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# ON GENERALIZED LOCALIZABILITY

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The localizability of a physical system $S$ in a space $X$ is defined as a homomorphism $h$ of the lattice $B(X)$ of the Borel sets in the space $X$ into the lattice $\mathcal{E}$ of yes – no experiments which can be performed on $S$. In each state $s \in \mathcal{S}$ on $\mathcal{E}$ and $\Delta \in B(X)$, $s(h(\Delta))$ is the probability of finding $S$ in $\Delta$. In [1] Jauch and Piron introduced generalized localizability; the condition that $h$: $B(X) \to \mathcal{E}$ is a homomorphism is weakened. They demand

$$h(\emptyset) = 0 \,, \quad h(X) = I \,, \tag{1}$$

$$\Delta_1 \cap \Delta_2 = \emptyset \Rightarrow h(\Delta_1) \perp h(\Delta_2) \,, \tag{2}$$

$$h(\Delta_1 \cap \Delta_2) = h(\Delta_1) \wedge h(\Delta_2) \,, \tag{3}$$

but they do not demand $h(\Delta_1 \cup \Delta_2) = h(\Delta_1) \vee h(\Delta_2)$ for $\Delta_1 \cap \Delta_2 = \emptyset$. In their paper [1] $\mathcal{E}$ is the lattice of projections in a Hilbert space $\mathfrak{H}$.

A transformation $h$: $B(X) \to \mathcal{E}$ satisfying (1)–(3) can be constructed in a simple manner. We take a homomorphism $h'$: $B(X) \to \mathcal{E}$, a projection $P$ non-commuting with all $h'(\Delta)$, $\Delta \in B(X)$, and then

$$h : h(\Delta) = P \wedge h'(\Delta)$$

satisfies (1)–(3) in $P\mathfrak{H}$.

This is the situation which is described in a theorem due to Neumark [2]: For every positive operator valued measure $T$ in a Hilbert space $\mathfrak{H}$ one can construct an extension $\mathfrak{H}' \supset \mathfrak{H}$ and a projection valued measure $T'$ so that

$$T(\Delta) = P\, T'(\Delta)\, P$$

where $P$ is the projection $\mathfrak{H}' \to \mathfrak{H}$. The converse that $T$ constructed in that manner is a POV-measure is also true.

Now Jauch and Piron made the following conjecture: For every generalized spectral measure $h$: $B(X) \to \mathcal{E}(\mathfrak{H})$ which satisfies (1)–(3) there exists an extension

$\mathfrak{H}' \supset \mathfrak{H}$ and a spectral measure $h'\colon B(X) \to \mathscr{E}(\mathfrak{H}')$ so that $h(\Delta) = P \wedge h'(\Delta)$ where $P$ is the projection $\mathfrak{H}' \to \mathfrak{H}$.

By the theorem of Neumark there must be a POV-measure on $B(X)$.

The existence of this measure can be proved without the detour over $h$ satisfying (1)–(3). It suffices to define the localizability of a system $S$ in a space $X$ in the following manner:

(A1) for every $\Delta \in B(X)$ and every state $s \in \mathscr{S}$ of the system $S$ there is a probability $p(\Delta, s)$ for $S$ in $\Delta$. This function is additive in $\Delta$ and linear in $s$ so that $\Delta_1 \cap \Delta_2 = \emptyset \Rightarrow \Rightarrow p(\Delta_1 \cup \Delta_2, .) = p(\Delta_1, .) + p(\Delta_2, .)$ and $s_1, s_2 \in \mathscr{S}, \alpha, \beta \geqq 0, \alpha + \beta = 1 \Rightarrow \Rightarrow p(., \alpha s_1 + \beta s_2) = \alpha\, p(., s_1) + \beta\, p(., s_2)$.

In order that this probability may be experimentally determined, we must suppose

(A2) for every $\Delta$ there is an observable $P(\Delta)$ with the mean value $p(\Delta, s)$ in every state $s$.

The definition requires that the sum of the observables $P(\Delta_1), P(\Delta_2)$ should exist, at least for $\Delta_1 \cap \Delta_2 = \emptyset$ (cf. [4]). In the sequel we shall deal with the case that $\mathscr{E}$ is a Boolean algebra or the lattice of projections in a Hilbert space $\mathfrak{H}$, and so this condition will be satisfied.

In the second case the observables $P(\Delta)$ are operators on $\mathfrak{H}$ and, in order that the mean value be from the interval $\langle 0, 1 \rangle$ for every state, it is necessary and sufficient that $P(\Delta)$ be a positive operator and $P(\Delta) \leqq I$ ($I$ – the unit projection). The condition $P(X) = I$ is equivalent to $p(X, s) = 1$ for every state $s$.

In the classical case, where $\mathscr{E}$ is a Boolean algebra, we can consider the sets in a compact space $T$ in place of $\mathscr{E}$ (Theorem of Stone). $P(\Delta)$ are then measurable functions on $T$ and states are measures on $T$. The necessary and sufficient condition that the mean value lie in $\langle 0, 1 \rangle$ is that the values of $P$ lie in $\langle 0, 1 \rangle$. This follows from the fact that for every event $x \in \mathscr{E}$ it is possible to find a state $s \in \mathscr{S}$ that $s(x) = 1$ provided that the structure $(\mathscr{E}, \mathscr{S})$ is strongly-order – determining [3] (cf. [4]). In this case, therefore, the localizability is a system of functions $P(\Delta)$, $\Delta \in B(X)$, with values in $\langle 0, 1 \rangle$ and $P(\Delta_1 \cup \Delta_2) = P(\Delta_1) + P(\Delta_2)$ for $\Delta_1 \cap \Delta_2 = \emptyset$. $P(X) = 1$ is always equivalent to $S$ in $X$.

Now the following statement is valid (this statement is an analogue of the statement from [2]):

*It is possible to find for $\mathscr{E}$ an extension $\mathscr{E}'$ and a system of states $\mathscr{S}'$ such that there is an isomorphism $h\colon B(X) \to \mathscr{E}'$ and for every state $s \in \mathscr{S}$ there is a state $s' \in \mathscr{S}'$ so that the probability $S \in \Delta$ in the state $s$ is equal to the probability of $h(\Delta)$ in the state $s'$.*

Proof. We shall consider the representation of the algebra $\mathscr{E}$ in the space $T$. In the product $X \times T$ we have sets $\Delta \times M$, $\Delta \in B(X)$, $M \in \mathscr{E}$. Now we define the measure $s'$ on $X \times T$ for $s \in \mathscr{S}$

$$s'(\Delta \times M) = \int_M P(\Delta)\, \mathrm{d}s\,.$$

$s'$ is a measure because $P$ is additive in $\Delta$. We can extend $s'$ on the algebra $\mathscr{E}'$ generated by the sets of the form $\Delta \times M$. In this algebra $\mathscr{E}'$ the sets $X \times M$ form an subalgebra that is isomorphic to $\mathscr{E}$. For every $\Delta$ we define $h(\Delta) = \Delta \times T$ and $h$ is an isomorphism. Finally it is

$$s'(h(\Delta)) = \int_T P(\Delta)\,\mathrm{d}s = p(\Delta, s)\,.$$

This completes the proof.

The part about the states is valid for [2], too:

*It is possible to extend every state $s$ in $\mathfrak{H}$ to a state $s'$ in $\mathfrak{H}'$ so that we have* $s'(h(\Delta)) = s(P(\Delta))$ (in [2] there is $F$ in place of $h$ and $B$ in place of $P$).

Proof. When the state $s$ is irreducible then there is a vector $\varphi \in \mathfrak{H}$ that defines this state. $h(\Delta)$ is formed by all sums $\Sigma_k \Delta_k \,.\, x_k$ (cf. [2]) where $\Delta_k \subset \Delta$. It follows from this that $(X - \Delta)\,\varphi$ is orthogonal to $h(\Delta)$ and from the equality $\varphi = X\varphi = \Delta\varphi + (X - \Delta)\,\varphi$ it follows that $\Delta\varphi$ is the projection of $\varphi$ into $h(\Delta)$. From the definition of the scalar product in $\mathfrak{H}'$ the equation

$$(h(\Delta)\,\varphi, \varphi)_{\mathfrak{H}'} = (P(\Delta)\,\varphi, \varphi)_{\mathfrak{H}}$$

follows. The validity for composed states is a consequence of linearity.

### *References*


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## Souhrn

## O ZOBECNĚNÉ LOKALIZOVATELNOSTI

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Ukazuje se, že definice zobecněné lokalizovatelnosti podle Jaucha a Pirona může být přímo zdůvodněna. Je dokázána věta pro klasické systémy, která je analogem Neumarkovy věty o representaci POV měr.



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