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NEUTRON TRANSPORT INITIAL VALUE PROBLEM IN NON-MULTIPLYING MEDIUM

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INTRODUCTION

The problem of finding the neutron density in a medium as a function of spatial, angular, energetic and time coordinates provided the initial density distribution is known, frequently occurs in the theory of transport of neutrons. It is usually formulaated in the following form:

$$(1) \qquad \left\{\frac{\partial}{\partial t} + \sqrt{(2E)}\, \boldsymbol{\omega}\nabla + \sqrt{(2E)}\, \Sigma_u(\boldsymbol{x}, \boldsymbol{\omega}, E, t)\right\} \varphi(\boldsymbol{x}, \boldsymbol{\omega}, E, t) =$$

$$= \int_\Omega \mathrm{d}\omega' \int_0^\infty \mathrm{d}E' \sqrt{(2\mathrm{E})}\, \Sigma(\boldsymbol{x}, \boldsymbol{\omega}' \to \boldsymbol{\omega}, E' \to E, t)\, \varphi(\boldsymbol{x}, \boldsymbol{\omega}', E', l) + \sqrt{(2E)}\, S(\boldsymbol{x}, \boldsymbol{\omega}, E, t),$$

$$\varphi(\boldsymbol{x}, \boldsymbol{\omega}, E, t = 0) = \psi(\boldsymbol{x}, \boldsymbol{\omega}, E).$$

The particular symbols have the following meaning:

$\boldsymbol{x}, \boldsymbol{\omega}, E, t$	coordinates of location, angle, energy and time;
Ω	surface area of the unit sphere;
φ	neutron density;
ψ	given initial neutron density;
S	source term;
Σ	macroscopic differential effective cross-section of the medium for neutron scattering;
Σ_u	total macroscopic effective cross-section of the medium for neutrons.

The usual approach most frequently used for solving Problem (1), is the examination of spectral properties of a certain operator in a Banach space. This method requires certain simplifications and restrictions concerning the properties of the effective cross-sections and the source, such as, for example, boundedness in all variables, quadratic integrability, etc. (See [1], [2], [3].) Unfortunately, these conditions are often too restrictive to be used in practice. However, even more serious difficulties arise in concrete calculations by means of spectral decomposition. In most cases, it is very difficult even to find only several eigenvalues and eigenfunctions and, consequently, it is almost impossible to specify the behaviour of the density function. Some further difficulties appear also in such cases when the characteristics of the medium vary with time.

Examining the behaviour of the initial density distribution or the source function $\sqrt{(2E)}\, S$ with respect to the energetic variable in practical problems, we meet rather frequently with the following two cases: either the functions have the character of the Maxwell distribution, or they behave approximately like the Dirac δ-function. In such cases, it seems convenient to find the function of the neutron density in the form of a series of successive approximations ([4], [5]). The characteristics of the medium are considered to be functions of all variables $\boldsymbol{x}, \omega, E, t$. The method used in the sequel is that of transforming Problem (1) to the integral equation

$$
\begin{aligned}
(2) \qquad & \varphi(\boldsymbol{x}, \omega, E, t) = \\
&= \int_0^t \mathrm{d}t_1 \int_\Omega \mathrm{d}\omega' \int_0^\infty \mathrm{d}E' \Big[\sqrt{(2E)}\, \Sigma(\boldsymbol{x} - \sqrt{(2E)}\, \omega(t - t_1), \omega' \to \omega, E' \to E, t_1) \times \\
&\quad \times \exp\left(\int_t^{t_1} \mathrm{d}t_2 \sqrt{(2E)}\, \Sigma_u(\boldsymbol{x} - \sqrt{(2E)}\, \omega(t - t_2), \omega, E, t_2)\right) \times \\
&\quad \times \varphi(\boldsymbol{x} - \sqrt{(2E)}\, \omega(t - t_1), \omega', E', t_1)\Big] + F(\boldsymbol{x}, \omega, E, t)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
F(\boldsymbol{x}, \omega, E, t) &= \Big\{\psi(\boldsymbol{x} - \sqrt{(2E)}\, \omega t, \omega, E) + \\
&+ \int_0^t \mathrm{d}t_1 \sqrt{(2E)}\, S(\boldsymbol{x} - \sqrt{(2E)}\, \omega(t - t_1), \omega, E, t_1) \times \\
&\times \exp\left(\int_0^{t_1} \mathrm{d}t_2 \sqrt{(2E)}\, \Sigma_u(\boldsymbol{x} - \sqrt{(2E)}\, \omega(t - t_2), \omega, E, t_2)\right)\Big\} \times \\
&\times \exp\left(\int_t^0 \mathrm{d}t_1 \sqrt{(2E)}\, \Sigma_u(\boldsymbol{x} - \sqrt{(2E)}\, \omega(t - t_1), \omega, E, t_1)\right).
\end{aligned}
$$

Existence and uniqueness of solution of Problem (1) are proved in [5] by the standard iterative method (see [10]) in the case of a sufficiently effective absorption. The aim of the present paper is to generalize the results obtained to an arbitrary non-multiplying (absorbing or non-absorbing) medium.

In the sequel we shall deal with the transformed form of Problem (1), i.e., with equation (2), and the following assumptions and notation will be used:

The function of the neutron density and the characteristics of the medium will defined on the set M of quadruplets $(\boldsymbol{x}, \omega, E, t)$:

$$M \equiv E_3 \times \Omega \times (0, \infty) \times \langle 0, \infty)\,.$$

Further we suppose

a) a non-absorbing and non-multiplying medium

$$\Sigma_u(\boldsymbol{x}, \omega, E, t) = \int_\Omega d\omega' \int_0^\infty dE' \, \Sigma(\boldsymbol{x}, \omega \to \omega', E \to E', t) \,,$$

b) validity of the relation of the detailed balance

(3) $$E e^{-E/[kT(t)]} \, \Sigma(\boldsymbol{x}, \omega \to \omega', E \to E', t) = E' e^{-E'/[kT(t)]} \, \Sigma(\boldsymbol{x}, -\omega \to -\omega', E' \to E, t)$$

where T is the medium temperature and k the Boltzmann constant;

c) the differential effective cross-section depends on the angle only by means of the scalar product of the angular vectors:

$$\Sigma(\boldsymbol{x}, \omega \to \omega', E \to E', t) = \Sigma(\boldsymbol{x}, \omega \,.\, \omega', E \to E', t) \geqq 0 \,.$$

Finally, we denote by $C\{B; M\}$ the family of functions φ such that the function

$$g(\boldsymbol{x}, \omega, E, t) = \int_\Omega d\omega' \int_0^\infty dE' \, \frac{\Sigma(\boldsymbol{x}, \omega \,.\, \omega', E \to E', t)}{E' \Sigma_u(\boldsymbol{x}, \omega, E, t)} \, e^{E'/[kT(t)]} \, \varphi(\boldsymbol{x}, \omega', E', t)$$

is bounded on the set M (B being a constant, $T(t) \geqq B/k$ for all $t \in \langle 0, \infty))$.

EXISTENCE OF SOLUTION

For the sake of brevity, let us accept the following notation:

$$\begin{aligned}
K(\varphi) &= \int_0^t dt_1 \int_\Omega d\omega' \int_0^\infty dE' \, \sqrt{(2E)} \, \Sigma(\boldsymbol{x} - \sqrt{(2E)} \, \omega(t - t_1), \omega \,.\, \omega', E' \to E, t_1) \,. \\
&\quad \times \exp\left(\int_t^{t_1} dt_2 \, \sqrt{(2E)} \, \Sigma_u(\boldsymbol{x} - \sqrt{(2E)} \, \omega(t - t_2), \omega, E, t_2)\right) \varphi(\boldsymbol{x} - \sqrt{(2E)} \, \times \\
&\quad \times \omega(t - t_1), \omega', E', t_1) \,, \\
R(\varphi) &= \int_0^t dt_1 \int_\Omega d\omega' \int_0^\infty dE' \, \sqrt{(2E)} \, \Sigma(\boldsymbol{x} - \sqrt{(2E)} \, \omega(t - t_1), \omega \,.\, \omega', E \to E', t_1) \,. \\
&\quad \times \exp\left(\int_t^{t_1} dt_2 \, \sqrt{(2E)} \, \Sigma_u(\boldsymbol{x} - \sqrt{(2E)} \, \omega(t - t_2), \omega, E, t_2)\right) \times \\
&\quad \times \varphi(\boldsymbol{x} - \sqrt{(2E)} \, \omega(t - t_1), \omega', E', t_1) \,, \\
\chi &= \exp\left(\int_t^0 dt_1 \, \sqrt{(2E)} \, \Sigma_u(\boldsymbol{x} - \sqrt{(2E)} \, \omega(t - t_1), \omega, E, t_1)\right), \\
\Theta &= \int_0^t dt_1 \exp\left(\int_t^{t_1} dt_2 \, \sqrt{(2E)} \, \Sigma_u(\boldsymbol{x} - \sqrt{(2E)} \, \omega(t - t_2), \omega, E, t_2)\right).
\end{aligned}$$

Theorem 1. *Let the following conditions be satisfied:*

(a) *To each quadruplet* $(\boldsymbol{x}, \boldsymbol{\omega}, E, t) \in M$ *there exists a nondegenerate interval* U *in* M $(\boldsymbol{x}, \boldsymbol{\omega}, E, t) \in U$ *so that* $\Sigma(\boldsymbol{x}_1, \boldsymbol{\omega}_1 \,.\, \boldsymbol{\omega}', E_1 \to E', t_1)$ *as the function of the variables* $\boldsymbol{x}_1, \boldsymbol{\omega}_1, E_1, t_1$ *is continuous on* U *for almost all pairs* $(\boldsymbol{\omega}', E') \in \Omega \times (0, \infty)$ *and has an integrable majorant on* $\Omega \times (0, \infty)$.

(b) *The function* $T(t) \geqq B_1/k$ *is continuous and non-decreasing on the interval* $\langle 0, \infty)$ $(B_1$ *a positive constant).*

(c) *The functions* $\psi(\boldsymbol{x} - \sqrt{(2E)}\, \boldsymbol{\omega} t, \boldsymbol{\omega}, E)$, $\sqrt{(2E)}\, S(\boldsymbol{x}, \boldsymbol{\omega}, E, t)$ *belong to the class* $C\{B_1; M\}$ *and* $|K(F)| \leqq E e^{-E/kT} A(\chi + \Theta)$ $(A$ *a positive constant),* $K(F)$ *being continuous on* M.

Then the series $\sum_{m=0}^{\infty} K^m(F)$ *where* $K^0(F) = F$, $K^m(F) = K(K^{m-1}(F))$ *converges on the set* M *and it solves Problem* (2) *in the class* $C\{B_1; M\}$.

Note. Conditions (a) and (b) guarantee the continuity of the function $\Sigma_u(\boldsymbol{x}, \boldsymbol{\omega}, E, t)$ on M and justify the integration in particular iterations. Obviously these conditions are not the most general possible but we use them for our convenience.

Before proceeding to the proof, let us establish the following

Lemma 1. *Let* $f(t)$ *be a real function such that* $f(t)$ *as well as* $\mathrm{d}f/\mathrm{d}t$ *are continuous on* $\langle 0, \infty)$ *(continuity from the right being consiered at zero) and let conditions* (a) *and* (b) *of Theorem 1 be satisfied. Then the series* $\sum_{m=0}^{\infty} R^m(\Theta_1)$ *where*

$$\Theta_1 = \int_0^t \mathrm{d}t_1 \frac{\mathrm{d}f(t_1)}{\mathrm{d}t_1} \exp \int_t^{t_1} \mathrm{d}t_2 \sqrt{(2E)}\, \Sigma_u(\boldsymbol{x} - \sqrt{(2E)}\, \boldsymbol{\omega}(t - t_2), \boldsymbol{\omega}, E, t_2))\,,$$

$$R^0(\Theta_1) = \Theta_1\,, \quad R^m(\Theta_1) = R(R^{m-1}(\Theta_1))$$

converges absolutely on the set M.

Proof will be given in three steps; for the moment, we restrict ourselves to the set $M_1 = E_3 \times \Omega \times (0, \infty) \times \langle 0, D \rangle$, where $D < \infty$ is an arbitrary but fixed positive number.

(i) $f(t) \equiv 1$:

By direct calculations, we easily verify relations

$$\begin{aligned} 1 \geqq R(1) &= 1 - \chi \geqq 0 \\ R(1) \geqq R^2(1) &= 1 - \chi - R(\chi) \geqq 0 \\ &\vdots \\ R(1) \geqq \ldots \geqq R^n(1) &= 1 - \sum_{m=0}^{n-1} R^m(\chi) \geqq 0\,. \end{aligned}$$

As it can be seen, the sequence $\{R^n(1)\}$ is non-increasing and bounded from below. Therefore, the series $\sum_{m=0}^{\infty} R^m(\chi)$ is convergent on the set M_1.

(ii) $f(t) \geqq 0$ non-decreasing:

Again, the validity of the following system of inequalities can be verified:

$$\begin{aligned} f(t) \geqq R(f) &= f(t) - f(0)\,\chi - \Theta_1 \geqq 0 \\ R(f) \geqq R^2(f) &= f(t) - f(0)\,\chi - \Theta_1 - f(0)\,R(\chi) - R(\Theta_1) \geqq 0 \\ &\vdots \\ R(f) \geqq \ldots \geqq R^n(f) &= f(t) - f(0) \sum_{m=0}^{n-1} R^m(\chi) - \sum_{m=0}^{n-1} R^m(\Theta_1) \geqq 0\,. \end{aligned}$$

The convergence of the series $\sum_{m=0}^{\infty} R^m(\Theta_1)$ for M_1 follows from the fact that the sequence $\{R^m(f)\}_0^\infty$ is non-negative and non-increasing, and from item (i) of the proof.

(iii) $f(t)$ arbitrary (fulfilling the assumptions of Lemma 1):

The absolute convergence of the series $\sum_{m=0}^{\infty} R^m(\Theta_1)$ on M_1 follows from the inequalities

$$|\Theta_1| \leqq \int_0^t \mathrm{d}t_1 \exp\left(\int_t^{t_1} \mathrm{d}t_2 \sqrt{(2E)}\, \Sigma_u(\boldsymbol{x} - \sqrt{(2\varepsilon)}\, \boldsymbol{\omega}(t - t_2), \boldsymbol{\omega}, E, t_2)\right) \max_{t_3 \in \langle 0, D \rangle} \left|\frac{\mathrm{d}f(t_3)}{\mathrm{d}t_3}\right|,$$

$$|R^m(\Theta_1)| \leqq R^m(|\Theta_1|)$$

and from items (i) and (ii) of the proof.

As the constant D was chosen arbitrarily, Lemma 1 obviously holds for the whole set M.

Proof of Theorem 1. According to condition (c) of the theorem

$$|K(F)| \leqq AEe^{-E/[kT(t)]}(\chi + \Theta)\,.$$

Taking into account the relation of the detailed balance (3) we obtain

$$|K^2(F)| \leqq AEe^{-E/[kT(t)]}\,[R(\chi) + R(\Theta)]$$

and generally, it holds for an arbitrary n

$$|K^n(F)| \leqq AEe^{-E/kT}[R^{n-1}(\chi) + R^{n-1}(\Theta)]\,.$$

Applying Lemma 1 to $f(t) = t$ (then $\Theta_1 \equiv \Theta$) we can see readily that the series $\sum_{m=0}^{\infty} K^m(F)$ is absolutely convergent for each quadruplet $(\boldsymbol{x}, \boldsymbol{\omega}, E, t) \in M$. It remains to prove that this series of successive approximations solves Problem (2). To this purpose, we shall introduce without a proof a well known theorem from integral calculus, see e.g. [6].

Theorem B. *Let f_n, $n = 1, 2, \ldots$ be measurable in M. Let there exist a function $\varphi(x)$ integrable on M such that $|f_n(x)| \leqq \varphi(x)$ almost everywhere in M, $n = 1, 2, \ldots$ Let $f(x) = \lim_{n\to\infty} f_n(x)$ exist almost everywhere in M.*

Then f_n and f are integrable in M and

$$\int_M f \, d\mu = \lim_{n\to\infty} \int_M f_n \, d\mu$$

(*integrable = integrable in the sense of Lebesgue-Stieltjes*).

Now, if we put $f_n = \sum_{m=0}^{\infty} K^m(F)$, Theorem B may be applied to the series. All assumptions are fulfilled and hence we may write

$$\varphi(\boldsymbol{x}, \omega, E, t) = \sum_{m=0}^{\infty} K^m(F) = F + \sum_{m=1}^{\infty} K^m(F) = F + \sum_{m=1}^{\infty} K(K^{m-1}(F)) =$$
$$= F + K(\sum_{m=0}^{\infty} K^m(F)) = F + K(\varphi)$$

Consequently the series $\varphi = \sum_{m=0}^{\infty} K^m(F)$ solves equation (2). It is evident that $\varphi \in C\{B_1; M\}$.

UNIQUENESS OF SOLUTION

The following functions will be introduced for any material medium:

$$(4) \qquad \Sigma_A(E \to E') = \Sigma_f \frac{\Theta^2}{2E} \{e^{-\varepsilon'+\varepsilon}[\operatorname{erf}(\Theta\sqrt{\varepsilon} - \zeta\sqrt{\varepsilon'}) \pm \operatorname{erf}(\Theta\sqrt{\varepsilon} + \zeta\sqrt{\varepsilon'})] +$$
$$+ \operatorname{erf}(\Theta\sqrt{\varepsilon'} - \zeta\sqrt{\varepsilon}) \mp \operatorname{erf}(\Theta\sqrt{\varepsilon'} + \zeta\sqrt{\varepsilon})\},$$

$$\Sigma_A(E) = \int_0^\infty dE' \, \Sigma_A(E \to E')$$

where the upper sign holds for $\varepsilon < \varepsilon'$ and the lower one for $\varepsilon > \varepsilon'$;

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x'^2} dx', \quad \Theta = \frac{A+1}{2\sqrt{A}}, \quad \zeta = \frac{A-1}{2\sqrt{A}}, \quad \varepsilon = \frac{E}{kT},$$
$$\varepsilon' = \frac{E}{'kT}, \quad A = \frac{M}{m}$$

(M is the mass of a particle of the medium while m is the mass of the impinging particle which means a neutron in our case), Σ_f is a positive constant.

Theorem 2. *Let the following conditions be satisfied:*

(*i*) *Conditions* (*a*) *and* (*b*) *of Theorem 1; further, let* $\mathrm{d}T/\mathrm{d}t = \mathrm{d}\Sigma/\mathrm{d}t = \partial\Sigma/\partial x_i = 0$, $i = 1, 2, 3$.

(*ii*) *The function* $\sqrt{(2E)}\,\Sigma_u$ *is continuous on the set*

$$E_3 \times \Omega \times \langle 0, \infty) \times \langle 0, \infty)\,,$$

(*iii*) *the integral* $\int_\Omega \mathrm{d}\omega' \int_0^\infty \mathrm{d}E' \sqrt{(2E')}\, \Sigma(\boldsymbol{x}, \boldsymbol{\omega}\,.\,\boldsymbol{\omega}', E \to E', t)$ *is finite on* M.

(*iv*) *To an arbitrary* $\eta > 0$ *there exists* $N > 0$ *such that*

$$\left| \frac{\int_\Omega \mathrm{d}\omega'\, \Sigma(\boldsymbol{x}, \boldsymbol{\omega}\,.\,\boldsymbol{\omega}', E \to E', t)}{\Sigma_A(E \to E')} - 1 \right| < \eta$$

almost everywhere (*in the sense of condition* (*i*)) *on the set*

$$M_2 \equiv E_3 \times \Omega \times (N, \infty) \times \langle 0, \infty) \times (N, \infty)\,.$$

Then there exists in the class $C\{B_1; M\}$ *only one solution of Problem* (*2*).

Lemma 2. *Let all conditions of Theorem 2 be satisfied. Then*

$$\lim_{E \to \infty} \frac{R(\sqrt{(2E)}\,\Sigma_u)}{\sqrt{(2E)}\,\Sigma_u} \leqq \begin{cases} \dfrac{6A^2 + 2}{6A(A + 1)} & A \geqq 1 \\[2ex] \dfrac{A^2 + 3}{3(A + 1)} & A \leqq 1 \end{cases}.$$

Proof of Lemma 2.

a) Let $T \to 0$. We obtain from definition (4)

$$\Sigma_A(E \to E') = \begin{cases} \dfrac{\Sigma_f}{E} \dfrac{(A + 1)^2}{4A} & E' \in (\alpha E, E) \\[2ex] 0 & E' \in (0, \infty) \div (\alpha E, E) \end{cases},$$

$$\Sigma_A(E) = \Sigma_f \quad \text{where} \quad \alpha = \left(\frac{A - 1}{A + 1}\right)^2.$$

Let us denote

$$R_A(\varphi) = \int_0^t \mathrm{d}t_1 \int_0^\infty \mathrm{d}E' \sqrt{(2E)}\, \Sigma_A(E \to E')\, e^{(t_1 - t)\sqrt{(2E)}\Sigma_A(E)}\, \varphi(E', t_1)$$

for every φ defined on $(0, \infty) \times \langle 0, \infty)$. Simple modifications then yield

$$R_A(\sqrt{(2E)}\,\Sigma_f) = \sqrt{(2E)}\,\Sigma_f(1 - e^{-t\sqrt{(2E)}\Sigma_1}) \times \begin{cases} \dfrac{6A^2 + 2}{6A(A + 1)} & A \geqq 1 \\ \dfrac{A^2 + 3}{3(A + 1)} & A \leqq 1 \end{cases}$$

which establishes immediately Lemma 2 (for $R = R_A$).

b) Now, let $T > 0$. If we introduce independent variables ε, ε' and T instead of the independent variables E, E' and T in relations (4), then it holds for fixed T:

$$\Sigma_A(E \to E') = \frac{F_A(\varepsilon, \varepsilon')}{kT}, \quad \Sigma_A(E) = \int_0^\infty \mathrm{d}\varepsilon'\, F_A(\varepsilon, \varepsilon') = F_A(\varepsilon)$$

where $F_A(\varepsilon, \varepsilon')$, $F_A(\varepsilon)$ do not depend on T and

$$\lim_{E\to\infty} \frac{R_A(\sqrt{(2E)}\,\Sigma_A)}{\sqrt{(2E)}\,\Sigma_A} = \lim_{\varepsilon\to\infty} \frac{1 - e^{-t\sqrt{(2\varepsilon kT)}F_A(\varepsilon)}}{2\varepsilon\, F_A^2(\varepsilon)} \times$$

$$\times \int_0^\infty \mathrm{d}\varepsilon'\, \sqrt{(4\varepsilon\varepsilon')}\, F_A(\varepsilon')\, F_A(\varepsilon, \varepsilon') \leqq \lim_{\varepsilon\to\infty} \frac{\int_0^\infty \mathrm{d}\varepsilon'\, \sqrt{(4\varepsilon\varepsilon')}\, F_A(\varepsilon')\, F_A(\varepsilon, \varepsilon')}{2\varepsilon\, F_A^2(\varepsilon)}\,.$$

The last term obviously does not depend on T and hence (according to item a) of the proof)

$$\lim_{E\to\infty} \frac{R_A(\sqrt{(2E)}\,\Sigma_A)}{\sqrt{(2E)}\,\Sigma_A(E)} \leqq \begin{cases} \dfrac{6A^2 + 2}{6A(A + 1)} & A \geqq 1 \\ \dfrac{A^2 + 3}{3(A + 1)} & A \leqq 1 \end{cases}.$$

c) Let us choose some η. According to conditions (iii) and (iv) of Theorem 2 we can write on M_2

$$R(\sqrt{(2E)}\,\Sigma_u) \geqq \int_0^t \mathrm{d}t_1 \int_\Omega \mathrm{d}\omega' \int_0^N \mathrm{d}E' \Big\{ \sqrt{(2E)}\,\Sigma(\boldsymbol{x} - \sqrt{(2E)}\,\boldsymbol{\omega}(t - t_1), \boldsymbol{\omega}\,.\,\omega', E \to E', t_1) \times$$

$$\times \exp\left(\left(\int_t^{t_1} \mathrm{d}t_2\, \sqrt{(2E)}\,\Sigma_u(\boldsymbol{x} - \sqrt{(2E)}\,\boldsymbol{\omega}(t - t_2), \boldsymbol{\omega}, E, t_2)\right)\right) \times$$

$$\times \sqrt{(2E')}\,\Sigma_u(\boldsymbol{x} - \sqrt{(2E)}\,\boldsymbol{\omega}(t - t_1), \boldsymbol{\omega}', E', t_1)\Big\} +$$

$$+ \frac{(1 - \eta)^2\,(1 - \exp(-t\sqrt{(2E)}\,\Sigma_A(E)(1 + \eta)))}{(1 + \eta)\,\Sigma_A(E)} \times$$

$$\times \int_N^\infty \mathrm{d}E'\, \Sigma_A(E \to E')\, \sqrt{(2E')}\,\Sigma_A(E'),$$

$$R(\sqrt{(2E)}\,\Sigma_u) \leqq \int_0^t \mathrm{d}t_1 \int_\Omega \mathrm{d}\omega' \int_0^N \mathrm{d}E' \Big\{\sqrt{(2E)}\,\Sigma(x - \sqrt{(2E)}\,\omega(t - t_1), \omega \cdot \omega', E \to E', t_1) \times$$

$$\times \exp\left(\int_t^{t_1} \mathrm{d}t_2 \sqrt{(2E)}\,\Sigma_u(x - \sqrt{(2E)}\,\omega(t - t_2), \omega, E, t_2)\right) \times$$

$$\times \sqrt{(2E')}\,\Sigma_u(x - \sqrt{(2E)}\,\omega(t - t_1), \omega', E', t_1)\Big\} +$$

$$+ \frac{(1 + \eta)^2 (1 - \exp(-t\sqrt{(2E)}\,\Sigma_A(E)\,(1 - \eta)))}{(1 - \eta)\,\Sigma_A(E)} \times$$

$$\times \int_N^\infty \mathrm{d}E'\,\Sigma_A(E \to E')\sqrt{(2E')}\,\Sigma_A(E') .$$

According to conditions (i) and (ii) of Theorem 2 we then have

$$\left(\frac{1 - \eta}{1 + \eta}\right)^2 \lim_{E \to \infty} \frac{R_A(\sqrt{(2E)}\,\Sigma_A)}{\sqrt{(2E)}\,\Sigma_A(E)} \leq \lim_{E \to \infty} \frac{R(\sqrt{(2E)}\,\Sigma_u)}{\sqrt{(2E)}\,\Sigma_u} \leqq \left(\frac{1 + \eta}{1 - \eta}\right)^2 \lim_{E \to \infty} \frac{R_A(\sqrt{(2E)}\,\Sigma_A)}{\sqrt{(2E)}\,\Sigma_A(E)} .$$

Items a) and b) of the proof together with the fact that η may be chosen arbitrarily small complete the proof of Lemma 2.

Proof of Theorem 2. Let us suppose that two different solutions $\varphi_1, \varphi_2 \in C\{B_1; M\}$ of Problem (2) exist. Then also $\varphi_3 = \varphi_1 - \varphi_2 \in C\{B_1; M\}$ and

(5) $$\varphi_3 = K(\varphi_3)$$

on M. Hence we obtain easily the inequalities

(6) $$|\varphi_3| \leqq K(|\varphi_3|) ,$$

(7) $$|\varphi_3| \leqq CEe^{-E/kT}$$

where $C < \infty$ is a non-negative constant. Substituting (7) into inequality (6) we obtain as the second approximation

$$|\varphi_3| \leqq CEe^{-E/kT}(1 - \chi)$$

(see Theorem 1 and Lemma 1). Using (6) recursively we obtain finally $|\varphi_3| \leqq CEe^{-Ek/T}$ χ_1 on M, where

$$\chi_1 = 1 - \sum_{m=0}^{\infty} R^m(\chi) \geqq 0 .$$

Applying formally the operator R and considering Lemma 1 and Theorem B we obtain

(8) $$\chi_1 = R(\chi_1) .$$

However, χ_1 is then also a solution of the problem

$$(9) \quad \left\{\frac{\partial}{\partial t} + \sqrt{(2E)}\,\Sigma_u\right\}\chi_1 = \int_\Omega \mathrm{d}\omega' \int_0^\infty \mathrm{d}E'\,\sqrt{(2E)}\,\Sigma(\boldsymbol{x}, \omega\,.\,\omega', E \to E', t)\,\chi_1(\boldsymbol{x}, \omega', E', t)$$

$$\chi_1(\boldsymbol{x}, \omega, E, t = 0) = 0$$

which can be verified by differentiating equation (8). Further it can be seen from the conditions of Theorem 2 that the function

$$\chi_2 = \frac{\partial \chi_1}{\partial t}$$

is also a solution of problems (8) and (9) on M. The equation of Problem (9) makes it easy to estimate

$$|\chi_2| \leqq \sqrt{(2E)}\,\Sigma_u$$

and hence, according to (8),

$$(10) \qquad |\chi_2| \leqq R(\sqrt{(2E)}\,\Sigma_u)\,.$$

For every real medium, the constant A satisfies the inequalities $A > 0$, $A < \infty$ and, consequently, the terms

$$\frac{6A^2 + 2}{6A(A + 1)}\,, \quad \frac{A^2 + 3}{3(A + 1)}$$

are always less than one.

Lemma 2 then implies the existence of $E_0 < \infty$ such that

$$\int_\Omega \mathrm{d}\omega' \int_0^\infty \mathrm{d}E'\,\frac{\Sigma(\omega\,.\,\omega', E \to E')\,\sqrt{(2E')}\,\Sigma_u(\omega', E')\,\sqrt{(2E)}}{(\sqrt{(2E)}\,\Sigma_u(\omega, E))^2} \leqq q < 1$$

(q a constant) for all $E > E_0$ and an arbitrary ω. (In accordance with condition (i) of Theorem 2 we do not write the arguments $\boldsymbol{x}$, t in the functions Σ and Σ_u.)

The last inequality together with condition (iv) guarantee the existence of a constant $D \geqq 0$ such that for each $t_0 \geqq 0$,

(11)

$$\int_{t_0}^t \mathrm{d}t_1 \int_\Omega \mathrm{d}\omega' \int_0^\infty \mathrm{d}E'\,\frac{\Sigma(\omega\,.\,\omega', E \to E')\,\sqrt{(2E')}\,\Sigma_u(\omega, E')}{\Sigma_u(\omega, E)}\,e^{-\int_{t_1}^t \mathrm{d}t_2 \sqrt{(2E)}\Sigma_u(\omega, E)} \leqq q_1 < 1$$

(q_1 being a constant) everywhere on the set $M_3^{t_0} \equiv E_3 \times \Omega \times (0, \infty) \times \langle t_0, t_0 + D\rangle$.

Let us put first $t_0 = 0$. Inequalities (10) and (11) yield the estimate

$$|\chi_2| \leqq q_1\,\sqrt{(2E)}\,\Sigma_u(\omega, E)\,.$$

Applying once more these inequalities we obtain

$$|\chi_2| \leqq q_1^2\,\sqrt{(2E)}\,\Sigma_u$$

and, generally after n steps

$$|\chi_2| \leqq q_1^n \sqrt{(2E)}\, \Sigma_u \, .$$

Consequently $\chi_2 = 0$ on M_3^0.

Let us consider the set M_3^D. In this case, the right-hand side of inequality (10) coincides with the left-hand side of inequality (11). Hence

$$|\chi_2| \leqq q_1 \sqrt{(2E)}\, \Sigma_u$$

and analogously as above, $\chi_2 = 0$ on M_3^D.

The same argument proves the equality $\chi_2 = 0$ on the set M_3^{2D}, M_3^{3D}, ... Therefore the function χ_2 is equal to zero everywhere on the set M. From the equations $\partial\chi_1/\partial t = = \chi_2$ and (9) we have then $\chi_1 = 0$, hence $\varphi_3 = 0$ and the proof is complete.

FINAL REMARKS

a) The set $C\{B; M\}$ is defined in such a way to provide the possibility of working with generalized functions. In this sense one can make the same remarks and present examples similar to those presented in [5].

b) If the neutron absorption cannot be neglected, i.e.

$$\Sigma_u = \int_0^\infty \mathrm{d}E'\, \Sigma(E \to E') + \Sigma_a \, , \quad \Sigma_a \geqq 0$$

and Σ_a is continuous on M, Theorems 1 and 2 evidently remain true. To show this, it is sufficient to observe that

$$K^n_{\Sigma_a \geqq 0}(|\varphi|) \leqq K^n_{\Sigma_a \equiv 0}(|\varphi|)$$

for any positive integer n.

c) Let us present one interesting mathematical consequence of Theorems 1 and 2.

Let f be a real function such that $f(t)$ as well as $\mathrm{d}f/\mathrm{d}t$ are continuous on the interval $\langle 0, \infty)$ (continuity from the right being condidered at zero) and let all assumptions of Theorem 2 be fulfilled. Then

$$f(t) = f(0) \sum_{m=0}^{\infty} R^m(\chi) + \sum_{m=0}^{\infty} R^m(\Theta_1) \, .$$

Proof. According to Lemma 1, the function

$$\psi = f(t) - f(0) \sum_{m=0}^{\infty} R^m(\chi) - \sum_{m=0}^{\infty} R^m(\Theta_1)$$

is a solution of the problem $\psi = R(\psi)$. However, it can be seen from the proof of Theorem 2 that this problem has only the trivial solution.

d) Knudsen's gas. Let us consider a set of non-interacting particles having a mass m, which are present in a material medium (a thermal bath) with particles of a mass M. The particles of the set are scattered by the medium and their distribution varies with time.

If we put – without any loss of generality – $m = 1$ (the mass is measured in the units m), the problem of finding the density of particles of the set for a given initial density ψ may be formulated in the following way [7] (we follow the notation used above and assume the same conditions as above for Σ and Σ_u):

$$\text{(12a)} \qquad \left\{\frac{\partial}{\partial t} + \sqrt{(2E)}\, \omega\nabla + \sqrt{(2E)}\, \Sigma_u\right\} \varphi(x, \omega, E, t) =$$

$$= \int_\Omega \mathrm{d}\omega' \int_0^\infty \mathrm{d}E' \sqrt{(2E)}\, \Sigma(x, \omega' \to \omega, E' \to E, t)\, \varphi(x, \omega', E', t)$$

$$\text{(12b)} \qquad \varphi(x, \omega, E, t = 0) = \psi(x, \omega, E)\,.$$

However, it can be seen that Problem (12) is formally identical with Problem (1). Therefore, Theorems 1 and 2 fully apply.

Problem (12) has not yet been solved generally. The most usual approximate method is the approximation of equation (12a) by a second order differential equation (see [7], [8], [9]). In papers [7] and [8] the cases

$$\psi = \delta(E - E_0)\,, \quad \psi = Ee^{-E/kT}$$

are solved in this way.

Theorems proved in the present paper make it possible to obtain accurate results.

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Souhrn

POČÁTEČNÍ ÚLOHA V TRANSPORTU NEUTRONŮ PRO NENÁSOBÍCÍ PROSTŘEDÍ

JAN KYNCL

V článku je diskutována transportní rovnice pro funkci hustoty neutronů v nenásobícím prostředí, je-li známo jejich počáteční rozložení. Charakteristiky prostředí a zdroje jsou obecně uvažovány jako funkce času. Je dokázána existence a jednoznačnost počáteční úlohy a zobecněny některé autorovy předchozí výsledky. Jako důsledek je provedena stručná diskuse chování Knudsenova plynu v termální lázni.

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