AXIOMATIC THEORY OF INVESTMENT EVALUATING

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One of the basic problems in the theory of investment evaluating is the choice of a criterion which is used to decide whether one investment is better then another. Usualy a untility function is chosen determining the ordering of investments, i.e. making it possible to decide which of two given investments is to be prefered.

The aim of this paper is to find a general form of such utility functions. Therefore we shall proceed in the reverse order: we shall suppose that an ordering of the set of all considered investments is given and we shall ask whether there exists a utility function for this ordering and what is its analytical expression. The answer is in the affirmative: if the ordering satisfies some conditions ("axioms"), then there exists the corresponding utility function. The axioms formally express some of the basic properties of the ordering of investments.

First of all it is necessary to state exactly what we shall understand under the term "investment". We shall limit our considerations to financial investments only, and we shall suppose that such an investment is fully characterized by its cash-flow. In the sequel we shall identify the investment and its cash-flow.

In the discrete case (i.e. the expenditures and revenues occur at discrete times) the appropriate mathematical representation of the cash-flow is an $m + 1$ dimensional vector $(p_0, p_1, \ldots, p_m)$, each component of which represents either revenue (if positive) or expenditure (if negative); the $m + 1$ dimensional vector space of such cash-flows can be called the "investment" space.

The currently used utility function has the general form:

$$u(p) = \sum_{i=0}^{m} p_i \alpha^i \tag{1}$$

where $0 < \alpha < 1$.

In the continuous case the cash-flow in the time interval $\langle 0, T \rangle$ is represented by a continuous function, which is defined on the interval $\langle 0, T \rangle$ and the space of all continuous functions on the interval $\langle 0, T \rangle$ can be taken as the investment space.

The most general case covers both discrete and continuous investments. The cash-flow can be then written in the form

$$p(t) = v(t) + \sum_{i=0}^{m} p_i \, \delta(t - t_i)$$

where δ means the Dirac delta-function. The utility function currently used in this case has the following form:

$$u(p) = \int_0^T p(t) \, e^{-\varrho t} \, \mathrm{d}t \,, \quad \varrho > 0$$

i.e.

$$u(p) = \int_0^T v(t) \, e^{-\varrho t} \, \mathrm{d}t + \sum_{i=0}^{m} p_i e^{-\varrho t_i}$$

(see e.g. [8]).

As we have mentioned above these utility functions are not the starting point for our consideration. On the contrary, we wish to demonstrate that if a given ordering satisfies some axioms, then there exists a number α (or ϱ) so that the utility function of the ordering has the form mentioned above.

In the discrete case the problem has been completely solved by Williams and Nassar [1]. They proposed four axioms and they proved that if the axioms are fulfilled for a given ordering, then there exists a number α so that (1) is the utility function of this ordering.

1. LINEARLY ORDERED TOPOLOGICAL VECTOR SPACES

Let V be a linearly ordered separable topological vector space in which the ordering relation $\lesssim$ satisfies the following requirements:

(i) V is fully ordered by $\lesssim$,

(ii) $x \lesssim y \Rightarrow x + z \lesssim y + z$ for all $z \in V$,

(iii) $x \lesssim y \Rightarrow \lambda x \lesssim \lambda y$ for all $\lambda \in R$ (real numbers), $\lambda > 0$,

(iv) if $x_n \to x$ in V and $x_n \lesssim y$ then $x \lesssim y$.

The relation $x \approx y$ means that both relations $x \lesssim y$ and $y \lesssim x$ are satisfied. $\approx$ is an equivalence and its kernel is closed. Therefore the canonical (algebraic) homomorphism v of the space V on the factor space V/N is continuous and V/N is a separable topological vector space (see [3]). The ordering $\lesssim$ on V induces canonically the ordering on V/N. V/N is a linearly ordered separable topological space and the homomorphism v is continuous and monotone.

Further we shall suppose that the ordering is Archimedean, i.e. we shall suppose that

(v) if $0 \precsim x$ and $nx \precsim y$ for all positive integers n, then $x \approx 0$.

Under this condition the induced ordering on V/N is also Archimedean. As every fully ordered Archimedean group is isomorphic to a subgroup of the additive group of real numbers (see [2] or [4] Chap. XIV, §7, or [5]), there is an isomorphism w of the commutative additive group of the factor space V/N on the subgroup of R. It is easy to see that $w(rx) = r\, w(x)$ holds for all real r.

Theorem 1. *Let V be an Archimedean fully ordered separable topological vector space. Then there exists a linear continuous functional u on V such that for all $x, y \in V$ it holds:* $x \prec y \Leftrightarrow u(x) < u(y)\,, \quad x \approx y \Leftrightarrow u(x) = u(y) \quad (u = w \circ v)\,.$

2. THE INVESTMENTS

We shall suppose use the same notation of the theory of distributions as e.g. in [6] or [7].

Definition. *An investment in the time interval $\langle 0, T\rangle$ is a distribution from $\mathscr{D}'(R)$, the support of which is contained in the interval $\langle 0, T\rangle$. $\mathscr{I}$ denotes the space of all investments.*

If an ordering on $\mathscr{I}$ satisfying the conditions (i)–(v) is given, we shall speak about a *preference* in the investment space. If $x, y \in \mathscr{I}$ and $x \succ y$, we shall say that the investment x is preferd to the investment y, or that x is better than y etc.

If a preference is given on the space $\mathscr{I}$, then according to the previous discussion there is linear continuous functional which is the utility function of that preference. It is possible to extend (linearly and continuously) the functional to the whole space $\mathscr{D}'(R)$ (Hahn-Banach's Theorem). The space $\mathscr{D}(R)$ is reflexive (a relatively simple proof can be found e.g. in [7]). Hence we deduce that there is a function $\varphi \in \mathscr{D}(R)$ such that

$$u(p) = \langle p, \varphi\rangle \quad \text{for all} \quad p \in \mathscr{D}'(R)\,.$$

3. THE AXIOMS

The foregoing discussion is too general and the conditions too weak to give reasonable results for the investment evaluation.

The first axiom joins the „natural" ordering of investments with the preference given on the investment space.

Axiom I. *If $p \in \mathscr{I}$ and $p > 0$, then $p \succ 0$.*

It is of course necessary to add the explication what is the meaning of $p > 0 : p > 0$ means that for all $\varphi \in \mathscr{D}$, $\varphi > 0$ (on $\langle 0, T \rangle$) it is $\langle p, \varphi \rangle > 0$. It is clear that this definition is natural: if the distribution p is a locally integrable function, i.e. if there is a function $p(t)$ such that

$$\langle p, \varphi \rangle = \int_R p(t)\, \varphi(t)\, \mathrm{d}t$$

for all $\varphi \in \mathscr{D}$, then $\langle p, \varphi \rangle > 0$ is equivalent to the fact that $p(t) \geqq 0$ holds for almost all $t \in R$ and $p(t) > 0$ on a set of positive measure. (The proof can be done in a similar way as e.g. in [7] Chapter I, §8, the note after second definition.) As an example of a positive distribution which is not a function we can take the δ-function.

Theorem 2. *If $\precsim$ is a preference in the investment space $\mathscr{I}$ satisfying Axiom I, then there exists a positive (on $\langle 0, T \rangle$) function $\varphi \in \mathscr{D}$ such that for all $p, q \in \mathscr{I}$*

$$p \prec q \Leftrightarrow \langle p, \varphi \rangle < \langle q, \varphi \rangle\,,$$

$$p \approx q \Leftrightarrow \langle p, \varphi \rangle = \langle q, \varphi \rangle\,.$$

Proof. The theorem was actually proved in the previous section. Axiom I guarantees only that φ is positive (in fact Axiom I is equivalent to the positivity of φ). It is sufficient to take the delta function $\delta(t - s) > 0$ and according to Axiom I

$$\langle \delta(t - s), \varphi \rangle > 0\,, \quad \text{i.e.} \quad \varphi(s) > 0 \quad \text{for all} \quad s \in \langle 0, T \rangle\,.$$

The explicit form of φ can be described in this way: We know that if $x \prec 0$ and $0 \prec y$ there is an α, $0 < \alpha < 1$ such that $\alpha x + (1 - \alpha)\, y \approx 0$.
(It is sufficient to take $\alpha = \sup \{\beta : \beta x + (1 - \beta)\, y \precsim 0\}$.) When applied to the investments v_t, w_t defined by

$$v_t(\tau) = \begin{cases} -1 & \tau \in \langle 0, t \rangle \\ 0 & \text{else} \end{cases}$$

$$w_t(\tau) = \delta(\tau - t)$$

for which $v_t < 0$, $w_t > 0$ and so $v_t \prec 0$, $w_t \succ 0$ it yields the existence of the number $\alpha(t)$, $0 < \alpha(t) < 1$ such that $\alpha(t)\, v_t + (1 - \alpha(t))\, w_t \approx 0$. If we denote

$$\beta(t) = \frac{1 - \alpha(t)}{\alpha(t)}$$

we have $v_t + \beta(t)\, w_t \approx 0$, which rewritten in terms of the utility functional, means

$$-\int_0^t \varphi(\tau)\, \mathrm{d}\tau + \beta(t)\, \varphi(t) = 0\,.$$

Hence we have first of all $\beta \in C^\infty$ and differentiating we obtain the differential equation $\beta'\varphi + \beta\varphi' = \varphi$ the solution of which is

$$\varphi(t) = \exp\left(\int_0^t \frac{1 - \beta'(\tau)}{\beta(\tau)}\, d\tau\right).$$

If we denote

$$\varrho(t) = -\int_0^t \frac{1 - \beta'(\tau)}{\beta(\tau)}\, d\tau\,,$$

we have

$$\varphi(t) = e^{-\varrho(t)}\,.$$

(The multiplicative constant is irrelevant for the utility function and so we omit it.) The utility function has then this analytical form: $u(p) = \langle p(t), e^{-\varrho(t)}\rangle$.

If the investment p is a locally integrable function, it is possible to write

$$u(p) = \int_0^T p(t)\, e^{-\varrho(t)}\, dt\,.$$

The function ϱ can be described more precisely if we introduce

Axiom II. *Let v_1, v_2 be the investments defined by the relations*

$$v_1(t) = \begin{cases} 1 & t \in \langle t_1, t_1 + \tau\rangle \\ 0 & \textit{else} \end{cases}$$

$$v_2(t) = \begin{cases} 1 & t \in \langle t_2, t_2 + \tau\rangle \\ 0 & \textit{else} \end{cases}$$

where $t_1, t_2 \in \langle 0, T\rangle$, $\tau > 0$, $t_1 + \tau < t_2$. Then $v_1 \succ v_2$.

Theorem 3. *If Axioms I and II are satisfied for the preference $\precsim$ on the investment space $\mathscr{I}$, then $\varrho(t)$ is an increasing function.*

Proof. If $\varrho(t)$ were not increasing, then there would exist t_1, t_2 such that $t_1 < t_2$ and $\varrho(t_1) \geqq \varrho(t_2)$. From the continuity of ϱ we can deduce that there exists $\tau > 0$ such that $t_1 + \tau < t_2$ and for $t \in \langle t_1, t_1 + \tau\rangle$, $s \in \langle t_2, t_2 + \tau\rangle$ it is $\varrho(t) \geqq \varrho(s)$. Then of course

$$\int_{t_1}^{t_1+\tau} e^{-\varrho(t)}\, dt \leqq \int_{t_2}^{t_2+\tau} e^{-\varrho(t)}\, dt\,,$$

which means that $v_1 \precsim v_2$ and this is a contradiction.

Example. If the investment is discrete, i.e. if it has the form

$$p(t) = \sum_{i=0}^{m} p_i\, \delta(t - t_i)$$

where $t_0 < t_1 < \ldots < t_m$, then the value of the utility function for this investment is

$$u(p) = \sum_{i=0}^{m} p_i e^{-\varrho(t_i)}$$

and if we put

$$\alpha_0 = e^{-\varrho(t_0)}$$
$$\alpha_i = e^{-[\varrho(t_i) - \varrho(t_{i-1})]}, \quad i = 1, 2, \ldots, m,$$

we have

$$u(p) = \alpha_0 p_0 + \alpha_0 \alpha_1 p_1 + \alpha_0 \alpha_1 \alpha_2 p_2 + \ldots + \alpha_0 \alpha_1 \ldots \alpha_m p_m$$

where $0 < \alpha_i < 1$. And this is exactly the form obtained by Williams and Nassar who started from similar axioms.

If we add the third axiom (which is not always satisfied in reality because it expresses the „time uniformity"), we can write the explicit expression for $\varrho(t)$:

Axiom III. (*Time consistency.*) *Let p be such an investment (it is even sufficient to consider only integrable functions), that p is zero in the interval $\langle T - \tau, T\rangle$, where $\tau \in \langle 0, T\rangle$. Let us denote by $\bar{p}(t)$ the shift of p:*

$$\bar{p}(t) = p(t - \tau) \quad t \geqq \tau,$$
$$\bar{p}(t) = 0 \qquad t < \tau$$

If $p \succ 0$ then $\bar{p} \succ 0$.

Theorem 4. *If the preference $\precsim$ is given on the investment space and if it satisfies Axioms I, II and III, then there exists such a positive number ϱ that the utility function of this preference is the following functional:*

$$u(p) = \langle p(t), e^{-\varrho t}\rangle .$$

(*I.e., if Axiom III is fulfilled the function ϱ is linear.*)

Proof. The axiom of consistency means that

$$\int_0^{T-\tau} p(t)\, e^{-\varrho(t)}\, \mathrm{d}t > 0 \Rightarrow \int_\tau^T p(t - \tau)\, e^{-\varrho(t)} \mathrm{d}t = \int_0^{T-\tau} p(t)\, e^{-\varrho(t+\tau)}\, \mathrm{d}t > 0 .$$

Let us define the function of two variables $\delta(t, \tau)$:

$$\delta(t, \tau) = \varrho(t + \tau) - \varrho(t) - \varrho(\tau) .$$

δ is continuous and we shall prove that it is constant:

$$\int_0^{T-\tau} p(t)\, e^{-\varrho(t)}\, \mathrm{d}t > 0 \Rightarrow \int_0^{T-\tau} p(t)\, e^{-\varrho(t)} \,.\, e^{-\delta(t,\tau)}\, \mathrm{d}t > 0\,.$$

$p(t)\, e^{-\varrho(t)}$ is actually an arbitrary integrable function and according to the lemma below, $\delta(t, \tau)$ is constant for fixed τ. We have

$$\varrho(\tau) = \varrho(0 + \tau) = \varrho(\tau) + \varrho(0) + \delta(0, \tau)$$

and so $\delta(0, \tau) = -\varrho(0)$ which means that δ is constant. If this constant value is c, then the function $\sigma(x) = \varrho(x) + c$ satisfies the functional equation

$$\sigma(t + \tau) = \sigma(t) + \sigma(\tau)\,,$$

which implies the existence of a number ϱ such that $\sigma(t) = \varrho t$, i.e. $\varrho(t) = \varrho t - c$. Without any loss of generality we can take $c = 0$.

Lemma. *Let the function φ be positive and continuous in the interval $\langle a, b\rangle$. If the following implication holds for all integrable functions $\varphi(t)$ on the interval $\langle a, b\rangle$:*

$$\int_a^b p(t)\, \mathrm{d}t > 0 \Rightarrow \int_a^b p(t)\, \varphi(t)\, \mathrm{d}t > 0\,,$$

then the function φ is constant.

Proof. Let us suppose that φ is not constant. Then there are certainly two intervals $i_1, i_2 \subseteq \langle a, b\rangle$, $i_1 \cap i_2 = \emptyset$,

$$\mu(i_1) = \mu(i_2) = d > 0 \quad (\mu \text{ is the length})$$

such that

$$I_1 = \int_{i_1} \varphi(t)\, \mathrm{d}t < \int_{i_2} \varphi(t)\, \mathrm{d}t = I_2\,.$$

This follows from the fact that there exists two points $t_1 \neq t_2$ such that $\varphi(t_1) < \varphi(t_2)$, and the continuity of φ implies the existence of two neighborhoods of t_1 and t_2 such that the values of φ in the neighborhood of t_1 are all less than the values of φ in the neighborhood of the point t_2; it is sufficient to reduce both neighborhoods to disjoint intervals of the same length. Since it is $I_2/I_1 > 1$ there is q such that $1 < q < I_2/I_1$. Let us define the function

$$p(t) = \begin{cases} q & t \in i_1 \\ -1 & t \in i_2 \\ 0 & t \notin i_1 \cup i_2 \end{cases}$$

We have

$$\int_a^b p(t)\,\mathrm{d}t = d(q - 1) > 0\,,$$

but

$$\int_a^b p(t)\,\varphi(t)\,\mathrm{d}t = qI_1 - I_2 < 0\,,$$

which contradicts the assumption. Hence φ is constant.

Example. Let us suppose that a preference satisfying Axioms I, II and III is given. There is $\varrho > 0$ such that

$$u(p) = \langle p(t), e^{-\varrho t}\rangle$$

is the corresponding utility function. If p is now an investment which is a linear combination of continuous and discrete investments, i.e. it has the form

$$p(t) = v(t) + \sum_{i=0}^{m} p_i\,\delta(t - iT/m)$$

where $v(t)$ is continuous, then the value of the utility function for this investment is

$$u(p) = \int_0^T v(t)\,e^{-\varrho t}\,\mathrm{d}t + \sum_{i=0}^{m} \alpha^i p_i\,,$$

where we put $\alpha = \exp(-T/m)$. This expression coincides with the expression used in the theory of investment evaluation (see e.g. [8]).

References

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Souhrn

AXIOMATICKÁ TEORIE OCEŇOVÁNÍ INVESTIC

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Jedním ze základních problémů v teorii oceňování investic je volba kriteria, na základě kterého se rozhoduje, zda je nějaká investice lepší než druhá. Obvykle se předpokládá nějaká užitková funkce, která uspořádává investice. Zde postupujeme obráceně: dáno je uspořádání splňující jisté požadavky a k němu se vyhledává odpovídající užitková funkce.

Investice se pokládá za plně určenou odpovídajícím peněžním tokem. Matematickým vyjádřením je distribuce s nosičem v nějakém předem daném časovém intervalu. Prostor všech investic označme $\mathscr{I}$.

Předpokládá se, že na $\mathscr{I}$ je dáno archimedovské úplné uspořádání, které souhlasí s algebraickou i topologickou strukturou $\mathscr{I}$. Takové uspořádání nazýváme preferencí. Formulují se tři axiomy, jejichž obsah lze vyjádřit zhruba takto: I. Je-li nějaká investice kladná (v normálním smyslu), je i lepší než nulová investice. II. Máme-li investici, která je charakteristickou funkcí nějakého intervalu, pak je tato investice lepší než investice, která je charakteristickou funkcí stejně dlouhého intervalu, který je celý napravo od prvního intervalu (druhá investice se „realizuje" později). III. Vznikne-li investice $\bar{p}$ posunutím do prava investice p a je-li p lepší než nula, pak je i $\bar{p}$ lepší než nula (časová stejnoměrnost).

Jsou-li tyto axiomy splněny pro danou preferenci, pak existuje kladné číslo ϱ tak, že užitkovou funkcí této preference je tento funkcionál:

$$u(p) = \langle p(t), e^{-\varrho t} \rangle .$$

Je-li na příklad investice součtem spojité a čistě diskretní investice:

$$p(t) = v(t) + \sum_{i=0}^{m} p_i \delta(t - iT/m) ,$$

pak hodnota užitkové funkce pro tuto investici je vyjádřena v obvyklém tvaru

$$u(p) = \int_0^T v(t)\, e^{-\varrho t}\, \mathrm{d}t + \sum_{i=0}^{m} \alpha^i p_i$$

kde

$$\alpha = \exp\left(-T/m\right) .$$

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