k}{q}v \right). \tag{6}$$

a_k, $k = 0, 1, 2, \ldots$, b_k, $k = 1, 2, \ldots$, are the Fourier coefficients of the function $\zeta(v)$. (The Fourier series for the functions $\zeta'(v)$, $\zeta''(v)$ are obtained by means of term by term differentiation of the series on the right-hand side of (6).) By inserting the corresponding series into equation (1) the formulas

$$a_0 = 0\,, \tag{7}$$

$$a_k\left[\left(\frac{k}{q}\right)^2 - A\right] + \frac{e}{2}a_{|k-q|}\left[\left(\frac{k-q}{q}\right)^2 - 1\right] + \frac{e}{2}a_{k+q}\left[\left(\frac{k+q}{q}\right)^2 - 1\right] = 0\,,$$

$$b_k\left[\left(\frac{k}{q}\right)^2 - A\right] + [\operatorname{sgn}(k-q)]\,.\,\frac{e}{2}b_{|k-q|}\left[\left(\frac{k-q}{q}\right)^2 - 1\right] +$$

$$+ \frac{e}{2}b_{k+q}\left[\left(\frac{k+q}{q}\right)^2 - 1\right] = 0\,, \quad k = 1, \ldots, q, q+1, \ldots\,,$$

are obtained (since the system 1, $\cos(v/q)$, $\sin(v/q)$, $\cos(2v/q), \ldots$ is a complete orthogonal system in the Hilbert space $L_2(-\pi q, \pi q)$[1]) or $L_2(a - \pi q, a + \pi q)$, a any real number – see [3], pp. 80, 85). Hence for given coefficients $a_1, \ldots, a_q$, the coefficients a_{q+k}, $k = 1, 2, \ldots$, may be found explicitly in terms of $a_1, \ldots, a_q$.

Consider some b_k, $k = 1, 2, \ldots$, satisfying both the third equation in (7) and the requirement that the series $\sum_{k=1}^{+\infty} b_k \sin kv/q$ is convergent absolutely for $v \in (-\infty, +\infty)$. (Such b_k surely exist, e.g. $b_k = 0$, $k = 1, 2, \ldots$) First, assume – for those b_k – that each vector with q real components, $(a_1, \ldots, a_q) \neq (0, \ldots, 0)$, "generates" – by means of (7) – a solution of equation (1). Let V_q be the space of all real q-vectors. Let the vectors

$$a^{(j)} = (a_1^{(j)}, \ldots, a_q^{(j)}) \in V_q\,, \quad j = 1, \ldots, k\,, \quad 1 \leqq k \leqq q\,, \tag{8}$$

be linearly independent. Denote the "corresponding" solutions of (1) by

$$\zeta^{(j)}(v) = \sum_{m=1}^{+\infty} a_m^{(j)} \cos \frac{m}{q} v + b_m \sin \frac{m}{q} v\,. \tag{9}$$

Let α_j, $j = 1, \ldots, k$, $(k \leqq q)$ be real numbers such that

$$\sum_{j=1}^{k} \alpha_j \zeta^{(j)}(v) = 0\,. \tag{10}$$

Taking into account the absolute convergence mentioned above (6) we have

$$\begin{aligned} 0 &= \sum_{j=1}^{k} \alpha_j \left[\sum_{m=1}^{+\infty} a_m^{(j)} \cos \frac{m}{q} v + b_m \sin \frac{m}{q} v \right] = \\ &= \sum_{m=1}^{+\infty} \left[\cos \frac{m}{q} v \,.\, \Big(\sum_{j=1}^{k} \alpha_j a_m^{(j)} \Big) + \sin \frac{m}{q} v \,.\, \Big(\sum_{j=1}^{k} \alpha_j b_m \Big) \right]. \end{aligned} \tag{11}$$

Hence it follows (see the above remark on the system $\{1, \cos(v/q), \sin(v/q), \cos(2v/q), \sin(2v/q), \ldots\}$)

$$\sum_{j=1}^{k} \alpha_j a_m^{(j)} = 0\,, \quad m = 1, \ldots, q\,. \tag{12}$$

Hence, since the vectors $a^{(j)}$, $j = 1, 2, \ldots, k$, have been assumed to be linearly independent, we find

$$\alpha_j = 0\,, \quad j = 1, \ldots, k\,, \quad 1 \leqq k \leqq q\,. \tag{13}$$

Thus it is seen that the functions $\zeta^{(j)}(v)$ are linearly independent which is for $k > 2$ a contradiction with the assumption that $\zeta^{(j)}$ are solutions of equation (1). Hence the series on the right-hand side of (9) can converge for at most two linearly independent q-vectors $a^{(j)}$.

---

[1]) $L_2(a, b)$ denotes the space of functions square-integrable on the interval (a, b).

The vectors

(14) $$e_j = (\delta_{1j}, \ldots, \delta_{qj}) \in V_q\,,[1] \quad j = 1, \ldots, q\,,$$

form a basis of the vector space V_q. Now, we are going to prove that

(15) $$\zeta(0) = \sum_{m=1}^{+\infty} a_m$$

diverges if

(16) $a_m = \delta_{mi}\,, \quad m = 1 \ldots, q\,, \quad i$ an arbitrary (fixed) integer, $\quad 1 \leqq i \leqq q\,,$

(17) a_m given according to (7) and (16) for $m = q + 1, q + 2, \ldots$

It follows from (16) and (17) that

(18) $$\sum_{m=1}^{+\infty} a_m = 1 + a_{q+i} + a_{2q-i} + a_{2q+i} + q_{3q-i} + \ldots$$
$$\text{if} \quad i = 1, \ldots, q - 1\,, \quad i \neq q - i$$

and

(19) $$\sum_{m=1}^{+\infty} a_m = \sum_{n=0}^{+\infty} a_{nq+i}$$
$$\text{for} \quad i = q \quad \text{or} \quad i = q - i\,,$$

where by (7)

(20)
$$a_{(n+1)q+i} = \frac{\left[\frac{(n-1)\,q + i}{q}\right]^2 - 1}{1 - \left[\frac{(n+1)\,q + i}{q}\right]^2}\, a_{|(n-1)q+i|} + \frac{2}{e}\, \frac{\left(\frac{nq + i}{q}\right)^2 - A}{1 - \left[\frac{(n+1)\,q + i}{q}\right]^2}\, a_{nq+i}\,,$$

$$a_{(n+2)q-i} = \frac{\left(\frac{nq - i}{q}\right)^2 - 1}{1 - \left[\frac{(n+2)\,q - i}{q}\right]^2}\, a_{|nq-i|} + \frac{2}{e}\, \frac{\left[\frac{(n+1)\,q - i}{q}\right]^2 - A}{1 - \left[\frac{(n+2)\,q - i}{q}\right]^2}\, a_{(n+1)q-i}\,,$$

$n = 0, 1, 2, \ldots, \; i = 1, \ldots, q - 1,$ for $i = q$ only the first formula is valid.

---

[1]) δ_{ij} is the Kronecker symbol.

We know from the foregoing consideration that if series (15) converged it would converge absolutely. Thus with respect to (18) and (19) it is sufficient to study the convergence of the series

$$\sum_{n=0}^{+\infty} a_{nq+i}\,, \qquad \sum_{n=0}^{+\infty} a_{(n+1)q-i}\,. \tag{21}$$

For n great enough instead of recurrence formulas (20), the following ones may be written:

(22)

$$\frac{a_{(n+1)q+i}}{a_{nq+i}} + \frac{1-\left[\dfrac{(n-1)\,q+i}{q}\right]^2}{1-\left[\dfrac{(n+1)\,q+i}{q}\right]^2}\,\frac{1}{\dfrac{a_{nq+i}}{a_{(n-1)q+i}}} = \frac{2}{e}\,\frac{\left(\dfrac{nq+i}{q}\right)^2 - A}{1-\left[\dfrac{(n+1)\,q+i}{q}\right]^2}\,,$$

$$q \geqq 1\,, \quad 1 \leqq i \leqq q$$

and

(23)

$$\frac{a_{(n+2)q-i}}{a_{(n+1)q-i}} + \frac{1-\left(\dfrac{nq-i}{q}\right)^2}{1-\left[\dfrac{(n+2)\,q-i}{q}\right]^2}\,\frac{1}{\dfrac{a_{(n+1)q-i}}{a_{nq-i}}} = \frac{2}{e}\,\frac{\left[\dfrac{(n+1)\,q-i}{q}\right]^2 - A}{1-\left[\dfrac{(n+2)\,q-i}{q}\right]^2}\,,$$

$$q \geqq 1\,, \quad 1 \leqq i \leqq q-1\,.$$

It is easy to see that the system (23) is included in (22) (after inserting $j = q - i$, $i = 1, \ldots, q - 1$, in (23)).

Let us denote

$$\frac{a_{(n+1)q+i}}{a_{nq+i}} = b_n(q, i)\,, \tag{24}$$

$$\frac{1-\left[\dfrac{(n-1)\,q+i}{q}\right]^2}{1-\left[\dfrac{(n+1)\,q+i}{q}\right]^2} = d_n(q, i)\,. \tag{25}$$

Thus relation (22) implies the existence of

$$\lim_{n\to+\infty}\left[b_n(q, i) + \frac{d_n(q, i)}{b_{n-1}(q, i)}\right] = -\frac{2}{e} < -2 \tag{26}$$

with – according to (25) –

$$\lim_{n \to \infty} d_n(q, i) = 1 . \tag{27}$$

Using Lemma 1 (see Section 3) it follows from (26), (27) that there exists a finite limit

$$\lim_{n \to +\infty} |b_n(q, i)| \neq 1 . \tag{28}$$

Moreover, by Lemma 1 the value of this limit does not depend on i or q.

Hence – by means of the ratio test – either series (15) diverges for all i, q ($q \geqq 1$, $1 \leqq i \leqq q$) or, "on the contrary", series (15) converges absolutely for all these i, q. However, in our case, this convergence would cause existence of more than two linearly independent solutions of equation (1) — see (8), (9), (10), (13). Accordingly we are compelled to substitute in (6)

$$a_k = 0 , \quad k = 0, 1, \dots , \tag{29}$$

only.

Assume now the existence of a q-vector $(b_1, \dots, b_q) \neq (0, \dots, 0)$ such that the series

$$\sum_{k=1}^{+\infty} b_k \sin \frac{kv}{q} \tag{30}$$

converges absolutely and represents a solution of (1), when b_k ($k > q$) are determined on the basis of the corresponding recurrence relation in (7). In virtue of the preceding considerations – now applied to the Fourier coefficients b_k – the necessary conclusion reads

$$\sum_{k=1}^{+\infty} |b_k| = +\infty . \tag{31}$$

But this is a contradiction with the condition (see [3], p. 44)

$$|b_n| \leqq \frac{K}{n^2} , \quad n = 1, 2, \dots , \tag{32}$$

where K is a positive constant. Consequently the relation in (7) is necessarily satisfied by

$$b_k = 0 , \quad k = 1, 2, \dots , \tag{33}$$

only.

Thus it is seen from (29) and (33) that the only periodic solution of equation (1) is the trivial solution.

3. LEMMAS 1 AND 2 AND THEIR PROOFS

Lemma 1. *Consider sequences* $\{b_n\}$, $\{d_n\}$ *of real numbers such that there exist* (*i*)

$$\lim_{n\to+\infty}\left(b_n+\frac{d_n}{b_{n-1}}\right)=c\neq\pm\infty\,,\quad |c|>2$$

and (*ii*)

$$\lim_{n\to+\infty} d_n = 1\,.$$

Then there exist $\lim\limits_{n\to+\infty} b_n$, $\lim\limits_{n\to+\infty} 1/b_n$ *and we find either*

$$\lim_{n\to+\infty} b_n=\frac{c}{2}+\sqrt{\left[\left(\frac{c}{2}\right)^2-1\right]}\,,\quad \lim_{n\to+\infty}\frac{1}{b_n}=\frac{c}{2}-\sqrt{\left[\left(\frac{c}{2}\right)^2-1\right]}$$

or

$$\lim_{n\to+\infty} b_n=\frac{c}{2}-\sqrt{\left[\left(\frac{c}{2}\right)^2-1\right]}\,,\quad \lim_{n\to+\infty}\frac{1}{b_n}=\frac{c}{2}+\sqrt{\left[\left(\frac{c}{2}\right)^2-1\right]}\,.$$

Proof. (I) Consider the case $c>2$, put $c=2+\Delta$, $\Delta>0$. Let $0<\varepsilon_0<\frac{1}{2}\Delta$ and $\varepsilon_0<1$. Then according to (i) and (ii) there exists a number n_1 such that

$$0<1-\varepsilon_0<d_n<1+\varepsilon_0 \quad \text{for all} \quad n>n_1 \tag{34}$$

and

$$b_n+\frac{d_n}{b_{n-1}}>2+\Delta-(\Delta-2\varepsilon_0)=2+2\varepsilon_0 \quad \text{for all} \quad n>n_1\,. \tag{35}$$

(*a*) Let for some $n>n_1$

$$b_{n-1}<0\,. \tag{36}$$

From (34) and (35) it follows that

$$b_n>b_n+\frac{d_n}{b_{n-1}}>2+2\varepsilon_0>1\,. \tag{37}$$

(b) For some $n>n_1$ let there be a number δ such that

$$b_n\geqq 1-\delta>0\,,\quad \delta\geqq 0\,. \tag{38}$$

From this and by means of (34) we find

$$0<\frac{d_{n+1}}{b_n}\leqq\frac{d_{n+1}}{1-\delta}<\frac{1+\varepsilon_0}{1-\delta}\,. \tag{39}$$

Consequently, on the basis of (35) and (39), we have

$$(40)\quad b_{n+1} > 2 + 2\varepsilon_0 - \frac{d_{n+1}}{b_n} > 2 + 2\varepsilon_0 - \frac{1 + \varepsilon_0}{1 - \delta} = (1 + \varepsilon_0)\left(1 - \frac{\delta}{1 - \delta}\right).$$

If moreover

$$(41)\qquad \delta < \frac{\varepsilon_0}{1 + 2\varepsilon_0}$$

then it follows from (40) that

$$(42)\qquad b_{n+1} > 1 .$$

Provided

$$(43)\qquad \delta = 0$$

relation (40) yields

$$(44)\qquad b_{n+1} > 1 + \varepsilon_0 .$$

Resulting the considerations *a* and *b* we conclude that there exist both a number ε_0^* and (an integer) n_2,

$$(45)\qquad 0 < \varepsilon_0^* < 1 , \quad n_2 > n_1$$

such that it holds either

$$(46)\qquad b_n > 1 + \varepsilon_0^* \Leftrightarrow 0 < \frac{1}{b_n} < \frac{1}{1 + \varepsilon_0^*} < 1 , \quad \text{for all} \quad n \geqq n_2$$

or

$$(47)\qquad 0 < b_n < 1 - \varepsilon_0^* \Leftrightarrow \frac{1}{b_n} > \frac{1}{1 - \varepsilon_0^*} > 1 , \quad \text{for all} \quad n \geqq n_2 .$$

Next we are going to study the sequences

$$(48)\qquad \{b_n\}_{n=n_2+1}^{+\infty} , \quad \{d_n\}_{n=n_2+1}^{+\infty} , \quad \left\{b_n + \frac{d_n}{b_{n-1}}\right\}_{n=n_2+1}^{+\infty} .$$

First, we state and prove Lemma 2.

Lemma 2. *Let all the assumptions of Lemma 1 be satisfied. Then there exists*

$$(49)\qquad \lim_{n\to+\infty}\left(b_n + \frac{d_{n+1}}{b_n}\right) = c .$$

Proof. For our purpose it is sufficient to prove Lemma 2 for $c > 2$ only.

It holds

$$(50) \qquad \left| b_n + \frac{d_{n+1}}{b_n} - c \right| \leqq \left| b_{n+1} + \frac{d_{n+1}}{b_n} - c \right| + \left| b_n - b_{n+1} \right| .$$

Thus – by (i) – it is sufficient to prove that

$$(51) \qquad \lim_{n \to +\infty} \left| b_n - b_{n+1} \right| = 0 .$$

(A) First, consider the case characterized by the relation (46). Let ε be an arbitrary positive number such that

$$(52) \qquad \varepsilon_1 = \frac{\varepsilon}{2(1/\varepsilon_0^* + 1 + \varepsilon_0^*)} < \varepsilon_0^* .$$

From (i) by means of Cauchy's condition for convergence it follows that there exists an n_3, $n_3 > n_2$, such that

$$(53) \qquad \left| b_{n+1} - b_n + \frac{d_{n+1}}{b_n} - \frac{d_n}{b_{n-1}} \right| < \varepsilon_1 \quad \text{for all} \quad n > n_3 .$$

Further by (ii)

$$(54) \qquad \lim_{n \to +\infty} \frac{d_{n+2}}{d_{n+1}} = 1 .$$

Therefore there exists a number n_4, $n_4 > n_3$, such that

$$(55) \qquad 1 - \varepsilon_1 < d_{n+1} < 1 + \varepsilon_1 , \quad 1 - \frac{\varepsilon_1}{1 + \varepsilon_0^*} < \frac{d_{n+2}}{d_{n+1}} < 1 + \frac{\varepsilon_1}{1 + \varepsilon_0^*}$$

$$\text{for all} \quad n > n_4 .$$

Choose an arbitrary fixed integer n, $n > n_4$, and introduce a fixed number δ_1 such that

$$(56) \qquad \left| \frac{d_{n+1}}{b_n} - \frac{d_n}{b_{n-1}} \right| < \delta_1 .$$

Hence, as follows from (53),

$$(57) \qquad -\varepsilon_1 - \delta_1 < b_{n+1} - b_n < \varepsilon_1 + \delta_1 ,$$

from which we find by means of (46)

$$(58) \qquad \left| \frac{b_{n+1} - b_n}{b_n} \right| < \frac{\varepsilon_1 + \delta_1}{1 + \varepsilon_0^*} .$$

Consequently, using (46), (52) and (55), we have

$$\left|\frac{d_{n+2}}{b_{n+1}} - \frac{d_{n+1}}{b_n}\right| = \left|\frac{d_{n+1}}{b_{n+1}}\right| \cdot \left|\frac{b_n(d_{n+2}/d_{n+1}) - b_{n+1}}{b_n}\right| < \tag{59}$$

$$< \left|\frac{b_n(d_{n+2}/d_{n+1}) - b_{n+1}}{b_n}\right| \leqq \left|\frac{d_{n+2}}{d_{n+1}} - 1\right| + \left|\frac{b_n - b_{n+1}}{b_n}\right| < \frac{2\varepsilon_1 + \delta_1}{1 + \varepsilon_0^*}.$$

Put

$$\delta_2 = \frac{2\varepsilon_1 + \delta_1}{1 + \varepsilon_0^*}. \tag{60}$$

Continue in this way and find

$$|b_{n+j+1} - b_{n+j}| < \varepsilon_1 + \delta_{j+1}, \tag{61}$$

where

$$\delta_{j+1} = \frac{2\varepsilon_1 + \delta_j}{1 + \varepsilon_0^*}, \quad j = 1, 2, \ldots \tag{62}$$

or

$$\delta_{j+1} = \frac{2\varepsilon_1}{1 + \varepsilon_0^*}\left[1 + \frac{1}{1 + \varepsilon_0^*} + \ldots + \frac{1}{(1 + \varepsilon_0^*)^{j-1}}\right] + \frac{\delta_1}{(1 + \varepsilon_0^*)^j}. \tag{63}$$

For our ε_0^* (see (45)) we have

$$\sum_{j=0}^{+\infty}\left(\frac{1}{1 + \varepsilon_0^*}\right)^j = \frac{1 + \varepsilon_0^*}{\varepsilon_0^*}, \quad \lim_{j\to+\infty}\frac{\delta_1}{(1 + \varepsilon_0^*)^j} = 0 \tag{64}$$

and hence there exists j_0 such that

$$-\varepsilon_1 + \frac{1 + \varepsilon_0^*}{\varepsilon_0^*} < 1 + \ldots + \frac{1}{(1 + \varepsilon_0^*)^{j-1}} < \frac{1 + \varepsilon_0^*}{\varepsilon_0^*} + \varepsilon_1, \tag{65}$$

$$0 < \frac{\delta_1}{(1 + \varepsilon_0^*)^j} < \varepsilon_1$$

for all integers $j > j_0$.

Thus we have found an n_0,

$$n_0 = n + j_0 \tag{67}$$

such that for all integers $m > n_0$ we have

$$|b_{m+1} - b_m| < 2\varepsilon_1\left(1 + \frac{1}{\varepsilon_0^*} + \varepsilon_0^*\right) = \varepsilon. \tag{68}$$

With respect to (50) and (i) the statement (49) has been proved for the case (46).

(B) It remains to prove this statement for the case (47), *i.e.* the case when

$$\text{(69)} \qquad \frac{1}{b_n} > \frac{1}{1 - \varepsilon_0^*} = 1 + \tilde{\varepsilon}_0 > 1 \quad \text{for all} \quad n \geqq n_2 \, .$$

Let ε be an arbitrary positive number such that

$$\text{(70)} \qquad \varepsilon_1 = \frac{\varepsilon}{2/\tilde{\varepsilon}_0 + 3} < 1 \, .$$

According to (i) and (ii) – in virtue of Cauchy's condition for convergence – there exists an n_3, $n_3 > n_2$, so that

$$\text{(71)} \qquad \left| \frac{d_{n+1}}{b_n} - \frac{d_n}{b_{n-1}} + b_{n+1} - b_n \right| < \varepsilon_1$$

and

$$\text{(72)} \qquad |d_n - 1| < \tfrac{1}{2}\varepsilon_1$$

hold for all integers $n > n_3$.

Let n be an arbitrary positive integer, $n > n_3$. In virtue of (69), for every $n > n_3$ and for every positive integer j, it holds:

$$\text{(73)} \qquad |b_{n+j+1} - b_{n+j}| < 1 = \delta_1 \, .$$

Hence in virtue of (71) and (69) we have

$$\text{(74)} \qquad \left| b_{n+j} b_{n+j-1} \left(\frac{d_{n+j+1}}{b_{n+j}} - \frac{d_{n+j}}{b_{n+j-1}} \right) \right| < \frac{\varepsilon_1 + \delta_1}{1 + \tilde{\varepsilon}_0} \, .$$

Moreover by means of (69) and (72) we find

$$\text{(75)} \qquad |b_{n+j} - b_{n+j-1}| \leqq |b_{n+j-1}(d_{n+j+1} - 1)| + |b_{n+j}(d_{n+j} - 1)| + \\ + \left| b_{n+j} b_{n+j-1} \left(\frac{d_{n+j+1}}{b_{n+j}} - \frac{d_{n+j}}{b_{n+j-1}} \right) \right| < \frac{2\varepsilon_1 + \delta_1}{1 + \tilde{\varepsilon}_0} = \delta_2 \, .$$

Consequently

$$\text{(76)} \qquad |b_{n+j-k+2} - b_{n+j-k+1}| < \delta_k = \frac{2\varepsilon_1 + \delta_{k-1}}{1 + \tilde{\varepsilon}_0} \, , \quad k = 1, \ldots, j + 1 \, .$$

Particularly, for $k = j + 1$ we have – see (62), (63) and (73) –

$$\text{(77)} \qquad |b_{n+1} - b_n| < \delta_{j+1} = \\ = \frac{2\varepsilon_1}{1 + \tilde{\varepsilon}_0} \left[1 + \frac{1}{1 + \tilde{\varepsilon}_0} + \ldots + \frac{1}{(1 + \tilde{\varepsilon}_0)^{j-1}} \right] + \frac{1}{(1 + \tilde{\varepsilon}_0)^j} \, ,$$

where j is an arbitrary positive integer.

Thus it is seen on the basis of (64), (65), (70) that

$$|b_{n+1} - b_n| < \varepsilon_1 \left(\frac{2}{\tilde{\varepsilon}_0} + 3 \right) = \varepsilon \quad (78)$$

$$\text{for all} \quad n > n_3 .$$

Q.E.D.*)

We now come back to continue the proof of Lemma 1.

Put

$$c_n = b_n + \frac{d_{n+1}}{b_n} . \quad (79)$$

By Lemma 2 there exists

$$\lim_{n \to +\infty} c_n = c > 2 . \quad (80)$$

Therefore

$$\lim_{n \to +\infty} \frac{c_n^2}{4} = \frac{c^2}{4} > 1 . \quad (81)$$

Hence there exist numbers ε_2 and n_5, $0 < \varepsilon_2 < 1$, $n_5 > n_2$, such that the inequatities

$$\frac{c_n}{2} > 1 + \varepsilon_2 , \quad \frac{c_n^2}{4} > 1 + \varepsilon_2 , \quad (82)$$

are valid for every integer $n > n_5$.

It follows from (ii) that there exists n_6, $n_6 > n_5$, so that

$$-1 - \varepsilon_2 < -d_{n+1} < -1 + \varepsilon_2 \quad \text{for all integers} \quad n > n_6 . \quad (83)$$

Accordingly

$$\frac{c_n^2}{4} - d_{n+1} > 0 \quad \text{for all the} \quad n > n_6 . \quad (84)$$

Thus we get from relations (79), (82), (83) and (84) that (see also (46) and (47)) either

$$b_n = \frac{c_n}{2} + \sqrt{\left[\left(\frac{c_n}{2} \right)^2 - d_{n+1} \right]} > 1 \quad \text{for all the} \quad n > n_6 > n_2 \quad (85)$$

or

$$0 < b_n = \frac{c_n}{2} - \sqrt{\left[\left(\frac{c_n}{2} \right)^2 - d_{n+1} \right]} < 1 \quad \text{for all the} \quad n > n_6 > n_2 . \quad (86)$$

---

*) If $c < -2$ the proof would be analogous to part II of the proof of Lemma 1 (see below).

Consequently there exist $\lim_{n\to+\infty} b_n$ and $\lim_{n\to+\infty} 1/b_n$ and we have either

$$(87) \qquad \lim_{n\to+\infty} b_n = \frac{c}{2} + \sqrt{\left[\left(\frac{c}{2}\right)^2 - 1\right]} > 1\,,$$

$$0 < \lim_{n\to+\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to+\infty} b_n} = \frac{c}{2} - \sqrt{\left[\left(\frac{c}{2}\right)^2 - 1\right]} < 1$$

or

$$(88) \qquad 0 < \lim_{n\to+\infty} b_n = \frac{c}{2} - \sqrt{\left[\left(\frac{c}{2}\right)^2 - 1\right]} < 1\,,$$

$$\lim_{n\to+\infty} \frac{1}{b_n} = \frac{c}{2} + \sqrt{\left[\left(\frac{c}{2}\right)^2 - 1\right]} > 1\,.$$

Thus we have proved Lemma 1 for the case $c > 2$, $c \neq +\infty$.

(II) It remains to consider the case $c < -2$, $c \neq -\infty$. Put

$$(89) \qquad b_n^* = -b_n \quad \text{for every positive integer } n, \quad c^* = -c > 2\,.$$

Then in accordance with part I of the proof there exist $\lim_{n\to+\infty} b_n^*$, $\lim_{n\to+\infty} (1/b_n^*)$. Now on the basis of (87), (88) and (89) $\lim_{n\to+\infty} b_n$ and $\lim_{n\to+\infty} (1/b_n)$ easily can be found, viz. either

(90)

$$\lim_{n\to+\infty} b_n = \frac{c}{2} - \sqrt{\left[\left(\frac{c}{2}\right)^2 - 1\right]} < -1\,, \quad 0 > \lim_{n\to+\infty} \frac{1}{b_n} = \frac{c}{2} + \sqrt{\left[\left(\frac{c}{2}\right)^2 - 1\right]} > -1\,,$$

or

(91)

$$-1 < \lim_{n\to+\infty} b_n = \frac{c}{2} + \sqrt{\left[\left(\frac{c}{2}\right)^2 - 1\right]} < 0\,, \quad \lim_{n\to+\infty} \frac{1}{b_n} = \frac{c}{2} - \sqrt{\left[\left(\frac{c}{2}\right)^2 - 1\right]} < -1\,.$$

Q.E.D.

References:

[1] *Г. Н. Дубошин:* Небесная механика, Аналитические и качественные методы, Издательство Наука, Москва, 1964.

[2] *H. Hochstadt:* Differential Equations, A Modern Approach, Holt, Rinehart and Winston, New York, Chicago, San Francisco, Toronto, London, 1964.

[3] *A. Kufner, J. Kadlec:* Fourier Series, Academia, Prague, 1971.

Souhrn

O NEEXISTENCI PERIODICKÝCH ŘEŠENÍ JEDNÉ VÝZNAMNÉ DIFERENCIÁLNÍ ROVNICE

VLADIMÍR MATAS

Variační rovnice „příslušné" přímkovým libračním centrům trojrozměrného eliptického restringovaného problému tří těles jsou ekvivalentní systému dvou diferenciálních rovnic druhého řádu a Hillově rovnici

$$\frac{d^2\zeta}{dv^2} + \frac{A + e\cos v}{1 + e\cos v}\,\zeta = 0\,,$$

kde $0 < e < 1$, $A > 1$ jsou konstanty. V předložené práci je podán důkaz, že pro všechny uvedené hodnoty parametrů A, e daná Hillova rovnice nemá žádné netriviální periodické řešení.

Author's address: Dr. *Vladimír Matas*, CSc., Československá akademie věd, Astronomický ústav, Budečská 6, 120 23 Praha 2.