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RELATIONS BETWEEN DISTRIBUTIONS OF RANDOM VIBRATORY PROCESSES AND DISTRIBUTIONS OF THEIR ENVELOPES

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1. INTRODUCTION AND PROBLEM STATEMENT

Very often in the present technical practice vibratory processes having relatively high resonance frequencies and randomly modulated amplitudes are to be treated. For a great number of practical applications especially in the field of strength and reliability calculations, the knowledge of the amplitude alternations is satisfactory while the information about the phase relations is not required. It is then usual to consider only the envelope of the given processes. This approach involves some useful simplifications both in the analytical work and in the treatment of the experimentally acquired data.

Introduction of the envelope conception into the analytical considerations results in decreasing the order of the corresponding differential equations. In the case of a one-degree-of-freedom system, the second-order differential equation of the vibratory motion will be reduced to a first-order differential equation for the envelope. Such a simplification may be found to be very useful when solving some more complex problems. This approach has made it possible to solve e.g. one class of parametrically random excited nonlinear vibratory system, see [5], [6], [12].

With regard to the experimental analysis of random vibratory processes it has been shown (see [11]) that estimates of statistical parameters of a vibratory process may be found if the envelopes of this process are measured and statistical parameters of these envelopes are evaluated.

To allow the widest applicability of the method of envelopes, the relations between distribution functions of the vibratory processes and those of the corresponding envelopes are needed.

It will be assumed first that the vibratory random process $Y(t)$ is stationary with zero mean value $M[Y] = 0$. The probability density function $f_y(y)$ is then symmetrical with respect to the line $y = 0$. It is defined for real numbers $y \in (-\infty, +\infty)$ and its parameters are time-invariant.

For the vibratory random process $Y(t)$, the envelope process $A(t)$ may be defined in different ways (see e.g. [4], [13]). For the purposes of our study, the process $Y(t)$ is assumed to be expressible in the form

$$(1.1) \qquad Y(t) = A(t) \,.\, \cos \varphi(t) = A(t) \,.\, \cos (\omega_0 t + \Theta(t)) \,,$$

where both $A(t)$ and $\Theta(t)$ are random functions.

Considering the Hilbert transform of $Y(t)$

$$(1.2) \qquad X(t) = -\frac{1}{\pi} \int_0^{\infty} [Y(t + \tau) - Y(t - \tau)] \,.\, \frac{d\tau}{\tau} \,,$$

one may write the adjoint process $X(t)$ to be

$$(1.3) \qquad X(t) = A(t) \,.\, \sin \varphi(t) \,.$$

From Eqs. (1.1) and (1.3), the expression for the envelope $A(t)$ may be written in the form

$$(1.4) \qquad A(t) = \sqrt{(X^2(t) + Y^2(t))} \,.$$

Note that for slowly varying $A(t)$ and $\Theta(t)$, the adjoint process $X(t)$ may be approximately expressed by

$$(1.5) \qquad X(t) \approx -\frac{1}{\omega_0} \,.\, \dot{Y}(t)$$

so that the practical relation for the envelope $A(t)$ of the vibratory process $Y(t)$ may be written as follows:

$$(1.6) \qquad A(t) \approx \sqrt{\left(Y^2(t) + \frac{1}{\omega_0^2} \,.\, \dot{Y}^2(t)\right)} .$$

When solving the problem of finding the relationship between the statistical parameters of the process $Y(t)$ and those of the process $A(t)$, the supposition mentioned above is made, namely that the characteristics of the envelope are known and the characteristics of the corresponding vibratory process are to be estimated. Thus, it will be assumed that for the distribution function or for the probability density of the envelope, a suitable analytical expression may be found resulting either from a theoretical consideration or as a close approximation to the empirical distribution obtained experimentally.

The relation between $f_a(a)$ and $f_y(y)$ has been derived elsewhere, see e.g. [14]. For the convenience of the reader, a concise derivation of this relation will be given here.

The characteristic function of $Y(t)$ is defined by

$$(1.7) \qquad g_y(s) = \int_{-\infty}^{+\infty} f_y(y) \,.\, \exp(isy)\, \mathrm{d}y$$

which gives after substituting for y from Eq. (1.1)

$$g_y(s) = \int_0^\infty \int_0^{2\pi} f_2(a, \vartheta) \,.\, \exp\left[isa \,.\, \cos(\omega_0 t + \vartheta)\right] \,.\, \mathrm{d}a \,.\, \mathrm{d}\vartheta \,.$$

Let us express $f_2(a, \vartheta)$ using the conditional distribution of Θ, i.e. $f_2(a, \vartheta) = f_a(a)$ $. f_\vartheta(\vartheta \mid a)$ and the expression exp [ ] using the Fourier expansion

$$\exp\left[isa \,.\, \cos(\omega_0 t + \vartheta)\right] = \sum_{n=-\infty}^{+\infty} i^n \,.\, J_n(sa) \,.\, \exp\left[in(\omega_0 t + \vartheta)\right]$$

(where $J_0(\,)$ is the Bessel function of the first kind of order n). Thus,

$$(1.8) \quad g_y(s) = \sum_{n=-\infty}^{+\infty} i^n \,.\, \int_0^\infty J_n(sa) \,.\, f_a(a) \,.\, da \,.\int_0^{2\pi} \exp\left[in(\omega_0 t + \vartheta)\right] . f_\vartheta(\vartheta \mid a) \,.\, \mathrm{d}\vartheta \,.$$

The supposition that $Y(t)$ is stationary implies the independence of $g_y(s)$ on t which may be fulfilled only if $f_\vartheta(\vartheta \mid a) = \text{const} = 1/2\pi$. It follows that $f_2(a, t) = (1/2\pi)$. $. f_a(a)$, i.e. the random processes $A(t)$ and $\Theta(t)$ are both stationary and statistically independent.

The expression for $g_y(s)$ looks then like this:

$$(1.9) \qquad g_y(s) = \int_0^\infty J_0(sa) \,.\, f_a(a) \,.\, \mathrm{d}a \,.$$

From $g_y(s)$, the probability density may be obtained by the Fourier transform:

$$(1.10) \qquad f_y(y) = \frac{1}{2\pi} \int_{-\infty}^\infty g_y(s) \,.\, \exp(-isy)\, \mathrm{d}s \,,$$

which after substituting from Eq. (1.9) gives

$$(1.10a) \qquad f_y(y) = \frac{1}{2\pi} \int_0^\infty f_a(a) \,.\, \mathrm{d}a \int_{-\infty}^\infty J_0(sa) \,.\, \exp(-isy)\ \mathrm{d}s \,.$$

It is known that

$$\frac{1}{2\pi}\int_{-\infty}^\infty J_0(sa) \,.\, \exp(-isy) \,.\, \mathrm{d}s = \frac{1}{\pi\sqrt{(a^2 - y^2)}} \quad \text{foı } a \geqq |y| \,,$$

$$= 0 \qquad \text{for } a < |y| \,.$$

The final expression giving the relation between $f_y(y)$ and $f_a(a)$ is then

$$f_y(y) = \frac{1}{\pi} \int_y^\infty \frac{f_a(a) \, . \, \mathrm{d}a}{\sqrt{(a^2 - y^2)}} \quad (\text{for } y \geqq 0) \, . \tag{1.11}$$

The analytical solution of the relation (1.11) in a closed form or one using some special functions is known only for some simple and analytically suitable functions f_a. Nevertheless, a set of functions f_a may be selected which cover a great part of technically important problems or which at least may be used as certain limiting cases.

If we do not succeed in finding the analytical solution of $f_y(y)$ from Eq. (1.11), some simple relations valid for even moments (with respect to the origin) of the probability densities f_a and f_y may be used, see e.g. [3]:

It follows from the properties of the characteristic functions that the n-th moment $\mu_n(y)$ of $f_y(y)$ is given by

$$\mu_n(y) = i^{-n} \, . \, \frac{\mathrm{d}^n g_y(s)}{\mathrm{d}s^n}\bigg|_{s=0}$$

which after substituting for $g_y(s)$ from Eq. (1.9) gives

$$\mu_n(y) = i^{-n} \, . \, J_0^{(n)}(0) \, . \int_0^\infty a^n \, . \, f_a(a) \, . \, \mathrm{d}a = i^{-n} \, . \, J_0^{(n)}(0) \, . \, \mu_n(a) \, .$$

It is known that

$$J_0^{(n)}(0) = \frac{(-1)^k \, . \, (2k)!}{(k!)^2 \, . \, 2^{2k}} \quad \text{for} \quad n = 2k \, ,$$

$$= 0 \qquad\qquad \text{for} \quad n = 2k + 1 \, ,$$

so that the final expression for the relation between the moments has the form:

$$\mu_{2k}(y) = \frac{(2k)!}{(k!)^2 \, . \, 2^{2k}} \, . \, \mu_{2k}(a) \, . \tag{1.12}$$

Owing to the symmetry of $f_y(y)$, i.e. $f_y(-y) = f_y(y)$,

$$\mu_{2k+1} = 0 \, .$$

After a detailed rewriting of Eq. (1.12) we have:

for the second moments

$$\mu_2(y) = \tfrac{1}{2} \, . \, \mu_2(a) \, , \tag{1.12a}$$

for the fourth moments

$$\mu_4(y) = \tfrac{3}{8} \, . \, \mu_4(a) \, , \tag{1.12b}$$

and for the sixth moments

$$\mu_6(y) = \tfrac{5}{16} \cdot \mu_6(a) . \tag{1.12c}$$

It may be expected that only the simplest one-parametric distributions will be adequate for the analytical treatment following Eq. (1.11). Therefore, in the next chapter, ten one-parametric distributions of the envelopes will be given, for which the analytical forms of the probability densities of the vibratory processes may be established without difficulties. In the third chapter, analytical expressions for distributions with threshold values are given. In the fourth and fifth chapters, some two-parametric and generalized gamma-distributions of the envelope are considered and even moments of the probability densities of the corresponding vibratory processes are evaluated. These even moments are then used for approximate expressions of the probability densities by means of the Gram-Charlier series (see Chapter 6).

2. ONE-PARAMETRIC DISTRIBUTIONS

The probability densities and their moments of ten selected one-parametric distributions of envelopes and the corresponding data of the vibratory processes distributions are summarized in Tables 1.1 to 1.10. Here some additional remarks are added both of general meaning and those related to the individual distributions.

The functions f_y according to Eq. (1.11) were computed either directly or by means of tables of integrals ([7], [8]). The moments of f_a were calculated from the definition

$$\mu_k(a) = \int_{-\infty}^{\infty} a^k \,.\, f_a(a) \,.\, \mathrm{d}a \tag{2.1}$$

and again tables were used whenever it was found useful. To make a comparison of different shapes of distributions possible, some further characteristics of symmetric distributions of envelopes are also given in Tables 1.1 to 1.10, viz. the central moments $\bar{\mu}_k$ according to the relation

$$\bar{\mu}_k = \sum_{j=0}^{k} \binom{k}{j} \cdot \mu_{k-j} \cdot (-\mu_1)^j , \tag{2.2}$$

i.e.

$$\begin{aligned}
\bar{\mu}_2 &= \mu_2 - \mu_1^2 , \\
\bar{\mu}_3 &= \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3 , \\
\bar{\mu}_4 &= \mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4 , \\
\bar{\mu}_5 &= \mu_5 - 5\mu_4\mu_1 + 10\mu_3\mu_1^2 - 10\mu_2\mu_1^3 + 4\mu_1^5 , \\
\bar{\mu}_6 &= \mu_6 - 6\mu_5\mu_1 + 15\mu_4\mu_1^2 - 20\mu_3\mu_1^3 + 15\mu_2\mu_1^4 - 5\mu_1^6 ,
\end{aligned}$$

and characteristic invariants

$$I_4 = \bar{\mu}_4/\bar{\mu}_2^2 - 3 \quad \text{(excess)}, \tag{2.3a}$$

$$I_6 = \bar{\mu}_6/\bar{\mu}_2^3 - 15(\bar{\mu}_4/\bar{\mu}_2^2) + 30 \tag{2.3b}$$

(invariant of the sixth degree).

The even moments of f_y were determined by means of Eqs. (1.12) and the corresponding invariants I_4 and I_6 from Eqs. (2.3).

2.1. Dirac unit impulse function (deterministic or causal distribution)

This is a limiting case where the random function becomes deterministic, the envelope $A(t)$ becoming constant and the vibratory process $Y(t)$ thus being harmonic with constant amplitude A_0. The resolution of the characteristics and the evaluation of f_y is evident.

2.2. Uniform distribution

This is another simple type of limiting cases, corresponding e.g. to alternated increase and decrease of amplitudes in the range $\langle 0, A_1 \rangle$.

2.3. Triangular distributions

Three types are considered, viz. triangular increasing, triangular decreasing and triangular combined (equiangular). In all these cases, the expressions for the distribution functions of the vibratory processes have the forms similar to those derived in § 2.2.

2.4. Parabolic distributions

Two types are considered, both leading to expressions similar to those given in §§ 2.2 and 2.3.

2.5. Exponential distribution

In the technical practice, the so-called gamma-distributions are often used. As a rule, the gamma-distribution is defined as a two-parametric one and it will be dealt with in this form in Chapter 4. For some particular values of the parameter m, the

Table 1.1.

ENVELOPE PROCESS $A(t)$

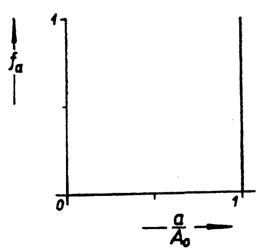
Dirac impulse function

Probability density function

$$f_a(a) = \delta(a - A_0)$$

Moments:

$\mu_1 = A_0$	
$\mu_2 = A_0^2$	$\bar{\mu}_2 = 0$
$\mu_3 = A_0^3$	$\bar{\mu}_3 = 0$
$\mu_4 = A_0^4$	$\bar{\mu}_4 = 0$
$\mu_5 = A_0^5$	$\bar{\mu}_5 = 0$
$\mu_6 = A_0^6$	$\bar{\mu}_6 = 0$



VIBRATORY PROCESS $Y(t)$

Probability density function

$$f_y(y) = \frac{1}{\pi\sqrt{(A_0^2 - y^2)}}\,; \quad |y| \leqq A_0$$
$$= 0\,; \quad |y| > A_0$$

Moments

$\mu_1 = \mu_3 = \mu_5 = 0$

$\mu_2 = \bar{\mu}_2 = \frac{1}{2}A_0^2$

$\mu_4 = \bar{\mu}_4 = \frac{3}{8}A_0^4$

$\mu_6 = \bar{\mu}_6 = \frac{5}{16}A_0^6$

Invariants:

$I_4 = -1.5$

$I_6 = 10$

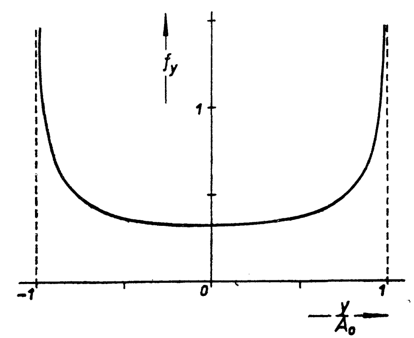


Table 1.2.

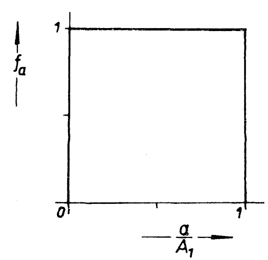
ENVELOPE PROCESS $A(t)$

Uniform distribution

Probability density function

$$f_a(a) = \frac{1}{A_1}\,; \quad 0 \leqq a \leqq A_1$$
$$= 0\,; \quad A_1 < a$$

Moments:		Invariants:
$\mu_1 = \frac{1}{2}A_1$		
$\mu_2 = \frac{1}{3}A_1^2$	$\bar{\mu}_2 = \frac{1}{12}A_1^2$	
$\mu_3 = \frac{1}{4}A_1^3$	$\bar{\mu}_3 = 0$	
$\mu_4 = \frac{1}{5}A_1^4$	$\bar{\mu}_4 = \frac{1}{80}A_1^4$	$I_4 = -1.2$
$\mu_5 = \frac{1}{6}A_1^5$	$\bar{\mu}_5 = 0$	$I_6 = 6.86$
$\mu_6 = \frac{1}{7}A_1^6$	$\bar{\mu}_6 = \frac{1}{448}A_1^6$	



VIBRATORY PROCESS $Y(t)$

Probability density function

$$f_y(y) = \frac{1}{2\pi A_1} \cdot \ln\left(\frac{A_1 + \sqrt{(A_1^2 - y^2)}}{A_1 - \sqrt{(A_1^2 - y^2)}}\right); \quad |y| \leqq A_1$$
$$= 0\,; \quad |y| > A_1$$

Moments:	Invariants:
$\mu_1 = \mu_3 = \mu_5 = 0$	
$\mu_2 = \bar{\mu}_2 = \frac{1}{6}A_1^2$	
$\mu_4 = \bar{\mu}_4 = \frac{3}{40}A_1^4$	$I_4 = -0.3$
$\mu_6 = \bar{\mu}_6 = \frac{5}{112}A_1^6$	$I_6 = -0.857$

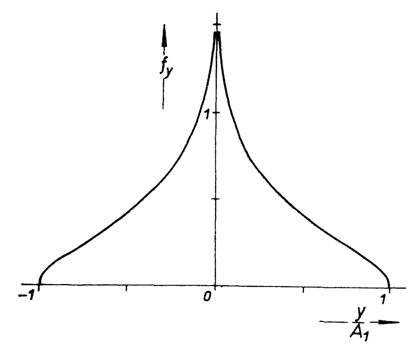


Table 1.3.

ENVELOPE PROCESS $A(t)$

Triangular increasing distribution

Probability density function

$$f_a(a) = \frac{2a}{A_2^2}\,; \quad 0 \leqq a \leqq A_2$$
$$= 0\,; \quad A_2 < a$$

Moments:

$\mu_1 = \frac{2}{3}A_2$

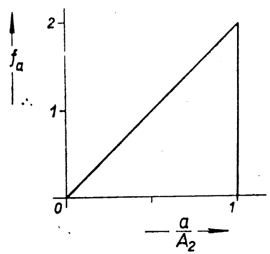
$\mu_2 = \frac{1}{2}A_2^2$

$\mu_3 = \frac{2}{5}A_2^3$

$\mu_4 = \frac{1}{3}A_2^4$

$\mu_5 = \frac{2}{7}A_2^5$

$\mu_6 = \frac{1}{4}A_2^6$



– VIBRATORY PROCESS $Y(t)$

Probability density function

$$f_y(y) = \frac{2}{\pi A_2^2}\sqrt{(A_2^2 - y^2)}\,; \quad 0 \leqq |y| \leqq A_2$$
$$= 0\,; \quad A_2 < |y|$$

Moments:

$\mu_1 = \mu_3 = \mu_5 = 0$

$\mu_2 = \bar{\mu}_2 = \frac{1}{4}A_2^2$

$\mu_4 = \bar{\mu}_4 = \frac{1}{8}A_2^4$

$\mu_6 = \bar{\mu}_6 = \frac{5}{64}A_2^6$

Invariants:

$I_4 = -1$

$I_6 = \;\;5$

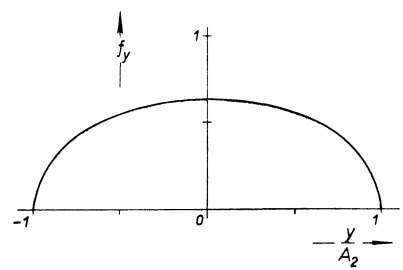


Table 1.4.

ENVELOPE PROCESS $A(t)$

Triangular decreasing distribution

Probability density function

$$f_a(a) = \frac{2}{A_2}\left(1 - \frac{a}{A_2}\right); \quad 0 \leqq a \leqq A_2$$

$$= 0; \quad A_2 < a$$

Moments:

$\mu_1 = \frac{1}{3}A_2$

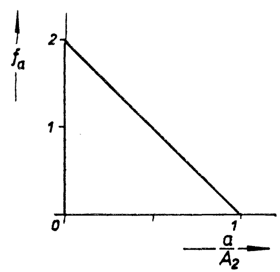
$\mu_2 = \frac{1}{6}A_2^2 \qquad \bar{\mu}_2 = \frac{1}{18}A_2^2$

$\mu_3 = \frac{1}{10}A_2^3 \qquad \bar{\mu}_3 = \frac{7}{45}A_2^3$

$\mu_4 = \frac{1}{15}A_2^4 \qquad \bar{\mu}_4 = \frac{1}{135}A_2^4$

$\mu_5 = \frac{1}{21}A_2^5 \qquad \bar{\mu}_5 = \frac{4}{1701}A_2^5$

$\mu_6 = \frac{1}{28}A_2^6 \qquad \bar{\mu}_6 = \frac{31}{20412}A_2^6$



VIBRATORY PROCESS $Y(t)$

Probability density function

$$f_y(y) = \frac{1}{\pi A_2}\left[\ln\left(\frac{A_2 + \sqrt{(A_2^2 - y^2)}}{A_2 - \sqrt{(A_2^2 - y^2)}}\right) - \frac{2}{A_2}\sqrt{(A_2^2 - y^2)}\right]; \quad 0 \leqq |y| \leqq A_2$$

$$= 0; \quad |y| > A_2$$

Moments:

$\mu_1 = \mu_3 = \mu_5 = 0$

$\mu_2 = \bar{\mu}_2 = \frac{1}{12}A_2^2$

$\mu_4 = \bar{\mu}_4 = \frac{1}{40}A_2^4$

$\mu_6 = \bar{\mu}_6 = \frac{5}{448}A_2^6$

Invariants:

$I_4 = 0.6$

$I_6 = -4.68$

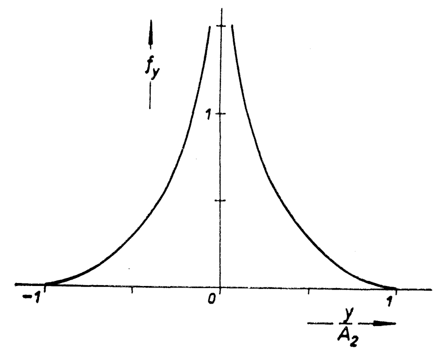


Table 1.5.

ENVELOPE PROCESS $A(t)$

Triangular symmetrical distribution

Probability density function

$$f_a(a) = \frac{a}{A_2^2}\,; \qquad 0 \leqq a \leqq A_2$$
$$= \frac{2}{A_2} - \frac{a}{A_2^2}\,; \qquad A_2 \leqq a \leqq 2A_2$$
$$= 0\,; \qquad 2A_2 < a$$

Moments:

$\mu_1 = A_2$

$\mu_2 = \frac{7}{6}A_2^2 \qquad \bar{\mu}_2 = \frac{1}{6}A_2^2$

$\mu_3 = \frac{3}{2}A_2^3 \qquad \bar{\mu}_3 = 0$

$\mu_4 = \frac{31}{15}A_2^4 \qquad \bar{\mu}_4 = \frac{1}{15}A_2^4$

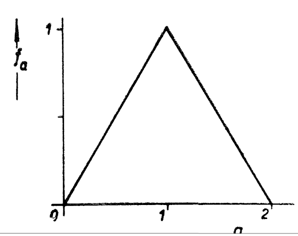
$\mu_5 = 3A_2^5 \qquad \bar{\mu}_5 = 0$

$\mu_6 = \frac{127}{28}A_2^6 \qquad \bar{\mu}_6 = \frac{1}{28}A_2^6$

Invariants:

$I_4 = -0.6$

$I_6 = 1.72$



VIBRATORY PROCESS $Y(t)$

Probability density function

$$f_y(y) = \frac{1}{\pi A_2}\,.\,\ln\left[\frac{2A_2 + \sqrt{(4A_2^2 - y^2)}}{2A_2 - \sqrt{(4A_2^2 - y^2)}}\,.\,\frac{A_2 - \sqrt{(A_2^2 - y^2)}}{A_2 + \sqrt{(A_2^2 - y^2)}}\right] +$$
$$+ \frac{2\sqrt{(A_2^2 - y^2)}}{\pi A_2^2} - \frac{\sqrt{(4A_2^2 - y^2)}}{\pi A_2^2}\,; \quad |y| < A_2$$
$$= \frac{1}{\pi A_2}\,.\,\ln\frac{2A_2 + \sqrt{(4A_2^2 - y^2)}}{2A_2 - \sqrt{(4A_2^2 - y^2)}} - \frac{\sqrt{(4A_2^2 - y^2)}}{\pi A_2^2}\,;$$
$$A_2 < |y| < 2A_2$$
$$= 0\,; \quad 2A_2 < |y|$$

Moments:

$\mu_1 = \mu_3 = \mu_5 = 0$

$\mu_2 = \bar{\mu}_2 = \frac{7}{12}A_2^2$

$\mu_4 = \bar{\mu}_4 = \frac{31}{40}A_2^4$

$\mu_6 = \bar{\mu}_6 = \frac{635}{448}A_2^6$

Invariants:

$I_4 = -0.72$

$I_6 = 2.95$

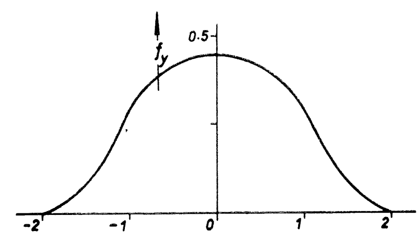


Table 1.6.

ENVELOPE PROCESS $A(t)$

Parabolic distribution I

Probability density function

$$f_a(a) = \frac{3}{2A_3^3}(A_3^2 - a^2)\,; \quad 0 \leqq a \leqq A_3$$

$$= 0\,; \quad A_3 < a$$

Moments:

$\mu_1 = \frac{3}{8}A_3$

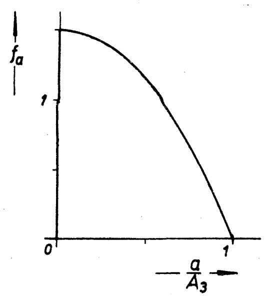
$\mu_2 = \frac{1}{5}A_3^2$

$\mu_3 = \frac{1}{8}A_3^3$

$\mu_4 = \frac{3}{35}A_3^4$

$\mu_5 = \frac{1}{16}A_3^5$

$\mu_6 = \frac{1}{21}A_3^6$



VIBRATORY PROCESS $Y(t)$

Probability density function

$$f_y(y) = \frac{3}{4\pi A_3}\cdot\left[1 - \frac{1}{2}\left(\frac{y}{A_3}\right)^2\right].\ln\frac{A_3 + \sqrt{(A_3^2 - y^2)}}{A_3 - \sqrt{(A_3^2 - y^2)}} - \frac{3\sqrt{(A_3^2 - y^2)}}{4\pi A_3^2}\,;$$

$$0 \leqq |y| \leqq A_3$$

$$= 0\,; \quad A_3 < |y|$$

Moments:

$\mu_1 = \mu_3 = \mu_5 = 0$

$\mu_2 = \bar{\mu}_2 = \frac{1}{10}A_3^2$

$\mu_4 = \bar{\mu}_4 = \frac{9}{280}A_3^4$

$\mu_6 = \bar{\mu}_6 = \frac{5}{336}A_3^6$

Invariants:

$I_4 = 0.215$

$I_6 = -3.33$

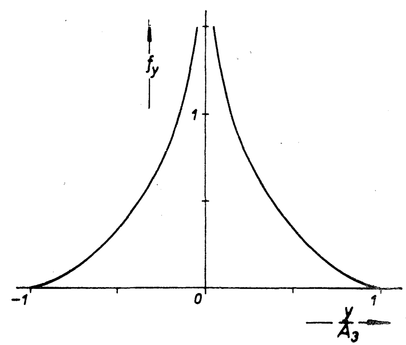


Table 1.7.

ENVELOPE PROCESS $A(t)$

Parabolic distribution II

Probability density function

$$f_a(a) = \frac{6}{A_3^3}(A_3 a - a^2)\,; \quad 0 \leqq a \leqq A_3$$

$$= 0\,; \quad A_3 < a$$

Moments:

$\mu_1 = \frac{1}{2}A_3$

$\mu_2 = \frac{3}{10}A_3^2 \qquad \bar{\mu}_2 = \frac{1}{20}A_3^2$

$\mu_3 = \frac{1}{5}A_3^3 \qquad \bar{\mu}_3 = 0$

$\mu_4 = \frac{1}{7}A_3^4 \qquad \bar{\mu}_4 = \frac{3}{560}A_3^4$

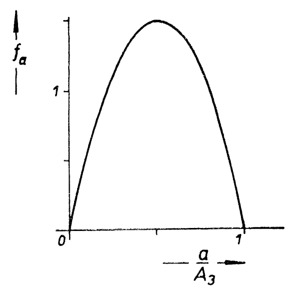
$\mu_5 = \frac{3}{28}A_3^5 \qquad \bar{\mu}_5 = 0$

$\mu_6 = \frac{1}{12}A_3^6 \qquad \bar{\mu}_6 = \frac{1}{1344}A_3^6$

Invariants:

$I_4 = -0.857$

$I_6 = \ \ 3.81$



VIBRATORY PROCESS $Y(t)$

Probability density function

$$f_y(y) = \frac{3\sqrt{(A_3^2 - y^2)}}{\pi A_3^2} - \frac{3y^2}{2\pi A_3^3} \cdot \ln \frac{A_3 + \sqrt{(A_3^2 - y^2)}}{A_3 - \sqrt{(A_3^2 - y^2)}}\,; \quad 0 \leqq |y| \leqq A_3$$

$$= 0\,; \quad A_3 < |y|$$

Moments:

$\mu_1 = \mu_3 = \mu_5 = 0$

$\mu_2 = \bar{\mu}_2 = \frac{3}{20}A_3^2$

$\mu_4 = \bar{\mu}_4 = \frac{3}{56}A_3^4$

$\mu_6 = \bar{\mu}_6 = \frac{5}{192}A_3^6$

Invariants:

$I_4 = -0.615$

$I_6 = \ \ 1.94$

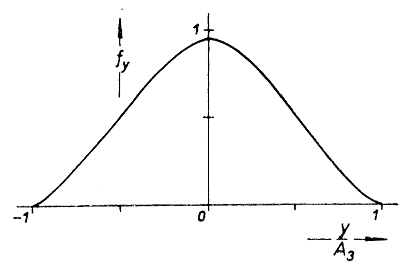


Table 1.8.

ENVELOPE PROCESS $A(t)$

Exponential distribution

Probability density function

$$f_a(a) = \alpha \, . \, \exp(-\alpha a)\,; \quad 0 \leqq a < \infty$$

Moments:

$\mu_1 = 1/\alpha$

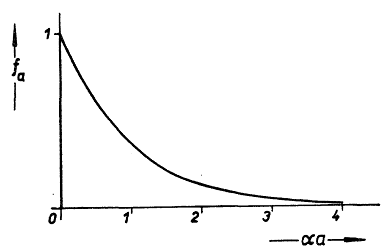
$\mu_2 = 2/\alpha^2 \qquad \bar{\mu}_2 = 1/\alpha^2$

$\mu_3 = 6/\alpha^3 \qquad \bar{\mu}_3 = 2/\alpha^3$

$\mu_4 = 24/\alpha^4 \qquad \bar{\mu}_4 = 9/\alpha^4$

$\mu_5 = 120/\alpha^5 \qquad \bar{\mu}_5 = 44/\alpha^5$

$\mu_6 = 720/\alpha^6 \qquad \bar{\mu}_6 = 265/\alpha^6$



VIBRATORY PROCESS $Y(t)$

Probability density function

$$f_y(y) = \frac{\alpha}{\pi} \, . \, K_0(\alpha|y|)\,; \quad 0 \leqq |y| < \infty$$

Moments:

$\mu_1 = \mu_3 = \mu_5 = 0$

$\mu_2 = \bar{\mu}_2 = 1/\alpha^2$

$\mu_4 = \bar{\mu}_4 = 9/\alpha^4$

$\mu_6 = \bar{\mu}_6 = 225/\alpha^6$

Invariants:

$I_4 = 6$

$I_6 = 120$

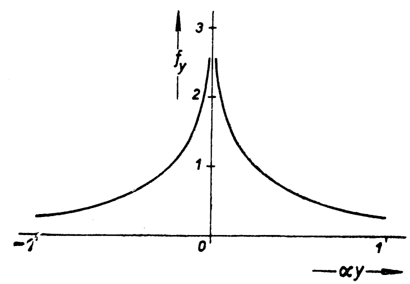


Table 1.9.

ENVELOPE PROCESS $A(t)$

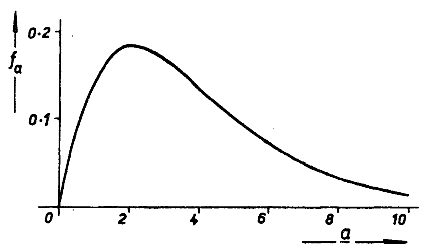
Gamma-distribution with $m = 2$

Probability density function

$$f_a(a) = \frac{a}{c^2} \cdot \exp\left(-\frac{a}{c}\right); \quad 0 \leqq a < \infty$$

Moments:

$$\begin{aligned} \mu_1 &= 2c & & \\ \mu_2 &= 6c^2 & \bar{\mu}_2 &= 2c^2 \\ \mu_3 &= 24c^3 & \bar{\mu}_3 &= 4c^3 \\ \mu_4 &= 120c^4 & \bar{\mu}_4 &= 24c^4 \\ \mu_5 &= 720c^5 & \bar{\mu}_5 &= 128c^5 \\ \mu_6 &= 5040c^6 & \bar{\mu}_6 &= 880c^6 \end{aligned}$$



VIBRATORY PROCESS $Y(t)$

Probability density function

$$f_y(y) = \frac{|y|}{\pi c^2} \cdot K_1\left(\frac{|y|}{c}\right); \quad 0 \leqq |y| < \infty$$

Moments:

$$\begin{aligned} \mu_1 &= \mu_3 = \mu_5 = 0 \\ \mu_2 &= \bar{\mu}_2 = 3c^2 \\ \mu_4 &= \bar{\mu}_4 = 45c^4 \\ \mu_6 &= \bar{\mu}_6 = 1575c^6 \end{aligned}$$

Invariants:

$$I_4 = 2$$
$$I_6 = 13.33$$

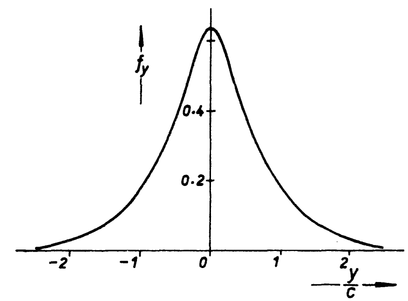


Table 1.10.

ENVELOPE PROCESS $A(t)$

Rayleigh distribution

Probability density function

$$f_a(a) = \frac{a}{\sigma^2} \cdot \exp\left(-\frac{a^2}{2\sigma^2}\right); \quad 0 \leqq a < \infty$$

Moments:

$\mu_1 = 1.2533\sigma$

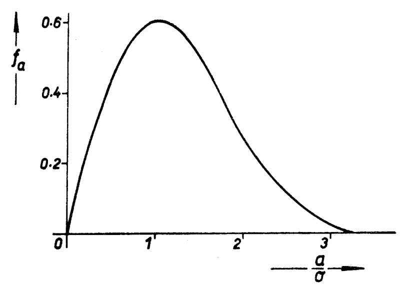
$\mu_2 = 2\sigma^2$ $\quad \bar{\mu}_2 = 0.43\sigma^2$

$\mu_3 = 3.76\sigma^3$ $\quad \bar{\mu}_3 = 0.18\sigma^3$

$\mu_4 = 8\sigma^4$ $\quad \bar{\mu}_4 = 0.60\sigma^4$

$\mu_5 = 18.8\sigma^5$ $\quad \bar{\mu}_5 = 0.73\sigma^5$

$\mu_6 = 48\sigma^6$ $\quad \bar{\mu}_6 = 1.753\sigma^6$



VIBRATORY PROCESS $Y(t)$

Normal (Gaussian) distribution

Probability density function

$$f_y(y) = \frac{1}{\sqrt{(2\pi)}\,\sigma} \cdot \exp\left(-\frac{y^2}{2\sigma^2}\right); \quad 0 \leqq |y| < \infty$$

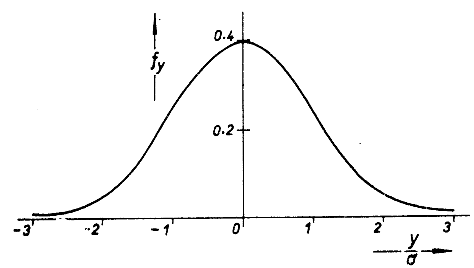
Moments: Invariants:

$\mu_1 = \mu_3 = \mu_5 = 0$

$\mu_2 = \bar{\mu}_2 = \sigma^2$

$\mu_4 = \bar{\mu}_4 = 3\sigma^4$ $\quad I_4 = 0$

$\mu_6 = \bar{\mu}_6 = 15\sigma^6$ $\quad I_6 = 0$



analytical solution of f_y may be found and these cases are shown in this part of the paper. These solutions are expressed by means of modified Bessel functions of pure imaginary argument of the second kind K_n. Tables of these functions see e.g. [1], [9]. Other names of the K_n functions also used are: modified Hankel function or MacDonald function. In this case we deal with the gamma function with $m = 1$, which is called the exponential distribution. The probability density f_y for this case is given by the MacDonald function of zero order, K_0.

2.6. Gamma function with $m = 2$

The probability density is given by the MacDonald function of the first order, K_1.

2.7. Rayleigh distribution

The couple given by the Gaussian distribution of the vibratory random process and the Rayleigh distribution of its envelope forms the basis for all considerations related to the envelope method. The Gaussian distribution corresponds to a typical random process occuring in practical applications and at the same time, the analytical solution of some more complex problems of statistical dynamics is, as a rule, possible only for this distribution function. The derivation of both distributions was made in both directions, the evaluation according to Eq. (1.11) becoming very easy by the substitution $a^2 - y^2 = x^2$. Tables of the Gaussian distribution and its derivatives are given in [15]. Note that the Rayleigh distribution differs from the first derivative of the Gaussian one only by a multiplier.

3. DISTRIBUTIONS WITH THRESHOLD VALUES AND THE PIECEWISE — CONTINUOUS APPROXIMATION OF THE ENVELOPE PROBABILITY DENSITY

In the technical practice the distributions with the so-called threshold values are of a considerable importance. Random processes with the threshold value probability density are typical e.g. for envelopes of parametrically excited vibratory processes with small variance ratio.

The threshold value A_0 has the following meaning: for $a \in \langle 0, A_0 \rangle$, the probability density f_a is identically equal to zero, for $a \in \langle A_0, \infty)$, it has the form of a suitable distribution discussed above. Analytically, we have

$$(3.1) \qquad \begin{aligned} f_a(a) &= 0 \qquad\qquad ; \quad a < A_0 \\ &= h(a - A_0)\,; \quad A_0 \leqq a < \infty\,. \end{aligned}$$

By substituting Eq. (3.1) into Eq. (1.11), we obtain

$$(3.2)\qquad f_y(y) = \frac{1}{\pi}\int_{A_0}^{\infty} \frac{h(a - A_0)\,.\,\mathrm{d}a}{\sqrt{(a^2 - y^2)}}\,;\quad |y| < A_0$$

$$= \frac{1}{\pi}\int_{y}^{\infty} \frac{h(a - A_0)\,.\,\mathrm{d}a}{\sqrt{(a^2 - y^2)}}\,;\quad |y| \geqq A_0$$

In some simple cases when the derived expressions may be integrated without difficulty, the function $h(a - A_0)$ may be used directly in this form. If it is impracticable to integrate Eq. (3.2) directly after substituting $h(a - A_0)$, while the integral transforms for $h(a)$ and its derivatives $h'(a)$, $h''(a)$, etc. may be easily found, the Taylor expansion of $h(a - A_0)$ at a may be used so that $h(a - A_0) = h(a) - A_0\,.\,h'(a) + \frac{1}{2}A_0^2\,.\,h''(a) - \ldots$

3.1. Uniform distribution with the threshold value

This is some generalization of the uniform distribution treated in § 2.2 in the sense that the nonzero values of f_a are defined in $\langle A_0, A_1 \rangle$ where $A_0 > 0$. For $A_0 = 0$ this distribution changes into the uniform one-parametric distribution (§ 2.2), for $A_0 \to A_1$ it approaches the Dirac impulse function (§ 2.1). For the generalized uniform distribution, the probability density f_y may be derived by direct evaluation of Eq. (3.1):

(3.4)

$$f_y(y) = \frac{1}{2\pi(A_1 - A_0)}\,.\,\ln\left[\frac{A_1 + \sqrt{(A_1^2 - y^2)}}{A_1 - \sqrt{(A_1^2 - y^2)}}\,.\,\frac{A_0 - \sqrt{(A_0^2 - y^2)}}{A_0 + \sqrt{(A_0^2 - y^2)}}\right];\quad 0 \leqq |y| < A_0\,,$$

$$= \frac{1}{2\pi(A_1 - A_0)}\,.\,\ln\frac{A_1 + \sqrt{(A_1^2 - y^2)}}{A_1 - \sqrt{(A_1^2 - y^2)}}\,;\quad A_0 \leqq |y| \leqq A_1\,,$$

$$= 0\,;\quad A_1 < |y|\,.$$

3.2. Triangular distribution with the threshold value

Let be

$$(3.5)\qquad f_a(a) = 0\,;\qquad 0 < a \leqq A_0\,,$$

$$= \frac{2(a - A_0)}{(A_1 - A_0)^2}\,;\quad A_0 \leqq a \leqq A_1\,,$$

$$= 0\,;\qquad A_1 < a\,.$$

This distribution has been chosen for demonstrating the derivation of $f_y(y)$ using both ways mentioned in § 3.

By direct substitution into Eq. (3.1), we get

for $|y| < A_0$:

$$(3.6a)\quad f_y(y) = \frac{2}{\pi(A_1 - A_0)^2}\left[\int_{A_0}^{A_1} \frac{a\,\mathrm{d}a}{\sqrt{(a^2 - y^2)}} - A_0 \int_{A_0}^{A_1} \frac{\mathrm{d}a}{\sqrt{(a^2 - y^2)}}\right] =$$

$$= \frac{2}{\pi(A_1 - A_0)^2}\left[\sqrt{(A_1^2 - y^2)} - \sqrt{(A_0^2 - y^2)}\right] -$$

$$- \frac{A_0}{\pi(A_1 - A_0)^2} \, . \, \ln\left[\frac{A_1 + \sqrt{(A_1^2 - y^2)}}{A_1 - \sqrt{(A_1^2 - y^2)}} \, . \, \frac{A_0 - \sqrt{(A_0^2 - y^2)}}{A_0 + \sqrt{(A_0^2 - y^2)}}\right];$$

for $|y| \geqq A_0$:

$$(3.6b)\quad f_y(y) = \frac{2}{\pi(A_1 - A_0)^2}\left[\int_{y}^{A_1} \frac{a\,\mathrm{d}a}{\sqrt{(a^2 - y^2)}} - A_0 \int_{y}^{A_1} \frac{\mathrm{d}a}{\sqrt{(a^2 - y^2)}}\right] =$$

$$= \frac{2\sqrt{(A_1^2 - y^2)}}{\pi(A_1 - A_0)^2} - \frac{A_0}{\pi(A_1 - A_0)^2} \, . \, \ln \frac{A_1 + \sqrt{(A_1^2 - y^2)}}{A_1 - \sqrt{(A_1^2 - y^2)}} \, .$$

Using the Taylor series for $f_a(a)$ and taking $h(a) = 2a/B^2$ for $0 \leqq a \leqq B$, $h'(a) = 2/B^2$, we have for $f_y(y)$, $|y| < A_0$:

$$f_y(y) = \frac{1}{\pi}\int_{A_0}^{A_0+B} \frac{g(a)\,.\,\mathrm{d}a}{\sqrt{(a^2 - y^2)}} - \frac{A_0}{\pi}\int_{A_0}^{A_0+B} \frac{g'(a)\,.\,\mathrm{d}a}{\sqrt{(a^2 - y^2)}} =$$

$$= \frac{2}{\pi B^2}\sqrt{(a^2 - y^2)}\Bigg|_{A_0}^{A_0+B} - \frac{A_0}{\pi B^2} \, . \, \ln \frac{a + \sqrt{(a^2 - y^2)}}{a - \sqrt{(a^2 - y^2)}}\Bigg|_{A_0}^{A_0+B}$$

Denoting $A_0 + B = A_1$, i.e. $B = A_1 - A_0$ we have

$$(3.7)\quad f_y(y) = \frac{2}{\pi(A_1 - A_0)^2}\sqrt{(A_1^2 - y^2)} - \sqrt{(A_0^2 - y^2)} -$$

$$- \frac{A_0}{\pi(A_1 - A_0)^2} \, . \, \ln\left[\frac{A_1 + \sqrt{(A_1^2 - y^2)}}{A_1 - \sqrt{(A_1^2 - y^2)}} \, . \, \frac{A_0 - \sqrt{(A_0^2 - y^2)}}{A_0 + \sqrt{(A_0^2 - y^2)}}\right]$$

wh'ch is identical with Eq. (3.6a). Similarly, for $|y| \geqq A_0$ the expression identical with that of Eq. (3.6b) is obtained.

The total coincidence of the expressions derived in both indicated ways results of course from the fact that the derivatives h'', h''', etc. of the function $h(a) = 2a/B^2$ are equal to zero so that by the first two terms of the Taylor series the function $h(a - A_0) = 2(a - A_0)/(A_1 - A_0)^2$ is not only approximated but exactly described.

3.3. General form of the probability density of the envelope given by continuous parts whose analytical expressions allow the evaluation of the integral transform

In is clear from the above discussion that the number of functions $f_a(a)$ for which the analytical solution of the integral transform Eq. (1.11) is known is not very great. The discussion concerning the threshold values suggests the possibility of an analytical expression of Eq. (1.11) transform for a function f_a given by piecewise-continuous parts for which the transform Eq. (1.11) is known.

Let the function $f_a(a)$ be given as follows:

$$(3.8)\qquad \begin{aligned} f_a(a) &= f_0(a)\;; & 0 &\leqq a < A_0\,,\\ &= f_1(a)\;; & A_0 &\leqq a < A_1\,,\\ &= f_k(a)\;; & A_{k-1} &\leqq a < A_k\,,\\ &= f_n(a)\;; & A_{n-1} &\leqq a < A_n\,,\\ &= f_\infty(a)\;; & A_n &\leqq a < \infty\,. \end{aligned}$$

Applying mathematical induction to the preceding considerations, we get the following expressions for $f_y(y)$:

$$(3.9)\quad f_y(y) = \frac{1}{\pi}\int_y^{A_0}\frac{f_0(a)\,.\,\mathrm{d}a}{\sqrt{(a^2-y^2)}} + \frac{1}{\pi}\sum_{i=1}^{n}\int_{A_{i-1}}^{A_i}\frac{f_i(a)\,.\,\mathrm{d}a}{\sqrt{(a^2-y^2)}} + \frac{1}{\pi}\int_{A_n}^{\infty}\frac{f_\infty(a)\,.\,\mathrm{d}a}{\sqrt{(a^2-y^2)}}\;;$$

$$0 \leqq |y| \leqq A_0\,,$$

$$= \frac{1}{\pi}\int_y^{A_k}\frac{f_k(a)\,.\,\mathrm{d}a}{\sqrt{(a^2-y^2)}} + \frac{1}{\pi}\sum_{i=k+1}^{n}\int_{A_{i-1}}^{A_i}\frac{f_i(a)\,.\,\mathrm{d}a}{\sqrt{(a^2-y^2)}} + \frac{1}{\pi}\int_{A_n}^{\infty}\frac{f_\infty(a)\,.\,\mathrm{d}a}{\sqrt{(a^2-y^2)}}\;;$$

$$A_{k-1} \leqq |y| \leqq A_k\,,$$

$$= \frac{1}{\pi}\int_y^{\infty}\frac{f_\infty(a)\,.\,\mathrm{d}a}{\sqrt{(a^2-y^2)}}\;;\quad A_k \leqq |y| < \infty\,.$$

It is clear that the requirements imposed on the functions $f_k(a)$ are more restrictive than those for the functions defined on the whole interval $\langle 0, \infty)$. For f_a, it is sufficient to know the definite integral in the range $\langle y, \infty)$, but for each f_k, the indefinite integral of the transform Eq. (1.11) must be known. This is fulfilled e.g. for a polynomial in a, i.e. $f_a = \sum_{i=1}^{n} k_i\,.\,a^i$. Odd powers lead then to expressions containing powers of $\sqrt{(a^2 - y^2)}$, even powers lead to expressions containing $\ln \{[a + \sqrt{(a^2 - y^2)}] : [a - \sqrt{(a^2 - y^2)}]\}$. It follows from Eq. (3.7) that with a greater number of sections, the expression for f_y becomes cumbersome. On the other hand, when intervals $A_{j+1} - A_j$ are chosen to be equal and the probability density function f_a is approximated by a polygon, a theoretical basis is built up for writing an algorithm for a digital computer.

4. TWO-PARAMETRIC DISTRIBUTIONS

The distributions given in this chapter were obtained by a generalization of some distributions given in Chapter 2. In Tables 2.1 to 2.4, moments of the probability densities of the envelopes and the corresponding densities of the vibratory processes up to the sixth order and the invariants I_4 and I_6 are given. When evaluating these quantities, the same approach was adapted as that in Chapter 2. It may be shown that in the invariants defined by Eqs. (2.3) the number of independent parameters is reduced by one, i.e., for two-parametric distributions these invariants depend only on one characteristic quantity. These relationships are given graphically in Fig. 1 instead of distribution functions f_y whose analytical expressions are not known.

4.1. Gaussian distribution of the envelope

This is some kind of generalization of the case given in § 2.1 in the sense that instead of a deterministic and constant value A_0, a random variable with the mean value μ and the variance σ^2 is assumed. For f_a which is defined for $a \geqq 0$ only, one may assume only the cases with $\mu/\sigma > 3.2$ for which the part of the distribution corresponding to $a < 0$ is negligible (it forms only approx. 0.1% of the whole area of the probability integral).

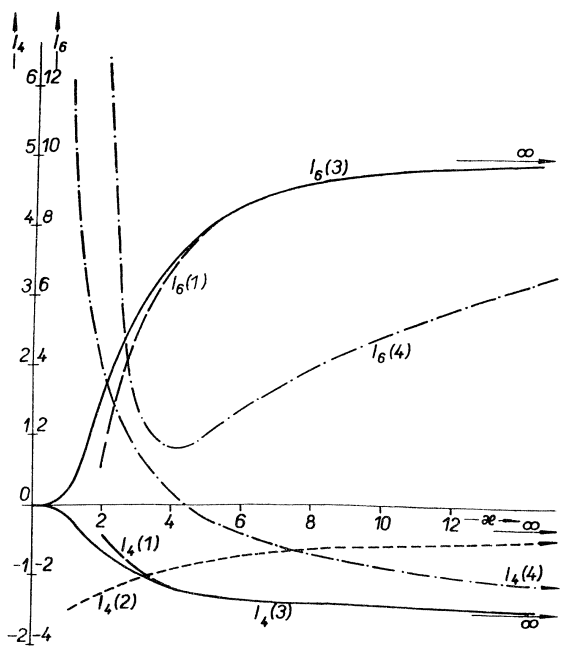


Fig. 1. Invariants I_4 and I_6 of the vibratory processes $Y(t)$ treated in Chapter 4 plotted against the dimensionless parameters $\varkappa$ where

$\varkappa = \dfrac{\mu}{\sigma}$ for the distribution function treated in § 4.1

$\varkappa = \dfrac{A_1}{A_0}$ for the distribution function treated in § 4.2

$\varkappa = \dfrac{\alpha}{\sigma}$ for the distribution function treated in § 4.3

$\varkappa = m$ for the distribution function treated in § 4.4

Table 2.1.

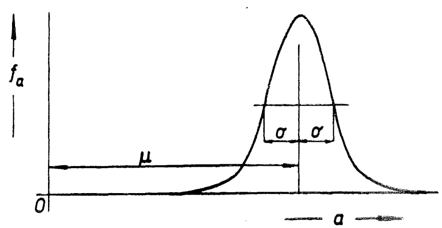
ENVELOPE PROCESS $A(t)$

Gaussian distribution of envelope

Probability density function

$$f_a(a) = \frac{1}{\sqrt{(2\pi)}\,\sigma} \cdot \exp\left\{-\frac{(a-\mu)^2}{2\sigma^2}\right\}; \quad 0 \leqq a < \infty$$

Moments:		Invariants:
$\mu_1 = \mu$		
$\mu_2 = \mu^2 + \sigma^2$	$\bar{\mu}_2 = \sigma^2$	
$\mu_3 = \mu^3 + 3\mu\sigma^2$	$\bar{\mu}_3 = 0$	
$\mu_4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$\bar{\mu}_4 = 3\sigma^4$	$I_4 = 0$
$\mu_5 = \mu^5 + 10\mu^3\sigma^2 + 15\mu\sigma^4$	$\bar{\mu}_5 = 0$	
$\mu_6 = \mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6$;	$\bar{\mu}_6 = 15\sigma^6$	$I_6 = 0$



VIBRATORY PROCESS $Y(t)$

Moments:

$$\mu_1 = \mu_3 = \mu_5 = 0$$
$$\mu_2 = \bar{\mu}_2 = \tfrac{1}{2}(\mu^2 + \sigma^2)$$
$$\mu_4 = \bar{\mu}_4 = \tfrac{3}{8}(\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4)$$
$$\mu_6 = \bar{\mu}_6 = \tfrac{5}{16}(\mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6)$$

Invariants:

$$I_4 = 1.5 - 3\left(\frac{\varkappa^2}{\varkappa^2 + 1}\right)^2 \qquad \varkappa = \mu/\sigma$$

$$I_6 = \frac{10\varkappa^4(\varkappa^2 - 3)}{(\varkappa^2 + 1)^3}; \quad \varkappa \to \infty \Rightarrow I_4 = -1.5\,; \quad I_6 = 10$$

Table 2.2.

ENVELOPE PROCESS $A(t)$

Uniform twoparametric distribution

Probability density function

$$f_a(a) = \begin{cases} 0\,; & a < A_0 \\ \dfrac{1}{A_1 - A_0}\,; & A_0 \leqq a \leqq A_1 \\ 0\,; & A_1 < a \end{cases}$$

Moments:

$$\mu_1 = \frac{1}{2}(A_1 + A_0)$$

$$\mu_2 = \frac{1}{3}\cdot\frac{A_1^3 - A_0^3}{A_1 - A_0} \qquad \bar{\mu}_2 = \frac{1}{12}(A_1 - A_0)^2$$

$$\mu_3 = \frac{1}{4}\cdot\frac{A_1^4 - A_0^4}{A_1 - A_0} \qquad \bar{\mu}_3 = 0$$

$$\mu_4 = \frac{1}{5}\cdot\frac{A_1^5 - A_0^5}{A_1 - A_0} \qquad \bar{\mu}_4 = \frac{1}{80}(A_1 - A_0)^4$$

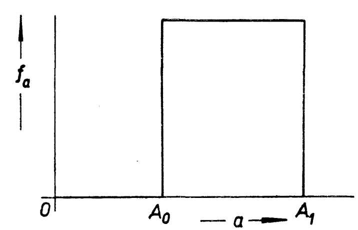
$$\mu_5 = \frac{1}{6}\cdot\frac{A_1^6 - A_0^6}{A_1 - A_0} \qquad \bar{\mu}_5 = 0$$

$$\mu_6 = \frac{1}{7}\cdot\frac{A_1^7 - A_0^7}{A_1 - A_0} \qquad \bar{\mu}_6 = \frac{1}{448}(A_1 - A_0)^6$$

Invariants:

$I_4 = -1.2$

$I_6 = 6.86$



VIBRATORY PROCESS $Y(t)$

Moments:

$$\mu_1 = \mu_3 = \mu_5 = 0$$

$$\mu_2 = \bar{\mu}_2 = \frac{1}{6}\cdot\frac{A_1^3 - A_0^3}{A_1 - A_0}$$

$$\mu_4 = \bar{\mu}_4 = \frac{3}{40}\cdot\frac{A_1^5 - A_0^5}{A_1 - A_0}$$

$$\mu_6 = \bar{\mu}_6 = \frac{5}{112}\cdot\frac{A_1^7 - A_0^7}{A_1 - A_0}$$

Invariants:

$$I_4 = 2.7\frac{(\varkappa^5 - 1)(\varkappa - 1)}{(\varkappa^3 - 1)^2} - 3\,; \quad \varkappa = \frac{A_1}{A_0}$$

$$I_6 = \frac{1}{14(\varkappa^3 - 1)^3}\left[-12(\varkappa^9 - 1) + \right.$$
$$+ 297\varkappa(\varkappa^7 - 1) + 135\varkappa^2(\varkappa^5 - 1) -$$
$$\left. - 693\varkappa^3(\varkappa^3 - 1) - 567\varkappa^4(\varkappa - 1)\right]$$

$\varkappa \to 1 \Rightarrow I_4 = -1.5\,; \quad I_6 = 10$

$\varkappa \to \infty \Rightarrow I_4 = -0.3\,; \quad I_6 = -0.857$

Table 2.3.

ENVELOPE PROCESS $A(t)$

Rayleigh-Rice distribution

Probability density function

$$f_a(a) = \frac{a}{\sigma^2} \cdot \exp\left[-\frac{a^2 + \alpha^2}{2\sigma^2}\right] \cdot I_0\left(\frac{\alpha a}{\sigma^2}\right)$$

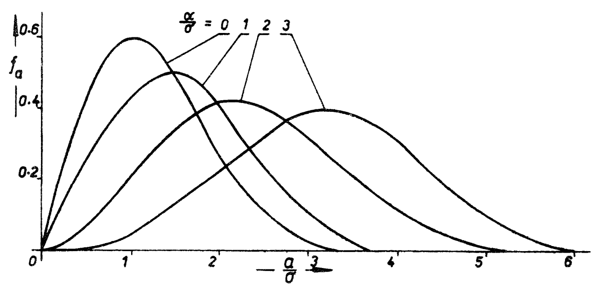
Moments:

$$\mu_1 = \sigma\sqrt{\frac{\pi}{2}} \cdot \left[\left(1 + \frac{\alpha^2}{2\sigma^2}\right) \cdot I_0\left(\frac{\alpha^2}{4\sigma^2}\right) + \frac{\alpha^2}{2\sigma^2} \cdot I_1\left(\frac{\alpha^2}{4\sigma^2}\right)\right] \cdot \exp\left(-\frac{\alpha^2}{4\sigma^2}\right)$$

$$\mu_2 = \sigma^2\left(2 + \frac{\alpha^2}{\sigma^2}\right)$$

$$\mu_4 = 8\sigma^4\left(1 + \frac{\alpha^2}{\sigma^2} + \frac{\alpha^4}{8\sigma^4}\right)$$

$$\mu_6 = 48\sigma^6\left(1 + \frac{3}{2}\frac{\alpha^2}{\sigma^2} + \frac{3}{8}\frac{\alpha^4}{\sigma^4} + \frac{1}{48}\frac{\alpha^6}{\sigma^6}\right)$$



VIBRATORY PROCESS $Y(t)$

Moments:

$$\mu_1 = \mu_3 = \mu_5 = 0$$

$$\mu_2 = \bar{\mu}_2 = \frac{1}{2}(2\sigma^2 + \alpha^2)$$

$$\mu_4 = \bar{\mu}_4 = 3\sigma^4\left(1 + \frac{\alpha^2}{\sigma^2} + \frac{\alpha^4}{8\sigma^4}\right)$$

$$\mu_6 = \bar{\mu}_6 = 15\sigma^6\left(1 + \frac{3}{2}\frac{\alpha^2}{\sigma^2} + \frac{3}{8}\frac{\alpha^4}{\sigma^4} + \frac{1}{48}\frac{\alpha^6}{\sigma^6}\right)$$

Invariants:

$$I_4 = -1.5\left(\frac{\varkappa^2}{2 + \varkappa^2}\right)^2 \qquad \varkappa = \alpha/\sigma$$

$$I_6 = 10\left(\frac{\varkappa^2}{2 + \varkappa^2}\right)^3$$

$$\varkappa \to \infty \Rightarrow I_4 = -1.5\,; \quad I_6 = 10$$

Table 2.4.

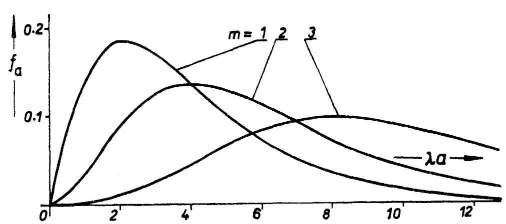
ENVELOPE PROCESS $A(t)$

Gamma distribution

Probability density function

$$f_a(a) = \frac{\lambda^m \,.\, a^{m-1}}{\Gamma(m)} \,.\, \exp(-\lambda a)\,; \quad 0 \leqq a < \infty$$

Moments:	for integer m:
$\mu_1 = \dfrac{\Gamma(m+1)}{\lambda \,.\, \Gamma(m)}$	$\dfrac{m}{\lambda}$
$\mu_2 = \dfrac{\Gamma(m+2)}{\lambda^2 \,.\, \Gamma(m)}$	$\dfrac{1}{\lambda^2}\, m(m+1)$
$\mu_3 = \dfrac{\Gamma(m+3)}{\lambda^3 \,.\, \Gamma(m)}$	$\dfrac{1}{\lambda^3}\, m(m+1)(m+2)$
$\mu_4 = \dfrac{\Gamma(m+4)}{\lambda^4 \,.\, \Gamma(m)}$	$\dfrac{1}{\lambda^4}\, m(m+1)(m+2)(m+3)$
$\mu_5 = \dfrac{\Gamma(m+5)}{\lambda^5 \,.\, \Gamma(m)}$	$\dfrac{1}{\lambda^5}\, m(m+1)(m+2)(m+3)(m+4)$
$\mu_6 = \dfrac{\Gamma(m+6)}{\lambda^6 \,.\, \Gamma(m)}$	$\dfrac{1}{\lambda^6}\, m(m+1)(m+2)(m+3)(m+4)(m+5)$



VIBRATORY PROCESS $Y(t)$

Moments:	for integer m:
$\mu_1 = \mu_3 = \mu_5 = 0$	
$\mu_2 = \bar{\mu}_2 = \dfrac{\Gamma(m+2)}{2\lambda^2 \,.\, \Gamma(m)}$	$\dfrac{m}{2\lambda^2}(m+1)$
$\mu_4 = \bar{\mu}_4 = \dfrac{3\Gamma(m+4)}{8\lambda^4 \,.\, \Gamma(m)}$	$\dfrac{3}{8\lambda^4} \prod_{i=0}^{3} (m+i)$
$\mu_6 = \bar{\mu}_6 = \dfrac{5\Gamma(m+6)}{16\lambda^6 \,.\, \Gamma(m)}$	$\dfrac{5}{16\lambda^6} \prod_{i=0}^{5} (m+i)$

Invariants:

$$I_4 = \frac{3}{2\Gamma^2(m+2)} \left[\Gamma(m+4)\,\Gamma(m) - 2\Gamma^2(m+2)\right]$$

$$I_6 = \frac{5}{2\Gamma^3(m+2)} \left[\Gamma(m+6)\,\Gamma^2(m) - 9\Gamma(m+4) \,.\, \Gamma(m+2) \,.\, \Gamma(m) + 12\Gamma^3(m+2)\right]$$

for integer m:

$$I_4 = \frac{3}{2} \cdot \frac{6 + 3m - m^2}{m + m^2}$$

$$I_6 = \frac{10}{m^2(m+1)^2} \,.\, (m^4 - 4m^3 - 4m^2 + 25m + 30)$$

$$m \to \infty \Rightarrow I_4 = -1.5\,; \quad I_6 = 10$$

It is evident from the evaluated moments that for $\sigma \to 0$ these characteristics turn into the distribution described in § 2.1 with $\mu \to A_0$ as it was expected when defining this distribution. As a characteristic quantity entering the invariants I_4 and I_6, the expression $\varkappa = \mu/\sigma$ is adequate.

4.2. Uniform two-parametric distribution (with a threshold value)

For this distribution, the analytical expression for the probability density f_y was derived in § 3.1. In this place, the moments of this distribution are added to point out the connections with both limiting cases $A_0 \to 0$ (giving one-parametric uniform distribution — § 2.2) and $A_0 \to A_1$ (giving Dirac impulse function — § 2.1). The characteristic quantity entering the invariants may be taken as $\varkappa = A_1/A_0$.

4.3. Rayleigh-Rice distribution

This is one possible generalization of the Rayleigh distribution which is very often used in technical applications. The parameter α characterizes the shift of the mean value $M[A]$ in the direction of the variable a. $I_0(.)$ is the modified Bessel function (of the imaginary argument) of the first kind of zero order, the values of which are given e.g. in [1] or [9].

It may be easily proved that for $\alpha = 0$ the Rayleigh-Rice distribution turns into that of Rayleigh. On the other hand, if $\alpha/\sigma \to \infty$, this distribution approaches the Dirac impulse function with $\alpha \to A_0$.

When deriving the moments of the Rayleigh-Rice distribution, the following relations for the moments with respect to the origin may be found from the integral tables

$$\mu_k = (2\sigma^2)^{\frac{1}{2}k} \,.\, \Gamma\left(1 + \frac{k}{2}\right) . \, {}_1F_1\left(-\frac{k}{2}, \quad 1\,; \quad -\frac{\alpha^2}{(2\sigma^2)}\right),$$

where ${}_1F_1$ is the confluent hypergeometric function which for the first argument with k odd leads to the modified Bessel function I_n and for even k (i.e. for even moments) leads owing to the relation ${}_1F_1(-n, 1; x) = L_n(x)$ (where $L_n(x)$ are Laguerre polynomials) to rational relations between parameters α and σ^2. As a characteristic quantity entering the invariants, the expression α/σ is suitable. Tables of integral distribution functions of the Rayleigh-Rice distribution are given in [2].

4.4. Gamma distribution

The last analytically defined two-parametric distribution of the envelopes given in this paragraph is the gamma distribution with parameters m and λ. The necessary moments may be derived without difficulty. One may also prove that by putting $m = 1$

the exponential distribution is obtained while by putting $m = 2$ the distribution described in § 2.6 is defined. It should be noted that the invariants of the vibratory process distribution f_y related to the gamma distribution of the envelopes contain the parameter m only.

5. GENERALIZED GAMMA DISTRIBUTIONS

In the technical practice, when analyzing some phenomena of statistical character, two parameters of a distribution function are not sufficient in some cases so that three- or even four-parametric distributions are to be taken into account. For problems connected with strength and reliability of machine parts or whole constructions, some generalizations of the gamma-distribution have proved to be very useful. One very important direction of generalization consists in introducing the threshold value as shown in Chapter 3. Another direction of generalization affects the slope and/or the shape of the distribution function, which in the case of the gamma distribution may be realized by introducing a suitable power for the independent variable entering the exponential function. For these two ways of generalization, moments for the distributions f_a and moments and invariants for the distributions f_y will be given.

5.1. Gamma distribution with the threshold value

The probability density function $f_a(a)$ has the form

$$(5.1) \qquad \begin{aligned} f_a(a) &= 0; && a < A_0 , \\ &= \frac{\lambda^m}{\Gamma(m)} . (a - A_0)^{m-1} . \exp\left[-\lambda(a - A_0)\right] ; && a \geqq A_0 , \end{aligned}$$

with parameters λ, m and A_0 the latter being the threshold value. It may be easily shown that the moments of $f_a(a)$ may be expressed in the form

$$(5.2) \qquad \mu_\nu(a) = \sum_{i=0}^{\nu} \binom{\nu}{i} . A_0^{\nu-i} . \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)} , \quad \nu = 1, 2, 3, \ldots$$

For the moments of the distribution $f_y(y)$, the relations

$$(5.3a) \qquad \mu_2(y) \equiv \bar{\mu}_2(y) = \frac{1}{2} . \sum_{i=0}^{2} \binom{2}{i} A_0^{2-i} . \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)} ,$$

$$(5.3b) \qquad \mu_4(y) \equiv \bar{\mu}_4(y) = \frac{3}{8} . \sum_{i=0}^{4} \binom{4}{i} A_0^{4-i} . \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)} ,$$

$$(5.3c) \qquad \mu_6(y) \equiv \bar{\mu}_6(y) = \frac{5}{16} . \sum_{i=0}^{6} \binom{6}{i} A_0^{6-i} . \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)}$$

may be derived according to Eq. (1.12) while for the invariants, the relations

(5.4a)

$$I_4(y) = \frac{3}{2} \cdot \frac{\sum_{i=0}^{4} \binom{4}{i} A_0^{4-i} \cdot \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)}}{\left[\sum_{i=0}^{2} \binom{2}{i} A_0^{2-i} \cdot \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)}\right]^2} - 3$$

(5.4b)

$$I_6(y) = \frac{5}{2} \cdot \frac{\sum_{i=0}^{6} \binom{6}{t} A_0^{6-i} \cdot \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)}}{\left[\sum_{i=0}^{2} \binom{2}{i} A_0^{2-i} \cdot \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)}\right]^3} - \frac{45}{2} \cdot \frac{\sum_{i=0}^{4} \binom{4}{i} A_0^{4-i} \cdot \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)}}{\left[\sum_{i=0}^{4} \binom{2}{i} A_0^{2-i} \cdot \frac{\Gamma(m+i)}{\lambda^i . \Gamma(m)}\right]^2} + 30$$

hold according to Eqs. (2.3).

5.2. Gamma-distribution with the power function of the argument

The probability density function of the envelope $f_a(a)$ will be expressed in the form

$$f_a(a) = \frac{k . \lambda^m}{\Gamma(m/k)} . a^{m-1} . \exp\left[-(\lambda a)^k\right] ; \quad a \geqq 0 \tag{5.5}$$

with parameters λ, m and k. For the moments the expression

$$\mu_\nu(a) = \frac{1}{\lambda^\nu} . \frac{\Gamma[(m+\nu)/k]}{\Gamma(m/k)} , \quad \nu = 1, 2, 3, \ldots \tag{5.6}$$

may be easily found yielding the moments and invariants of the distribution of the vibratory pı ocess in the form

(5.7a) $\mu_2(y) \equiv \bar{\mu}_2(y) = \frac{1}{2} \lambda^{-2} . \Gamma[(m+2)/k]/\Gamma(m/k)$,

(5.7b) $\mu_4(y) \equiv \bar{\mu}_4(y) = \frac{3}{8} \lambda^{-4} . \Gamma[(m+4)/k]/\Gamma(m/k)$,

(5.7c) $\mu_6(y) \equiv \bar{\mu}_6(y) = \frac{5}{16} \lambda^{-6} . \Gamma[(m+6)/k]/\Gamma(m/k)$,

(5.8a) $I_4(y) = \frac{3}{2} . \{\Gamma^2[(m+2)/k]\}^{-1} . \{\Gamma[(m+4)/k] . \Gamma(m/k) - 2\Gamma^2[(m+2)/k]\}$,

(5.8b) $I_6(y) = \frac{5}{2} . \{\Gamma^3[(m+2)/k]\}^{-1} . \{\Gamma[(m+6)/k] . \Gamma^2(m/k) -$

$- 9 . \Gamma[(m+4)/k] . \Gamma[(m+2)/k] . \Gamma(m/k) + 12 . \Gamma^3[(m+2)/k]\}$.

5.3. **Four-parametric gamma distribution**

Assuming both generalizations used in §§ 5.1 and 5.2, the four-parametric gamma-distribution

$$(5.9) \qquad f_a(a) = 0\,; \qquad\qquad a < A_0$$

$$= \frac{k\,.\,\lambda^m}{\Gamma(m/k)}\,.\,(a - A_0)^{m-1}\,.\,\exp\left\{-[\lambda(a - A_0)]^k\right\}\,; \quad a \geqq A_0$$

with parameters λ, m, k and A_0 may be defined.

For the moments of $f_a(a)$ we may derive

$$(5.10) \qquad \mu_\nu(a) = \sum_{i=0}^{\nu} \binom{\nu}{i}\,.\,A_0^{\nu-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}$$

and thus for the moments of the distribution $f_y(y)$, we have

$$(5.11a) \qquad \mu_2(y) \equiv \bar{\mu}_2(y) = \frac{1}{2}\,.\sum_{i=0}^{2}\binom{2}{i} A_0^{2-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}\,,$$

$$(5.11b) \qquad \mu_4(y) \equiv \bar{\mu}_4(y) = \frac{3}{8}\,.\sum_{i=0}^{4}\binom{4}{i} A_0^{4-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}\,,$$

$$(5.11c) \qquad \mu_6(y) \equiv \bar{\mu}_6(y) = \frac{5}{16}\,.\sum_{i=0}^{6}\binom{6}{i} A_0^{6-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}$$

and for its invariants accordingly

$$(5.12a) \qquad I_4(y) = \frac{3}{2}\,.\,\frac{\displaystyle\sum_{i=0}^{4}\binom{4}{i} A_0^{4-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}}{\left[\displaystyle\sum_{i=0}^{2}\binom{2}{i} A_0^{2-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}\right]^2} - 3\,,$$

$$(5.12b) \qquad I_6(y) = \frac{5}{2}\,.\,\frac{\displaystyle\sum_{i=1}^{6}\binom{6}{i} A_0^{6-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}}{\left[\displaystyle\sum_{i=0}^{2}\binom{2}{i} A_0^{2-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}\right]^3} -$$

$$- \frac{45}{2}\,.\,\frac{\displaystyle\sum_{i=0}^{4}\binom{4}{i} A_0^{4-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}}{\left[\displaystyle\sum_{i=0}^{2}\binom{2}{i} A_0^{2-i}\,.\,\frac{\Gamma[(m+i)/k]}{\lambda^i\,.\,\Gamma(m/k)}\right]^2} + 30\,.$$

The four-parameter gamma-distribution involves in the general formulation a very broad class of distribution functions used in the technical practice. For $m = k$ we get the so-called Weibull distribution function which is often used in fatigue and reliability studies.

As special cases some distributions already mentioned are also included. E.g. for $m = 2$, $k = 2$, $A_0 = 0$ and $\lambda = 1/(\sigma\sqrt{2})$, the Rayleigh distribution described in § 2.7 is obtained. Let us consider another special case, i.e. the Rayleigh distribution with nonzero threshold value $A_0 > 0$, which has also practical applications in technical problems, e.g. when analyzing random parametric vibrations. Putting $m = k = 2$ and $\lambda = 1/(\sigma\sqrt{2})$ into Eqs. (5.9) to (5.12), we get

$$\mu_\nu(a) = \sum_{i=0}^{\nu} \binom{\nu}{i} . A_0^{\nu-i} . (\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right), \tag{5.13}$$

$$\mu_2(y) \equiv \bar{\mu}_2(y) = \frac{1}{2} \sum_{i=0}^{2} \binom{2}{i} . A_0^{2-i}(\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right), \tag{5.14a}$$

$$\mu_4(y) \equiv \bar{\mu}_4(y) = \frac{3}{8} \sum_{i=0}^{4} \binom{4}{i} . A_0^{4-i}(\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right), \tag{5.14b}$$

$$\mu_6(y) \equiv \bar{\mu}_6(y) = \frac{5}{16} \sum_{i=0}^{6} \binom{6}{i} . A_0^{6-i}(\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right), \tag{5.14c}$$

$$I_4(y) = \frac{3}{2} . \frac{\sum_{i=0}^{4} \binom{4}{i} . A_0^{4-i}(\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right)}{\left[\sum_{i=0}^{2} \binom{2}{i} . A_0^{2-i}(\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right)\right]^2} - 3\,, \tag{5.15a}$$

$$I_6(y) = \frac{5}{2} . \frac{\sum_{i=0}^{6} \binom{6}{i} . A_0^{6-i}(\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right)}{\left[\sum_{i=0}^{2} \binom{2}{i}\; A_0^{2-i}(\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right)\right]^3} - \tag{5.15b}$$

$$- \frac{45}{2} . \frac{\sum_{i=0}^{4} \binom{4}{i} . A_0^{4-i}(\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right)}{\left[\sum_{i=0}^{2} \binom{2}{i} . A_0^{2-i}(\sigma\sqrt{2})^i . \Gamma\left(1 + \frac{i}{2}\right)\right]^2} + 30\,.$$

6. APPROXIMATION OF THE DISTRIBUTION FUNCTION USING THE GRAM-CHARLIER SERIES

For some purposes, especially for the analytical treatment of more complex problems, the knowledge of the numerical characteristics (moments) of a distribution function is not sufficient and the probability density function is to be expressed at least in an approximate form. In such cases an expansion of the probability density function in a series, the terms of which are normal (Gaussian) function and its derivatives which form an orthogonal set, is very useful.

Assuming the normal random variable $t = [y - \mu_1(y)]/[\sqrt{\mu_2(y)}]$, the best approximation of the distribution function $f(t)$ by means of the set $\varphi(t)$, $\varphi_1(t)$, $\varphi_2(t)$, ...

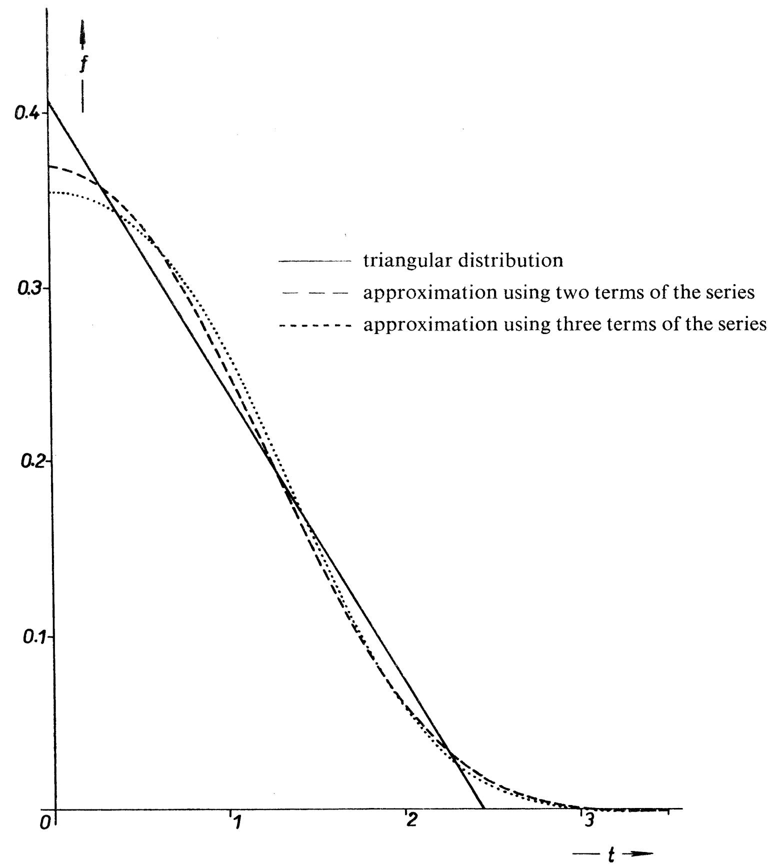


Fig. 2. Approximation of a symmetrical triangular distribution (only the right half is plotted) using the Gram-Charlier series

$\ldots, \varphi_i(t) = \mathrm{d}^i\varphi/\mathrm{d}t^i$ is given by the so-called Gram-Charlier series, (see [10])

$$(6.1) \qquad f_t(t) = \varphi(t) - \frac{1}{6}\mu_3(t) \cdot \varphi_3(t) + \frac{1}{24}[\mu_4(t) - 3] \cdot \varphi_4(t) -$$

$$- \frac{1}{120}[\mu_5(t) - 10\mu_3(t)] \cdot \varphi_5(t) + \frac{1}{720}[\mu_6(t) - 15\mu_4(t) + 30] \cdot \varphi_6(t) - \ldots,$$

where $\varphi(t)$ is the probability density function of the Gaussian distribution with parameters 0, 1. All the moments given in Eq. (6.1) are central ones owing to the introduced normalization.

The distribution functions of the vibratory processes are, as it was already told in the introduction, symmetrical functions so that in Eq. (6.1) only even derivatives occur. Analysing the constants belonging to the individual terms of the Gram-Charlier series, one may state that they are just the invariants introduced in Eqs. (2.3a) and (2.3b).

Hence the Gram-Charlier series if restricted up to the sixth derivatives $\varphi_6(t)$, is given in the form

$$(6.2) \qquad f_t(t) \approx \varphi(t) + \frac{I_4}{24} \cdot \varphi_4(t) + \frac{I_6}{720} \cdot \varphi_6(t) .$$

This expansion contains three independent terms, which are for most cases sufficient for a satisfactory approximation. In Fig. 2, a graphical example of an approximation of a symmetrical triangular distribution is shown. It is evident that a distribution with sharp changes in its shape (e.g. a uniform distribution over a short finite interval) would be approximated rather improperly.

The basic requirement imposed upon every probability density function according to the definition is that it cannot assume negative values. The applicability of Eq. (6.2) is thus restricted by the relation

$$(6.3) \qquad \varphi(t) + \frac{I_4}{24} \cdot \varphi_4(t) + \frac{I_6}{720} \cdot \varphi_6(t) \geqq 0$$

which should be valid over the whole interval $t \in (0, \infty)$. Putting a fixed value $t = t_i$ into Eq. (6.3), we get (for the "equal to" sign) an equation of a line in the variables I_4, I_6 which separates the area of permissible application of the Gram-Charlier approximation from the area of not permissible combinations of I_4 and I_6. Choosing t_i on some discrete levels (e.g. in steps 0.1 over the range (0, 3)), a set of lines may be drawn (see Fig. 3) which mark out the area of permissible combinations of I_4 and I_6, where Eq. (6.3) holds. It is evident that with increasing t the permissible area decreases. It may happen that some not very suitable forms of distribution functions are to be approximated using Gram-Charlier series. One must expect in such cases that the approximation formula will produce negative values for the probability density, as a rule for some higher values of t. In such situations, the negative values are put equal to zero and the distribution is normalized by means of the condition that the integral of the density function in the interval $(-\infty, +\infty)$ is equal to one.

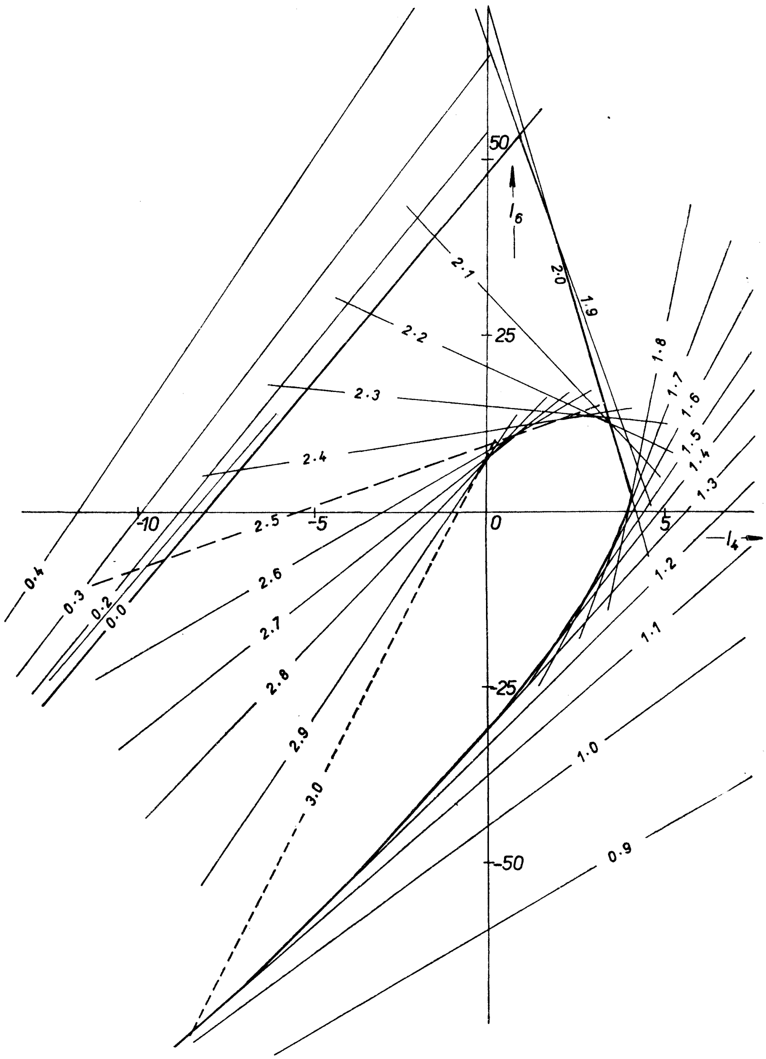


Fig. 3. Areas of permissible combinations for I_4 and I_6 based on the requirement for the Gram-Charlier approximation that $f(t) \geqq 0$

——— $t \leqq 2.0$ — — — $t \leqq 2.5$ ------- $t \leqq 3.0$

The values of the invariants I_4 and I_6 were given in Chapters 4 and 5 so that all quantities in the Gram-Charlier series are fully specified. The reduction from the normalized variable t back to the original variable y will be accomplished with regard to $\mu_1(t) = 0$ by means of the relation

$$f_y(y) = 1/\sqrt{\bar{\mu}_2} \cdot f_t(y/\sqrt{\bar{\mu}_2}) . \tag{6.3}$$

7. GENERALIZATION OF THE DESCRIBED METHOD TO SOME NON-STATIONARY PROCESSES

Under the assumption used in Chapter 1 for the conditional probability density $f_\vartheta(\vartheta \mid a)$ to be equal to $1/(2\pi)$, the probability density f_a may be taken as a time function, i.e. we write $f_a(a, t)$. Comparing with the procedure following Eqs. (1.7) to (1.10), we find that

$$f_y(y, t) = \frac{1}{\pi} \int_y^\infty \frac{f_a(a, t) \, . \, \mathrm{d}a}{\sqrt{(a^2 - y^2)}} . \tag{7.1}$$

Similarly, for the moments the relations

$$\mu_{2k}(y, t) = \frac{(2k)!}{(k!)^2 \, . \, 2^{2k}} \cdot \mu_{2k}(a, t) \tag{7.2}$$

may be developed, their interpretation being evident.

With regard to Eq. (7.1), let us consider $f_a(a, t)$ to be of the form $f_a(a, t) \equiv$ $\equiv f_a(a, \varrho(t), \sigma(t), \tau(t), \ldots)$, where $\varrho(t), \sigma(t), \tau(t), \ldots$ are time dependent parameters of the distribution. Then it is evident that $f_y(y, t)$ is of the form $f_y(y, t) \equiv f_y(y, \varrho(t), \sigma(t), \tau(t), \ldots)$.

Let us consider e.g. the Rayleigh distribution with $\sigma(t)$ being a function of time: $f_a(a, t) = a/[2\sigma^2(t)] \, . \, \exp[-a^2/(2\sigma^2(t))]$. Then the corresponding normal (Gaussian) distribution has the form

$$f_y(y, t) = 1/[\sqrt{(2\pi)} \, . \, \sigma(t)] \, . \, \exp[-y^2/(2\sigma^2(t))] .$$

8. SOME PRACTICAL REMARKS ON EVALUATING EXPERIMENTAL DATA

The ensembles of experimental data are processed, as a rule by means of standard programmes, on a digital computer. As a result of this treatment, the empirical probability density, the empirical distribution function and their first four moments are obtained. The type of the analytical distribution function which approaches best the empirical distribution will be found by plotting the empirical distribution on the probability paper of the expected analytical distribution function. For a treatment of envelopes, the most useful probability paper is that of Rayleigh, in which the Rayleigh distribution function is graphically represented as a straigh line. Other types of distributions are represented also in certain characteristic shapes, so that a first estimate of the analytical form may be drawn from this plot. To facilitate this process in practice, most of the distributions dealt with in this paper are plotted on the Rayleigh paper in Fig. 4.

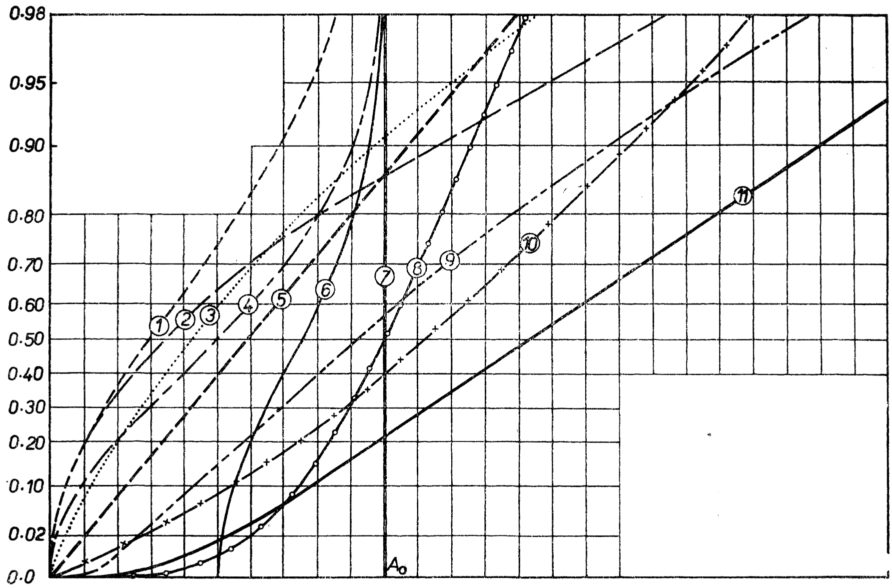


Fig. 4. Typical cumulative distribution functions of envelopes plotted in the Rayleigh probability paper:

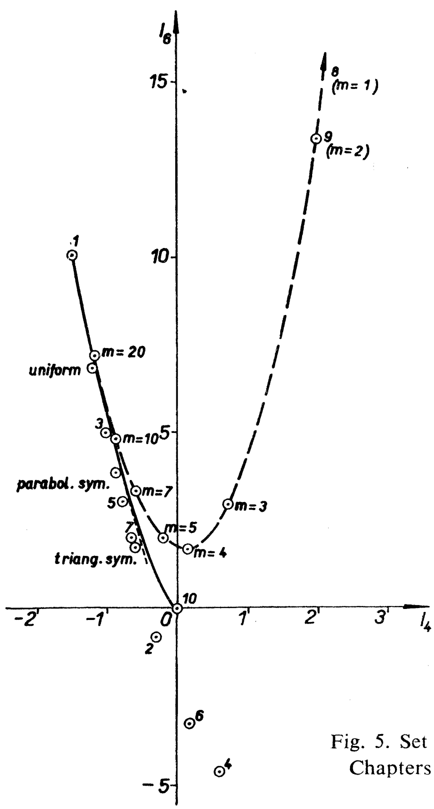
1 — triangular $\langle 0, A_0 \rangle$	5 — Rayleigh	9 — gamma, $m = 4$
2 — exponential	6 — uniform$\langle A_0/2, A_0 \rangle$	10 — Rayleigh-Rice, $\frac{\alpha}{\sigma} = 2$
3 — gamma, $m = 2$	7 — Dirac impulse function	11 — gamma, $m = 6$
4 — uniform $\langle 0, A_0 \rangle$	8 — normal (Gaussian), $\frac{\mu}{\sigma} = 5$	

After the analytical distribution function which is close enough to the empirical distribution has been selected, the estimates of the parameters of the analytical distribution function are calculated from the moments. If the number of unknown parameters is less than the number of evaluated moments, the lower order moments are to be preferred as the basis for the calculation. If the estimates of the same parameter obtained from moments of different order are too different, proof is to be made (e.g. by plotting into the corresponding probability paper) whether the analytical distribution was properly selected or whether it is not necessary to smooth the empirical distribution.

After the suitable expression for the distribution function of the envelope has been found, the analytical form of the corresponding distribution function of the vibratory process may be estimated, either directly, if known, or by means of the Gram-Charlier series.

9. CONCLUSION

The method of analyzing the envelope of a random vibratory process instead of the original process has some advantages both in the analytical and in the experimental research. The knowledge of the relations between the distribution function of the process and its envelope is evidently the basic supposition for a versatile use of this method. Assuming that the distribution function of the envelope is known, a selected set of distributions having expressive importance in technical practice especially in the field of strength and reliability problems is summarized in this paper together with the corresponding characteristics of the distribution functions of the related



1 . . . 10: numbers refer to distributions given in tables 1.1 to 1.10, respectively

——— transformed Rayleigh-Rice distributions

— — — transformed gamma-distributions

- - - - - - - transformed Gaussian distributions

Fig. 5. Set of distributions of vibratory processes treated in Chapters 2 and 4 characterized by their invariants I_4, I_6

vibratory processes. Where the analytical solution of the integral transform was inaccessible, the invariants of the fourth and sixth orders were evaluated allowing an approximate analytical expression by means of Gram-Charlier series. All necessary expressions are given in complete details facilitating the practical use for solving particular problems.

Although a set of quite different one- and two-parametric distributions of the envelopes was selected for the treatment, the corresponding distributions of the vibratory processes have shown some relationship expressed by the fact that their invariants I_4 and I_6 are lying close to the line $8 \, . \, I_4 + I_6 + 2{,}5 = 0$ (see Fig. 5) except the gamma distributions with lower values of the parameter m. Another interesting conclusion is that relatively great differences in envelope distribution functions result in much smaller differences in distribution functions of the corresponding vibratory processes. This effect is especially pronounced for low values of the variance ratio of the envelope distribution. A practical hint follows from this statement that the expression of the envelope distribution, e.g. when estimated experimentally, need not to be too precise. It is advissable better to use less close approximation, which is, however, justified by an easy analytical treatment either of the envelope probability density or of the distribution function of the corresponding vibratory process.

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Souhrn

VZTAHY MEZI ROZDĚLENÍMI NÁHODNÝCH KMITAVÝCH PROCESŮ A ROZDĚLENÍMI JEJICH OBÁLEK

Oldřich Kropáč

Za předpokladu, že kmitavý náhodný proces je úzkopásmový a rozdělení fáze je rovnoměrné na intervalu $(0,2\pi)$, lze najít integrální transformaci mezi rozdělením tohoto kmitavého náhodného procesu a rozdělením jeho obálky, přičemž se vychází z předpokladu, že je známo rozdělení obálky a hledá se rozdělení kmitavého procesu. Příspěvek obsahuje slovník dvojic přidružených rozdělení obálek a procesů nejčastěji používaných v technických aplikacích, a to 10 jednoparametrických, čtyři dvouparametrická, obecné rozdělení s prahovou hodnotou a zobecněná gama-rozdělení. Pro analyticky přístupné tvary rozdělení obálek jsou uvedeny analytické tvary rozdělení procesů, v ostatních případech jsou pro kmitavý proces uvedeny momenty hustoty pravděpodobnosti, které umožňují analytickou aproximaci Gram-Charlierovou řadou.

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