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SOME PROPERTIES OF LINEAR HOMOGENEOUS TRANSFORMATION OF INDEPENDENT VARIABLE IN ORDINARY DIFFERENTIAL LINEAR EQUATIONS

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INTRODUCTION

Let **C** denote the set of all complex numbers.

Let a self-adjoint equation of the 2nd order in the complex domain

$$[\Theta(t)\, y'(t)]' - B(t)\, y(t) = 0 \tag{0.1}$$

be given.

Put $z(t) = y(kt)$, where $k \in \mathbf{C}$. Then $z(t)$ satisfies the self-adjoint equation

$$[\Theta(kt)\, z'(t)]' - k^2 B(kt)\, z(t) = 0\,. \tag{0.2}$$

Let ${}_tLy(t)$ be an arbitrary linear homogeneous differential operator of the 2nd order with the property (for some $m \in \mathbf{C}$)

$${}_tLy(kt) = k^m\, {}_{kt}Ly(kt)\,. \tag{0.3}$$

Let $f(t) \neq 0$ be an arbitrary factor. Put

$${}_t\tilde{L}y(t) = f(t)\, {}_tLy(t)\,. \tag{0.4}$$

Then it holds

$${}_t\tilde{L}y(kt) = \frac{f(t)}{f(kt)}\, k^m\, {}_{kt}\tilde{L}y(kt)\,. \tag{0.5}$$

Hence it follows that the operator ${}_t\tilde{L}y(t)$ fulfils for some $r \in \mathbf{C}$

$${}_t\tilde{L}y(kt) = k^r\, {}_{kt}\tilde{L}y(kt) \tag{0.6}$$

iff (= if and only if) the factor $f(t)$ is a homogeneous function of degree $m - r$, i.e. if

$$f(kt) = k^{m-r} f(t) . \tag{0.7}$$

holds.

Let the self-adjoint operator of the 2nd order ${}_tLy(t) = [\Theta(t)\, y'(t)]' - B(t)\, y(t)$ fulfil (0.3). Let $A(t)$ be a homogeneous function of degree r, i.e. $A(kt) = k^r A(t)$. If $y(t)$ is a solution of the equation

$${}_tLy(t) = -A(t)\, y(t) , \tag{0.8}$$

then $z(t) = y(kt)$ is a solution of the equation

$$[\Theta(t)\, z'(t)]' + [\lambda\, A(t) - B(t)]\, z(t) = 0 , \tag{0.9}$$

where $\lambda = k^{m+r}$.

1. Let us have a real self-adjoint equation in a suitable interval (which will be determined later)

$$[\Theta(t)\, y'(t)]' - B(t)\, y(t) = -A(t)\, y(t) \tag{1.1}$$

where for suitable real numbers m, r it is

(i) $\Theta(kt) = k^{2-m}\, \Theta(t) , \quad \Theta \in C^1 , \quad \Theta(t) \neq 0 ,$

(ii) $B(kt) = k^{-m}\, B(t)$ or $B = 0 , \quad B \in C^0 ,$

(iii) $A(kt) = k^r\, A(t) , \quad A(t) > 0 , \quad A \in C^0 .$

For an arbitrary fixed non-trivial solution y of the equation (1.1) let us put $z(t) = y(kt)$ where k is a real parameter. Then for $\lambda = k^{m+r}$ the function $z(t)$ satisfies the differential equation

$$[\Theta(t)\, z'(t)]' + [\lambda\, A(t) - B(t)]\, z(t) = 0 . \tag{1.2}$$

If $m + r \neq 0$, $k > 0$, then $\lambda > 0$ and by the relation $\lambda = k^{m+r}$ a one-to-one correspondence is given between the values k, λ. Then $\lambda \uparrow \infty$ iff either $m + r > 0$ and $k \uparrow \infty$, or $m + r < 0$ and $k \downarrow 0$. (The symbols $\uparrow$, $\downarrow$ include the monotony of the convergence).

Suppose the equation (1.1), in case of $m + r > 0$, to be oscillatoric (with infinitely many roots) in an annular neighborhood O^*_∞ of the point ∞, while in case of $m + r < 0$ it is supposed to be oscillatoric (with infinitely many roots) in an annular neighborhood O^*_{0+} of the point 0 from the right side.

Let $k_1, k_2, k_3, \ldots$ range monotonically over all positive roots of the solution y of the equation (1.1) in such a way that the corresponding values $\lambda_1, \lambda_2, \lambda_3, \ldots$ are increasing to ∞. Put $z_i(t) = y(k_i t)$, $i = 1, 2, 3, \ldots$

I. The case $m + r > 0$. Let the equation (1.1) be defined in some interval $]a, \infty[$ where $0 \leqq a < k_1$ so that k_1 is still an inner point of the domain of the equation (1.1). Then for $k \geqq k_1$, $z(t) = y(kt)$ is defined in the interval $]b, 1]$ where $(0 \leqq)\ b = (1/k_1)\, a(<1)$, since for $k \geqq k_1$ the transformation $t \to kt$ maps the interval $]b, 1]$ onto the interval $]kb, k] \subseteq]a, \infty[$.

II. The case $m + r < 0$. Let the equation (1.1) be defined in some interval $]0, a[$ where $0 < k_1 < a \leqq \infty$ so that k_1 is still an inner point of the domain of the equation (1.1). Put $(\infty \geqq)\ b = (1/k_1)\, a(>1)$. For $0 < k \leqq k_1$ the transformation $t \to kt$ maps the interval $[1, b[$ onto the interval $[k, kb[\subseteq]0, a[$ and thus the function $z(t) = y(kt)$ is defined in the interval $[1, b[$.

In the case I the function $z_i(t)$, $i = 1, 2, 3, \ldots$ satisfies the differential equation (1.2) with the parameter $\lambda = \lambda_i$ in the interval $]b, 1]$, and thus for $\beta \in]b, 1]$ it holds

$$(1.3) \qquad [\Theta(z_i' z_j - z_i z_j')]_\beta^1 + (\lambda_i - \lambda_j) \int_\beta^1 A z_i z_j \, dt = 0\,, \quad i, j = 1, 2, 3, \ldots$$

Hence it follows that the sequence of functions $z_1, z_2, z_3, \ldots$ is in the interval $]b, 1]$ orthogonal with the weight $A(t)$ iff for any $i \neq j$

$$(1.4) \qquad \lim_{\beta \to b^+} \Theta(\beta)\, [z_i'(\beta)\, z_j(\beta) - z_i(\beta)\, z_j'(\beta)] = 0\,.$$

In the case II the function $z_i(t)$, $i = 1, 2, 3, \ldots$ satisfies the differential equation (1.2) with the parameter $\lambda = \lambda_i$ in the interval $[1, b[$, and thus for $\beta \in [1, b[$ it holds

$$(1.5) \qquad [\Theta(z_i' z_j - z_i z_j')]_1^\beta + (\lambda_i - \lambda_j) \int_1^\beta A z_i z_j \, dt = 0\,, \quad i = 1, 2, 3, \ldots$$

Hence it follows that the sequence of functions $z_1, z_2, z_3, \ldots$ is in the interval $[1, b[$ orthogonal with the weight $A(t)$ iff for any $i \neq j$

$$(1.6) \qquad \lim_{\beta \to b^-} \Theta(\beta)\, [z_i'(\beta)\, z_j(\beta) - z_i(\beta)\, z_j'(\beta)] = 0\,.$$

Example. For the Bessel equation

$$(1.7) \qquad (ty')' - \frac{n^2}{t} y = -ty\,, \quad n \geqq 0 \quad \text{fixed}$$

in the interval $]0, \infty[$ we have $\Theta(t) = t$, $B(t) = n^2/t$, $A(t) = t$, $m = 1$, $r = 1$. In the interval $]0, \infty[$ the solution $J_n(t)$ has infinitely many roots k_i, $i = 1, 2, 3, \ldots$ increasing to ∞ so that the case I occurs. The functions $J_n(k_i t)$, $i = 1, 2, 3, \ldots$ form in the interval $]0, 1]$ an orthogonal sequence with the weight t iff for $i \neq j$

$$(1.8) \qquad \lim_{t \to 0^+} \{t\, [k_i\, J_n'(k_i t)\, J_n(k_j t) - k_j\, J_n(k_i t)\, J_n'(k_j t)]\} = 0\,.$$

According to the formula $J_n'(x) = -J_{n+1}(x) + (n/x) J_n(x)$, the expression $p(t)$ following the limit symbol in (1.8) is reduced to $p(t) = p_1(t) - p_2(t)$ where $p_1(t) = tk_j J_{n+1}(k_j t) J_n(k_i t)$, $p_2(t) = tk_i J_{n+1}(k_i t) J_n(k_j t)$.

Consider that the following two rules hold for the asymptotic equality $\sim$:

$$1^\circ \quad a_i \sim b_i,\ i = 1, 2 \Rightarrow a_1 a_2 \sim b_1 b_2,$$
$$2^\circ \quad a \sim b,\ b \to 0 \quad \Rightarrow a \to 0.$$

From the formula $J_n(x) \sim x^n/(2^n \Gamma(1 + n))$ for $x \to 0$ we have then

$$p_1(t) \sim \frac{k_j}{2^{2n+1} \Gamma(2 + n) \Gamma(1 + n)} t^{2(n+1)},$$

$$p_2(t) \sim \frac{k_i}{2^{2n+1} \Gamma(2 + n) \Gamma(1 + n)} t^{2(n+1)}$$

so that $p_i(t) \to 0$, $i = 1, 2$ holds and thus $p(t) \to 0$ for $t \to 0$ iff $n + 1 > 0$.

2. Let the linear differential operator of the n-th order in the complex domain

(2.1) $$_tLy(t) = \sum_{i=1}^{n} a_i(t)\, y^{(i)}(t)$$

have the following property: After the linear substitution $t \to kt$, $k \in \mathbf{C}$, it fulfils for a suitable $m \in \mathbf{C}$ the relation

(2.2) $$_tLy(kt) = k^m\, {}_{kt}Ly(kt)\,.$$

Let the differential equation

(2.3) $$_tLy(t) = \lambda\, y(t)$$

have a solution $y(t)$ for a constant $\lambda \in \mathbf{C}$. Then the function $y(kt)$ satisfies the equation

(2.4) $$_tLy(kt) = \lambda k^m\, y(kt)\,.$$

Form an equation of the $2n$-th order with the operator $_tL^2 = {}_tL_tL$ and constants $p, q \in \mathbf{C}$

(2.5) $$_tL^2\, y(t) + 2p\, {}_tLy(t) + q\, y(t) = 0\,.$$

Look for its solution in the form $y(kt)$ where $y(t)$ is a solution of (2.3) and k is a suitable constant. We get „a charakteristic" equation for the unknown k

(2.6) $$(\lambda k^m)^2 + 2p(\lambda k^m) + q = 0\,.$$

For any k fulfilling (2.6) and for any $y(t)$ fulfilling (2.3) the function $y(kt)$ then fulfils (2.5).

In case of $p^2 = q$ the equation (2.6) has a double root $k^m = -p/\lambda$. The corresponding differential equation

$$({}_tL + p)^2\, y(t) = 0 \tag{2.7}$$

has a solution $y(t)$ iff $y(t)$ satisfies the equation

$$({}_tL + p)\, y(t) = z(t) \tag{2.8}$$

where $z(t)$ is a suitable solution of the equation

$$({}_tL + p)\, z(t) = 0\,. \tag{2.9}$$

The last mentioned assertions hold generally for any operator $A : M \to M$ on any set M: for $b \in M$ the equation $A^2 y = b$ is equivalent to the equations $Ay = z$, $Az = b$.

3. For arbitrary $n \in \mathbf{C}$ put, in the complex domain,

$$ {}^n_tE\, y(t) = y''(t) + \frac{1}{t}\, y'(t) - \frac{n^2}{t^2}\, y(t)\,. \tag{3.1}$$

Then the Bessel equation of the index n may be written in the form

$$ {}^n_tE\, y(t) = -y(t) \tag{3.2}$$

or

$$({}^n_tE + 1)\, y(t) = 0\,. \tag{3.3}$$

For an arbitrary $k \in \mathbf{C}$ and for an arbitrary solution $y(t)$ of the equation (3.3) the function $z(t) = y(kt)$ is a solution of the equation

$$({}^n_tE + k^2)\, z(t) = 0 \tag{3.4}$$

as the operator ${}^n_tE\, y(t)$ has the property (2.2) for $m = 2$. From this property it also follows that, if $y(t)$ is a solution of the equation

$$({}^n_tE + 1)\, y(t) = f(t) \tag{3.5}$$

where f is an arbitrary continuous function, then for arbitrary $k \in \mathbf{C}$ the function $z(t) = y(kt)$ is a solution of the equation

$$({}^n_tE + k^2)\, z(t) = k^2 f(kt)\,. \tag{3.6}$$

Consider the iterated equation ($p, q \in \mathbf{C}$)

$$({}^n_tE^2 + 2p\, {}^n_tE + q)\, y(t) = 0\,. \tag{3.7}$$

In case of $y(t)$ being a solution of the Bessel equation (3.3), $y(kt)$ is a solution of the equation (3.7) iff

$$k_{1,2}^2 = p \pm \sqrt{(p^2 - q)}\,. \tag{3.8}$$

Combinations of the four values $\pm k_1$, $\pm k_2 \in \mathbf{C}$ and of the two linearly independent solutions $J_n(t)$, $Y_n(t)$ of the equation (3.3) yield eight solutions of the equation (3.7). Since it holds for $m \in \mathbf{Z}$, $n \in \mathbf{C}$ ($\mathbf{Z}$ is the set of all integers)

$$J_n(t\mathrm{e}^{\mathrm{i}m\pi}) = \mathrm{e}^{\mathrm{i}m\pi n}\, J_n(t)\,, \tag{3.9}$$

$$Y_n(t\mathrm{e}^{\mathrm{i}m\pi}) = \mathrm{e}^{-\mathrm{i}m\pi n}\, Y_n(t) + 2\mathrm{i}\,\frac{\sin m\pi n}{\sin n\pi}\cos n\pi\, J_n(t)\,, \tag{3.10}$$

we can cancel the four solutions containing the arguments $-k_1t$, $-k_2t$, because they are linear combinations of the others. The remaining solutions $J_n(k_1t)$, $J_n(k_2t)$, $Y_n(k_1t)$, $Y_n(k_2t)$ are linearly independent iff $k_1 \neq k_2$.

Proof. Take $a\, J_n(k_1t) + b\, J_n(k_2t) + c\, Y_n(k_1t) + d\, Y_n(k_2t) = 0$. Put $y(t) = a\, J_n(k_1t) + c\, Y_n(k_1t) = -b\, J_n(k_2t) - d\, Y_n(k_2t)$. Then $y(t)$ is a solution of the equation (3.4) for $k = k_1$ and $k = k_2$ so that $k_1^2\, y(t) = k_2^2\, y(t)$. Hence in case of $k_1 \neq k_2$ we have $y(t) = 0$ and then $a = c = 0$, $b = d = 0$, Q.E.D.

In case of $k_1 = k_2 = k$, i.e. by $p^2 = q$, we get only two linearly independent solutions $J_n(kt)$, $Y_n(kt)$ of the equation (3.7), which is now of the form

$$({}^{n}_{t}E + p)^2\, y(t) = 0\,. \tag{3.11}$$

Since $k^2 = p$ it is

$$({}^{n}_{t}E + k^2)^2\, y(t) = 0\,. \tag{3.12}$$

Let $a, b \in \mathbf{C}$ be arbitrary fixed constants. Then the function $Z_n(t) = a\, J_n(t) + b\, Y_n(t)$ is called a „general" cylindrical function of the index n. Since it is a fixed linear combination of the functions $J_n(t)$, $Y_n(t)$ with coefficients independent of the index n, the same recurrent relations hold for $Z_n(t)$ as for $J_n(t)$ and $Y_n(t)$, e.g.

$$t\, Z_n'(t) - n\, Z_n(t) = -t\, Z_{n+1}(t)\,. \tag{3.13}$$

The Bessel equation of the index n in the self-adjoint form is

$$(ty')' + \left(t - \frac{n^2}{t}\right) y = 0\,. \tag{3.14}$$

Put $y(t) = t\, Z_{n+1}(t)$, [2]. From the relation (3.13) we get

$$y(t) = n\, Z_n(t) - t\, Z_n'(t)\,. \tag{3.15}$$

Differentiating, multiplying by t and differentiating once more we get

$$(3.16) \qquad (ty')' = n(tZ_n')' + (t^2 - n^2) Z_n' + 2tZ_n ,$$

and once more by (3.13) we find

$$(3.17) \qquad (ty')' + \left(t - \frac{n^2}{t}\right) y = 2tZ_n$$

or

$$(3.18) \qquad ({}_t^nE + 1)\, y(t) = 2Z_n(t) .$$

From the considerations concerning (2.8) it appears that $y(t) = t\, Z_{n+1}(t)$ is a solution of the equation

$$(3.19) \qquad ({}_t^nE + 1)^2\, y(t) = 0 .$$

According to (3.5), (3.6) it follows from (3.18) that the function $z(t) = y(kt) = kt\, Z_{n+1}(kt)$ is a solution of the equation

$$(3.20) \qquad ({}_t^nE + k^2)\, z(t) = 2k^2\, Z_n(kt)$$

so that $z(t) = y(kt) = kt\, Z_{n+1}(kt)$ satisfies the equation (3.12). So we find that the functions $t\, J_{n+1}(kt)$, $t\, Y_{r+1}(kt)$ are again solutions of the equation (3.12). At the same time the solutions $J_n(kt)$, $Y_n(kt)$, $t\, J_{n+1}(kt)$, $t\, Y_{n+1}(kt)$ of the equation (3.12) are linearly independent.

Proof. Consider a linear relation $a\, J_n(kt) + b\, Y_n(kt) + ct\, J_{n+1}(kt) + dt\, Y_{n+1}(kt) = 0$. Put $Z_n(t) = -(c/k)\, J_n(t) - (d/k)\, Y_n(t)$. Then the function $kt\, Z_{n+1}(kt) = -ct\, J_{n+1}(kt) - dt\, Y_{n+1}(kt) = a\, J_n(t) + b\, Y_n(t)$ is a solution of both equations (3.20) and (3.12). Hence it follows that $Z_n = 0$ so that $c = d = 0$ as well as $a = b = 0$; Q.E.D.

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Souhrn

NĚKTERÉ VLASTNOSTI LINEÁRNÍ HOMOGENNÍ TRANSFORMACE NEZÁVISLE PROMĚNNÉ V OBYČEJNÝCH DIFERENCIÁLNÍCH LINEÁRNÍCH ROVNICÍCH

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Odst. 1 obsahuje obecnou lineární diferenciální rovnici 2. řádu (1.1), jejíž řešení $y(t)$ vytváří orthogonální posloupnost $y(k_i t)$, kde k_i je vhodně uspořádaná posloupnost kladných kořenů řešení $y(t)$. Jde v podstatě o „Eulerovské" rovnice.

Odst. 3 obsahuje jisté zobecnění úvah [1] str. 105 a [2] o nalezení obecného řešení rovnice

$$\left(\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt}\right)^2 w - 2b_0\left(\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt}\right) w + w = 0$$

v tom smyslu, že je nalezeno *obecné* řešení rovnice (3.7).

Pozoruhodná věta: *je-li* $Z_n(t) = a\, J_n(t) + b\, Y_n(t)$ *libovolné řešení* (3.3), $k \in \mathbf{C}$ *libovolné, pak*

1° $Z_n(kt)$ *je řešení* (3.4),

2° $kt\, Z_{n+1}(kt)$ *je řešení* (3.20) *a tudíž* (3.12),

je rozšířením úvah [2] o nalezení řešení $t\, Z_{n+1}(t)$ rovnice (3.12) a skýtá důkaz lineární nezávislosti jejích řešení $J_n(kt)$, $Y_n(kt)$, $t\, J_{n+1}(kt)$, $t\, Y_{n+1}(kt)$.

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