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THERMODYNAMICS OF MONOPOLAR CONTINUUM OF GRADE n

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1. STATIC AND DEFORMATION QUANTITIES

In the three-dimensional Euclidean space provided with an orthogonal Cartesian coordinate system let us consider a body whose motion is described by the position $x_i(\tau)$ of its each particle at the time τ:

$$x_i(\tau) = x_i(X_\alpha, \tau) \quad (-\infty < \tau \leqq t)\,. \tag{1}$$

X_α are coordinates of the particle in a certain reference configuration of the body $\varkappa$, which need not be a position assumed by the body during the motion. For a real material, the function of motion (1) must satisfy the condition

$$\det\left|\frac{\partial x_i(\tau)}{\partial X_\alpha}\right| > 0\,.$$

The tensor indices corresponding to the general position of the material point at the time τ will be denoted by i, j, k; those corresponding to the position of the point at the time $\tau = t$ will be denoted by p, r, q and those concerning the configuration $\varkappa$ by α, β, γ.

Suppose that the functoin $\boldsymbol{x}(X_\alpha, \tau)$ has a number of continuous derivatives which is sufficient for further evaluations.

The k-th deformation gradient let be denoted by ${}_kG(\tau)$. Its components are

$${}_kG_{i\alpha_1\ldots\alpha_k}(\tau) = x_{i,\alpha_1\ldots\alpha_k}(\tau) \equiv \frac{\partial^k x_i(\tau)}{\partial X_{\alpha_1} \ldots \partial X_{\alpha_k}}\,, \quad k = 1, 2, \ldots\,.$$

Further denote ${}_0\boldsymbol{G}(\tau) = \boldsymbol{x}(\tau)$, ${}_k\boldsymbol{G} = {}_k\boldsymbol{G}(t)$.

The material derivative will be denoted by a dot over the quantity, e.g. ${}_k\dot{\boldsymbol{G}}(\tau) = = \mathrm{d}\, {}_k\boldsymbol{G}(\tau)/\mathrm{d}\tau \,|_{X\alpha=\mathrm{konst}}$.

In the theory of continuum of the grade n, deformation gradients of higher orders take part in the work of internal and external forces.

The rate of work $V_1(\tau)$ of the external forces ${}_k\boldsymbol{t}(\tau)$ at the point $x_i(\tau)$ which act on the surface of the body, related to the unit of surface area in the configuration $\varkappa$ ($\mathrm{d}s = 1$) may be expressed in the following way:

$$V_1(\tau) = \sum_{k=0}^{n-1} \operatorname{tr} \{{}_k\boldsymbol{t}(\tau), {}_k\dot{\boldsymbol{G}}^T(\tau)\} . \tag{2}$$

In terms of components this equation assumes the form

$$V_1(\tau) = {}_0t_i(\tau)\, \dot{x}_i(\tau) + \sum_{k=1}^{n-1} {}_kt_{i\alpha_1\ldots\alpha_k}(\tau)\, {}_k\dot{G}_{i\alpha_1\ldots\alpha_k}(\tau) .$$

Similarly the rate of work $V_2(\tau)$ of the external body forces ${}_k\boldsymbol{f}(\tau)$ which act at the point $x_i(\tau)$, related to the unit of mass in the configuration $\varkappa$ may be expressed in the form

$$V_2(\tau) = \sum_{k=0}^{n-1} \operatorname{tr} \{{}_k\boldsymbol{f}(\tau), {}_k\dot{\boldsymbol{G}}^T(\tau)\} . \tag{3}$$

If the scalar $V_1(\tau)$ in equation (2) has the dimension of the rate of work of forces and if ${}_k\boldsymbol{G}(\tau)$ is an arbitrary deformation gradient of the k-th order, then equation (2) defines the generalized external forces ${}_k\boldsymbol{t}(\tau)$. Generalized body forces ${}_k\boldsymbol{f}(\tau)$ may be defined analogously on the basis of equation (3).

2. KINETIC ENERGY

Consider a neighbourhood of the material point X_α with the mass M. Divide this neighbourhood into N particles with the masses $m^{(P)}$ and with the material coordinates of the centre of mass $X_\alpha^{(P)}$. The motion of the body given by equation (1) determines also the motion of the centres of mass of the individual particles $x_i^{(P)}(\tau) = x_i(X_\alpha^{(P)}, \tau)$. Denoting the position of the centre of mass of the neighbourhood of the point X_α at the time τ by $x_i(\tau)$, the specific kinetic energy k ($M = 1$) obviously fulfils

$$2\, k(\tau) = \dot{x}_i(\tau)\, \dot{x}_i(\tau) + \sum_{P=1}^{N} m^{(P)}\, \dot{y}_i^{(P)}(\tau)\, \dot{y}_i^{(P)}(\tau) , \tag{4}$$

$$y_i^{(P)}(\tau) = x_i^{(P)}(\tau) - x_i(\tau) , \quad \sum_{P=1}^{N} m^{(P)} = 1 .$$

The relative velocity of the particles $\dot{y}_i^{(P)}(\tau)$ can be evaluated from

$$\dot{y}_i^{(P)}(\tau) \doteq \sum_{k=1}^{l} \frac{1}{k!}\, \dot{x}_{i,\alpha_1\ldots\alpha_k}(X_\alpha, \tau)\, Y_{\alpha_1}^{(P)} \ldots Y_{\alpha_k}^{(P)} ,$$

$$Y_\alpha^{(P)} = X_\alpha^{(P)} - X_\alpha , \quad 1 \leqq l \leqq n - 1 .$$

Equation (4) can be arranged by means of these relations into the form

$$2\,k(\tau) = \dot{x}_i(\tau)\,\dot{x}_i(\tau) + \sum_{j,k=1}^{l} y_{\alpha_1\ldots\alpha_j:\beta_1\ldots\beta_k}\dot{x}_{i,\alpha_1\ldots\alpha_j}(\tau)\,\dot{x}_{i,\beta_1\ldots\beta_k}(\tau)\,,$$

$$y_{\alpha_1\ldots\alpha_j:\beta_1\ldots\beta_k} = \sum_{P=1}^{N} m^{(P)}\,\frac{1}{j!\,k!}\,Y^{(P)}_{\alpha_1}\ldots\,Y^{(P)}_{\alpha_j}\,Y^{(P)}_{\beta_1}\,\ldots\,Y^{(P)}_{\beta_k}\,.$$

Introducing a new quantity ${}_k\varphi_{i\beta_1\ldots\beta_k}(\tau) = \sum_{j=1}^{l} y_{\alpha_1\ldots\alpha_j:\beta_1\ldots\beta_k}\,\ddot{x}_{i,\alpha_1\ldots\alpha_j}(\tau)$, the material derivative of the specific kinetic energy is given by the equation

(5) $$\dot{k}(\tau) = \ddot{x}_i(\tau)\,\dot{x}_i(\tau) + \sum_{k=1}^{l} {}_k\varphi_{i\beta_1\ldots\beta_k}(\tau)\,{}_k\dot{G}_{i\beta_1\ldots\beta_k}(\tau)\,.$$

3. THE FIRST LAW OF THERMODYNAMICS

Denote the kinetic energy of the body by K and its internal energy by E. By the action of the outer medium the body receives the mechanical energy cauesd by external forces and the non-mechanical one caused by the heat flux and the heat supplies. If the mechanical power is denoted by $W$and the non-mechanical power by Q, then. the First Law of Thrermodynamics for the time $\tau = t$ may be written in the form

(6) $$\dot{K} + \dot{E} = W + Q\,,$$

$\dot{K} = \int_v \varrho \dot{k}\,dv$, where $\varrho = \varrho(X_\alpha, t)$ is the mass density,

$\dot{E} = \int_v \varrho \dot{e}\,dv$, where e is the specific internal energy,

$$W = \int_s V_1\,ds + \int_v \varrho V_2\,dv\,,$$

$$Q = -\int_s \boldsymbol{q}\,.\,\boldsymbol{n}\,ds + \int_v \varrho r\,dv\,,$$

where $\boldsymbol{q} = \boldsymbol{q}(X_\alpha, t)$ is the heat flux and $r = r(X_\alpha, \tau)$ the heat supply.

Substituting the above relations into equation (6), we obtain with regard to (5), (2), (3)

(7) $$\int_v \varrho\big({}_0\boldsymbol{f}\,.\,\dot{\boldsymbol{x}} + \sum_{k=1}^{n-1} \operatorname{tr}\{{}_k\boldsymbol{F}, {}_k\dot{\boldsymbol{G}}^T\} + r\big)\,dv +$$
$$+ \int_s \big({}_0\boldsymbol{t}\,.\,\dot{\boldsymbol{x}} + \sum_{k=1}^{n-1} \operatorname{tr}\{{}_k\boldsymbol{t}, {}_k\dot{\boldsymbol{G}}^T\} - \boldsymbol{q}\,.\,\boldsymbol{n}\big)\,ds = \int_v \varrho(\ddot{\boldsymbol{x}}\,.\,\dot{\boldsymbol{x}} + \dot{e})\,dv\,,$$

where

$$_k\boldsymbol{F} = {}_k\boldsymbol{f} - {}_k\boldsymbol{\varphi}\,, \quad k = 1, \dots, l$$

$$_k\boldsymbol{F} = {}_k\boldsymbol{f}\,, \qquad k = l + 1, \dots, n - 1\,.$$

Let us consider a motion which differs from that studied above by an arbitrary constant velocity $\boldsymbol{a}$:

$$\dot{\boldsymbol{x}}^*(\tau) = \dot{\boldsymbol{x}}(\tau) + \boldsymbol{a}\,.$$

Let us suppose that the quantities $_0\boldsymbol{t}$, $_0\boldsymbol{f}$, $_k\boldsymbol{t}$, $_k\boldsymbol{F}$ $(k = 1, 2, \dots, n - 1)$, $\dot{e}$, $\boldsymbol{q}$, r are invariant with respect to the velocity $\boldsymbol{a}$. A comparison of equations (7) written respectively for the motions $\dot{\boldsymbol{x}}(\tau)$ and $\dot{\boldsymbol{x}}^*(\tau)$ together with the relation $_i\dot{\boldsymbol{G}} = {}_i\dot{\boldsymbol{G}}^*$ $(i = 1, 2, \dots, n)$ yield the equation

$$\int_v \varrho\, {}_0\boldsymbol{f}\,\mathrm{d}v + \int_s {}_0\boldsymbol{t}\,\mathrm{d}s = \int_v \varrho\ddot{\boldsymbol{x}}\,\mathrm{d}v\,. \tag{8}$$

If we apply this equation to the elementary coordinate tetrahedron with the normal n_β and if we denote by $_0T_{p\beta}$ the force acting on an element of surface perpendicular to the coordinate axis X_β, we obtain the boundary condition

$$_0T_{p\beta}n_\beta = {}_0t_p\,, \quad p = 1, 2, 3$$

by means of which the equations of balance

$$_0T_{p\beta,\beta} + \varrho\, {}_0f_p = \varrho\ddot{x}_p\,, \quad p = 1, 2, 3 \tag{9}$$

follow from (8).

Using the preceding relations we arrange equation (7) to the form

$$\int_v \Big(\operatorname{tr}\{{}_0\boldsymbol{T}, {}_1\dot{\boldsymbol{G}}^T\} + \varrho\sum_{k=1}^{n-1}\operatorname{tr}\{{}_k\boldsymbol{F}, {}_k\dot{\boldsymbol{G}}^T\} - \operatorname{div}\boldsymbol{q} + \varrho r - \varrho\dot{e}\Big)\,dv = \tag{10}$$

$$= -\int_s \sum_{k=1}^{n-1}\operatorname{tr}\{{}_k\boldsymbol{t}, {}_k\dot{\boldsymbol{G}}^T\}\,\mathrm{d}s\,.$$

If we apply this equation to the elementary coordinate tetrahedron with the normal n_β and if we denote by $_kT_{p\alpha_1\dots\alpha_k\beta}$ the force acting on an element of surface perpendicular to the coordinate axis X_β, we obtain the boundary condition for the generalized forces $_kt_{p\alpha_1\dots\alpha_k}$

$$_kT_{p\alpha_1\dots\alpha_k\beta}n_\beta = {}_kt_{p\alpha_1\dots\alpha_k}\,, \quad k = 1, 2, \dots, n - 1\,. \tag{11}$$

If the notation

$$_kT_{p\alpha_1\dots\alpha_k\beta,\beta} \dots \operatorname{div}{}_k\boldsymbol{T}\,, \tag{12}$$

$$\operatorname{div}{}_k\boldsymbol{T} + \varrho\,{}_k\boldsymbol{F} + {}_{k-1}\boldsymbol{T} = {}_k\boldsymbol{\Pi}\,, \quad k = 1, 2, \dots, n - 1$$

$$_{n-1}\boldsymbol{T} = {}_n\boldsymbol{\Pi}\,,$$

is introduced, then with respect to equation (11) the First Law of Thermodynamics assumes the form

$$\varrho \dot{e} + \operatorname{div} \boldsymbol{q} - \varrho r - \sum_{k=1}^{n} \operatorname{tr} \{{}_k\boldsymbol{\Pi}, {}_k\dot{\boldsymbol{G}}^T\} = 0 . \tag{13}$$

Let us now consider a motion of the body which differs from that studied above by an arbitrary constant angular velocity ${}_0\dot{\boldsymbol{\omega}}$ supposing that the body assumes the original position at the time $\tau = t$. Evidently it holds

$${}_k\dot{\boldsymbol{G}}^* = {}_k\dot{\boldsymbol{G}} + {}_0\dot{\boldsymbol{\omega}} \,.\, {}_k\boldsymbol{G} , \quad k = 1, 2, \ldots, n .$$

Let us further assume that the quantities ${}_kT$, ${}_k\boldsymbol{F}$ $(k = 0, 1, 2, \ldots, n - 1)$, e, $\boldsymbol{q}$, r are invariant with respect to the angular velocity ${}_0\dot{\boldsymbol{\omega}}$. A comparison of equation (13) with the same equation written for the motion changed by ${}_0\dot{\boldsymbol{\omega}}$ yields $\sum_{k=1}^{n} \operatorname{tr} \{{}_k\boldsymbol{\Pi}, ({}_0\dot{\boldsymbol{\omega}} \,.\, {}_k\boldsymbol{G})^T\} = 0$.

Since ${}_0\dot{\boldsymbol{\omega}}$ is an arbitrary skew-symmetric tensor, the preceding equation implies the conditions of the balance of momentum

$$e_{spr}\big({}_{n-1}T_{p\alpha_1\ldots\alpha_n}\, x_{r,\alpha_1\ldots\alpha_n} + \sum_{l=1}^{n-1} \big({}_lT_{p\alpha_1\ldots\alpha_l\beta,\beta} + \varrho\, {}_lF_{p\alpha_1\ldots\alpha_l} + {}_{l-1}T_{p\alpha_1\ldots\alpha_l}\big)\, x_{r,\alpha_1\ldots\alpha_l} = 0 , \tag{14}$$

$$e_{spr} = \tfrac{1}{2}(s - p)(p - r)(r - s) ; \quad s, p, r = 1, 2, 3 .$$

4. THE SECOND LAW OF THERMODYNAMICS

Denote by $\vartheta(X_\alpha, \tau)$ the local temperature which is assumed always to be positive and by $\eta(X_\alpha, \tau)$ the specific entropy. Define the quantities Γ and γ in the following way

$$\Gamma \equiv \frac{d}{dt} \int_v \varrho \eta \, dv - \int_v \varrho \frac{r}{\vartheta} \, dv + \int_s \frac{1}{\vartheta} \boldsymbol{q} \,.\, \boldsymbol{n} \, ds , \tag{15}$$

$$\Gamma \equiv \int_v \varrho \gamma \, dv .$$

Transforming the relations we obtain

$$\gamma = \dot{\eta} - \frac{r}{\vartheta} + \frac{1}{\varrho \vartheta} \operatorname{div} \boldsymbol{q} - \frac{1}{\varrho \vartheta^2} \boldsymbol{q} \cdot {}_1\boldsymbol{g} , \tag{16}$$

where ${}_1g \equiv \vartheta_{,\alpha}$.

The Second Law of Thermodynamics asserts: The inequality $\Gamma \geqq 0$ must hold for any admissible thermodynamic process.

The necessary and sufficient condition that this inequality hold is $\gamma \geqq 0$.

By introducing the free energy $\psi \equiv e - \vartheta\eta$ and by means of the First Law of Thermodynamic (13), equation (16) may be arranged to the form

$$\vartheta\gamma = -\dot{\psi} + \frac{1}{\varrho}\sum_{k=1}^{n} \operatorname{tr}\{{}_k\boldsymbol{\Pi}, {}_k\dot{\boldsymbol{G}}^T\} - \eta\dot{\vartheta} - \frac{1}{\varrho\vartheta}\boldsymbol{q} \cdot {}_1\boldsymbol{g} \geqq 0 . \tag{17}$$

5. CONTINUUM OF GRADE n

Let us suppose in accordance with the principle of determinism and local action [1] that all physical quantities are determined by the deformation and temperature history of the material and that the values of these quantities at a point are affected by the deformation history only of a small neighbourhood of this point. The deformation history of the neighbourhood of the point is determined by the history of changes of the distances of the points $x_i(X_\alpha, \tau)$ from an arbitrary point of the neighbourhood $z_i(Z_\alpha, \tau)$.

This distance $\Delta\boldsymbol{x}(\tau)$ may be approximately expressed by the following formula:

$$\Delta\boldsymbol{x}(\tau) \doteq \sum_{k=1}^{n} \frac{1}{k!}(\boldsymbol{Z} - \boldsymbol{X})^k \cdot {}_k\boldsymbol{G}(\tau) ,$$

Where $\boldsymbol{Z}$, $\boldsymbol{X}$ are the radiusvectors of the points with coordinates Z_α, X_α.

On the basis of the presented hypotheses, there is a functional relation between the physical quantities and the gradients ${}_k\boldsymbol{G}(\tau)$ up to $k = n$ for the continuum of grade n.

Such a continuum is called *monopolar* since all deformation quantities are derived from one function (1).

Now let us have a physical quantity $\boldsymbol{A}(\tau)$; its history $\boldsymbol{A}^t(s)$ is defined by the relation

$$\boldsymbol{A}^t(s) \equiv \boldsymbol{A}(t - s) , \quad \tau = t - s .$$

Further let us define the difference history $\boldsymbol{A}_d^t(s)$ by the equation

$$\boldsymbol{A}_d^t(s) \equiv \boldsymbol{A}^t(s) - \boldsymbol{A}^t(0) .$$

Hence the knowledge of the history $\boldsymbol{A}^t(s)$ is equivalent to the knowledge of the quantities $\boldsymbol{A}_d^t(s)$ and $\boldsymbol{A}^t(0)$.

Let us suppose that the vector of the heat flux $\boldsymbol{q}$ is dependent on m ($m \geqq 1$) gradients of temperature. The principle of equipresence asserts that if one of the constitutive equations depends on a certain physical quantity, then also the other constitutive equations must depend on it, provided this is not in contradiction with the fundamental principles of the mechanics of continuum. Hence it is necessary to consider the dependence of all other physical quantities on the temperature gradients up to the order m.

Consequently, the constitutive equations of a material of grade n may be expressed in the following functional form:

$$(18)\qquad \psi = \mathop{p}_{s=0}^{\infty}\left({}_1\boldsymbol{G}_d^t(s), \ldots, {}_n\boldsymbol{G}_d^t(s), \vartheta_d^t(s);\ {}_1\boldsymbol{G}, \ldots, {}_n\boldsymbol{G}, \vartheta;\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right),$$

$$\eta = \mathop{h}_{s=0}^{\infty}\left({}_1\boldsymbol{G}_d^t(s), \ldots, {}_n\boldsymbol{G}_d^t(s), \vartheta_d^t(s);\ {}_1\boldsymbol{G}, \ldots, {}_n\boldsymbol{G}, \vartheta;\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right),$$

$${}_1\boldsymbol{\Pi} = \mathop{{}_1\pi}_{s=0}^{\infty}\left({}_1\boldsymbol{G}_d^t(s), \ldots, {}_n\boldsymbol{G}_d^t(s), \vartheta_d^t(s);\ {}_1\boldsymbol{G}, \ldots, {}_n\boldsymbol{G}, \vartheta;\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right),$$

$$\vdots$$

$${}_n\boldsymbol{\Pi} = \mathop{{}_n\pi}_{s=0}^{\infty}\left({}_1\boldsymbol{G}_d^t(s), \ldots, {}_n\boldsymbol{G}_d^t(s), \vartheta_d^t(s);\ {}_1\boldsymbol{G}, \ldots, {}_n\boldsymbol{G}, \vartheta;\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right),$$

$$\boldsymbol{q} = \mathop{\boldsymbol{q}}_{s=0}^{\infty}\left({}_1\boldsymbol{G}_d^t(s), \ldots, {}_n\boldsymbol{G}_d^t(s), \vartheta_d^t(s);\ {}_1\boldsymbol{G}, \ldots, {}_n\boldsymbol{G}, \vartheta;\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right),$$

where the components of ${}_k\boldsymbol{g}$ are $\vartheta(X_\alpha, t)_{,\alpha_1\ldots\alpha_k}$.

All physical quantities (dependent as well as independent) together form a thermodynamic process. It will be called admissible if these quantities do not contradict the equations of balance and the First Law of Thermodynamics.

Lemma 1. *A unique thermodynamic process corresponds to any choice of the functions* $\boldsymbol{x}(X_\alpha, \tau)$ *and* $\vartheta(X_\alpha, \tau)$.

Proof. All the gradients considered ${}_k\boldsymbol{G}(\tau)$ and ${}_j\boldsymbol{g}$ may be derived from the given functions, as well as the quantities ψ, η, ${}_k\boldsymbol{\Pi}$, $\boldsymbol{q}$ from equations (18). By a suitable choice of r, ${}_0\boldsymbol{f}$, ${}_k\boldsymbol{F}$ it can be guaranteed that equations (13), (9), (14) hold.

Lemma 2. *Let* $\alpha(\tau)$, ${}_j\boldsymbol{a}(\tau)$, ${}_k\boldsymbol{A}(\tau)$ $(j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$, $|\tau| < \infty$ *be arbitrary functions,* $\boldsymbol{Y}$ *the position of an arbitrary point whose coordinates are* Y_α. *Then there is at least one admissible thermodynamic process such that* $\vartheta(\boldsymbol{X}, \tau)$, ${}_j\boldsymbol{g}(\boldsymbol{X}, \tau)$ *and* ${}_k\boldsymbol{G}(\boldsymbol{X}, \tau)$ $(j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$ *assume the values* $\alpha(\tau)$, ${}_j\boldsymbol{a}(\tau)$, ${}_k\boldsymbol{A}(\tau)$ *for* $\boldsymbol{X} = \boldsymbol{Y}$.

Proof. Choose the functions $\boldsymbol{x}(\tau)$ and $\vartheta(\tau)$ so that

$$\boldsymbol{x}(\tau) = \boldsymbol{y}(\tau) + \sum_{k=1}^{n} \frac{1}{k!}\, {}_k\boldsymbol{A}(\tau)\,(\boldsymbol{X} - \boldsymbol{Y})^k,$$

$$\vartheta(\tau) = \alpha(\tau) + \sum_{j=1}^{m} \frac{1}{j!}\, {}_j\boldsymbol{a}(\tau)\,(\boldsymbol{X} - \boldsymbol{Y})^k.$$

The assertion of Lemma 2 follows from Lemma 1.

Let us now introduce a vector space $\mathscr{A}$ whose elements are ordered sets of tensor and scalar quantities $\mathbf{\Lambda} = ({}_1\boldsymbol{a}, \ldots, {}_n\boldsymbol{a}, b)$.

The following algebraic operations are defined in the vector space:

$$\alpha\mathbf{\Lambda} = (\alpha\, {}_1\boldsymbol{a}, \ldots, \alpha\, {}_n\boldsymbol{a}, \alpha b),$$

$$\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2 = ({}_1\boldsymbol{a}_1 + {}_1\boldsymbol{a}_2, \ldots, {}_n\boldsymbol{a}_1 + {}_n\boldsymbol{a}_2, b_1 + b_2),$$

$$\mathbf{\Lambda}_1 \,.\, \mathbf{\Lambda}_2 = \sum_{k=1}^{n} \operatorname{tr} \{{}_k\boldsymbol{a}_1, {}_k\boldsymbol{a}_2^T\} + b_1 b_2 .$$

The norm of an element is given by the formula $\|\mathbf{\Lambda}\| = \sqrt{(\mathbf{\Lambda} \,.\, \mathbf{\Lambda})}$.

In what follows deformation gradients ${}_k\boldsymbol{G}(\tau)$ are considered instead of the components ${}_k\boldsymbol{a}$, b is the absolute temperature $\vartheta(\tau)$. Denote the difference history of the element $\mathbf{\Lambda}^t(s)$ by

$$\mathbf{\Lambda}_d^t(s) = ({}_1\boldsymbol{G}_d^t(s), \ldots, {}_n\boldsymbol{G}_d^t(s), \vartheta_d^t(s)),$$

$$\mathbf{\Lambda}_d^t(0) = \mathbf{0} .$$

If the quantities ${}_k\boldsymbol{G}^t(s)$, $\vartheta^t(s)$ are constant in time for $\tau \leqq t$, then denote $\mathbf{\Lambda}_d^t(s) = \mathbf{0}^+(s)$.

Lemma 3. *Given $\boldsymbol{X}$, $\mathbf{\Lambda}^t(s)$ and the quantities ${}_j\boldsymbol{g}$, ${}_j\dot{\boldsymbol{g}}$ $(j = 1, 2, \ldots, m)$ may be chosen arbitrarily and there exists at least one admissible thermodynamic process which corresponds to the chosen values.*

Proof. The given vector $\mathbf{\Lambda}^t(s)$ implies the values ${}_k\boldsymbol{G}(\tau)$ $(k = 1, 2, \ldots, n)$, $\vartheta(\tau)$ for $\tau \leqq t$. If the temperature gradients are expressed in the form.

$${}_j\boldsymbol{g}(\tau) = {}_j\boldsymbol{g} + (\tau - t)\, {}_j\dot{\boldsymbol{g}} , \quad j = 1, 2, \ldots, m$$

then for every τ the quantities ${}_k\boldsymbol{G}(\tau)$, $\vartheta(\tau)$, ${}_j\boldsymbol{g}(\tau)$ are determined. According to Lemma 2 there is always a thermodynamic process corresponding to these values.

Let us introduce the vector $\mathbf{\Sigma} \in \mathscr{A}$

$$\varrho\mathbf{\Sigma} = ({}_1\boldsymbol{\Pi}, \ldots, {}_n\boldsymbol{\Pi}, -\varrho\eta),$$

which makes it possible to write the constitutive equations (18) more briefly in the form

$$\psi = \underset{s=0}{\overset{\infty}{p}} (\mathbf{\Lambda}_d^t(s);\; \mathbf{\Lambda}, {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}), \tag{19}$$

$$\mathbf{\Sigma} = \underset{s=0}{\overset{\infty}{\mathscr{S}}} (\mathbf{\Lambda}_d^t(s);\; \mathbf{\Lambda}, {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}),$$

$$\boldsymbol{q} = \underset{s=0}{\overset{\infty}{\boldsymbol{q}}} (\mathbf{\Lambda}_d^t(s);\; \mathbf{\Lambda}, {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}) .$$

Since $\dot{\Lambda}(\tau) = ({}_1\dot{G}(\tau), \ldots, {}_n\dot{G}(\tau), \dot{\vartheta}(\tau))$, the Second Law of Thermodynamics (17) may be expressed in the following form:

$$\vartheta\gamma = -\dot{\psi} + \Sigma \,.\, \dot{\Lambda} - \frac{1}{\varrho\vartheta}\, \boldsymbol{q} \,.\, {}_1\boldsymbol{g} \geqq 0 \,. \tag{20}$$

6. MATERIAL WITH FADING MEMORY

Let us assume the validity of the following principle of the fading memory: Deformations and temperature from the more distant past have less effect on the present physical quantities than those which occurred in the recent past.

The characteristic quantity of the fading is the function $h(s)$ $(0 \leqq s < \infty)$, which expresses the rate of the fading of memory. We assume that this function is continuous, positive, monotone decreasing and that for an arbitrarily small $\delta > 0$ the condition $\lim_{s \to \infty} s^{1/2+\delta}\, h(s) = 0$ is satisfied.

Let us now define a Hilbert space $\mathscr{H}_h$ as a space of functions $\Gamma(s)$ such that $\Gamma(s) \in \mathscr{A}$ for all s; the scalar product being defined as follows:

$$(\Gamma_1(s), \Gamma_2(s))_h = \int_0^\infty \Gamma_1(s) \,.\, \Gamma_2(s)\, h^2(s)\, ds \,.$$

The norm of $\Gamma(s) \in \mathscr{H}_h$ is given by the formula

$$\|\Gamma(s)\|_h = \left(\int_0^\infty \|\Gamma(s)\|^2\, h^2(s)\, ds \right)^{1/2} .$$

Let us assume that all functionals considered have a neighbourhood of zero history $\mathbf{0}^+(s)$ in the definition domain $\mathscr{D}$ of the functions $\Lambda_d^t(s)$ for arbitrary values of the parameters Λ, ${}_j\boldsymbol{g}$ and that they have in the domain continuous Frechet differentials with respect to the norm of the space $\mathscr{H}_h$.

Hence

$$\underset{s=0}{\overset{\infty}{f}}\,(\Lambda_d^t(s) + \Gamma(s);\ \Lambda, {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}) = \underset{s=0}{\overset{\infty}{f}}\,(\Lambda_d^t(s);\ \Lambda, {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}) +$$
$$+ \,\delta \underset{s=0}{\overset{\infty}{f}}\,(\Lambda_d^t(s);\ \Lambda, {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g} \mid \Gamma(s)) + 0(\|\Gamma(s)\|_h)$$

holds for any pair of functions $\Lambda_d^t(s)$ and $\Gamma(s)$ such that $\Lambda_d^t(s)$ as well as $\Lambda_d^t(s) + \Gamma(s)$ are in $\mathscr{D}$, δf being a linear functional with respect to $\Gamma(s)$, continuous in all variables.

Further let us assume that f is differentiable with respect to the variables Λ, ${}_1\boldsymbol{g}, \ldots$ $\ldots, {}_m\boldsymbol{g}$ for any function $\Lambda_d^t(s) \in \mathscr{D}$. This means that there exist functionals $\partial_\Lambda f$ and $\partial_g f$ such that the following relations hold for any Ω from a neighbourhood of the zero of $\mathscr{A}$ and for any $\boldsymbol{v}$ from a neighbourhood of the zero of $\mathscr{V}$:

$$(21) \qquad \underset{s=0}{\overset{\infty}{f}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda} + \boldsymbol{\Omega},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) = \underset{s=0}{\overset{\infty}{f}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) + $$
$$+ \partial_{\boldsymbol{\Lambda}} \underset{s=0}{\overset{\infty}{f}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) . \boldsymbol{\Omega} + 0(\|\boldsymbol{\Omega}\|) ,$$
$$\underset{s=0}{\overset{\infty}{f}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_j\boldsymbol{g} + \boldsymbol{v}, \ldots, {}_m\boldsymbol{g}\right) = \underset{s=0}{\overset{\infty}{f}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) + $$
$$+ \partial_{{}_j\boldsymbol{g}} \underset{s=0}{\overset{\infty}{f}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) . \boldsymbol{v} + 0(\|\boldsymbol{v}\|), \quad j = 1, 2, \ldots, m .$$

Any one of the functionals $p, \boldsymbol{q}, {}_k\pi$ $(k = 1, 2, \ldots, n)$ might be considered in the preceding equations instead of the functional f.

7. MATERIAL DERIVATIVE OF THE FREE ENERGY

The restrictions to which the functionals in the preceding chapter were subjected enable us to express the material derivative of the free energy in the form

$$\dot{\psi} = \frac{d}{dt} \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) = \delta \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m g \mid \dot{\boldsymbol{\Lambda}}_d^{t}(s)\right) + $$
$$+ \partial_{\boldsymbol{\Lambda}} \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) . \dot{\boldsymbol{\Lambda}} + \sum_{j=1}^{m} \partial_{{}_j\boldsymbol{g}} \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) . {}_j\dot{\boldsymbol{g}} .$$

Let us suppose that there exists the derivative $(d/ds)\, \boldsymbol{\Lambda}^t(s)$ and that it is continuous in the space $\mathscr{H}_h$. Then

$$\dot{\boldsymbol{\Lambda}}_d^{t}(s) = -\frac{d}{ds} \boldsymbol{\Lambda}^t(s) - \dot{\boldsymbol{\Lambda}} .$$

As $\dot{\boldsymbol{\Lambda}}$ is constant with respect to the variable s, we can write $\dot{\boldsymbol{\Lambda}} \equiv \dot{\boldsymbol{\Lambda}}^+(s)$.

Define the functional ∇p by the equation

$$(22) \qquad \nabla \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) . \dot{\boldsymbol{\Lambda}} = \delta \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g} \mid \dot{\boldsymbol{\Lambda}}^+(s)\right) .$$

Hence it holds for the material derivative of the free energy:

$$(23) \qquad \dot{\psi} = \left[\partial_{\Lambda} \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) - \nabla \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right)\right] . \dot{\boldsymbol{\Lambda}} - $$
$$- \delta \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g} \,\middle|\, \frac{d}{ds} \boldsymbol{\Lambda}^t(s)\right) + \sum_{j=1}^{m} \partial_{{}_j\boldsymbol{g}} \underset{s=0}{\overset{\infty}{p}} \left(\boldsymbol{\Lambda}_d^{t}(s);\ \boldsymbol{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}\right) . {}_j\dot{\boldsymbol{g}}_k .$$

If the expression (23) for the free energy is substituted into equation (20), then we obtain with regard to (19)

$$(24)\qquad \begin{aligned} \vartheta\gamma = \Big[\overset{\infty}{\underset{s=0}{\mathscr{S}}} (\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ _1\boldsymbol{g},\ \ldots,\ _m\boldsymbol{g}) - \partial_\Lambda \overset{\infty}{\underset{s=0}{p}} (\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ _1\boldsymbol{g},\ \ldots,\ _m\boldsymbol{g}) + \\ + \nabla \overset{\infty}{\underset{s=0}{p}} (\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ _1\boldsymbol{g},\ \ldots,\ _m\boldsymbol{g})\Big] \cdot \dot{\mathbf{\Lambda}} + \delta \overset{\infty}{\underset{s=0}{p}} \left(\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ _1\boldsymbol{g},\ \ldots,\ _m\boldsymbol{g} \,\middle|\, \frac{d}{ds}\mathbf{\Lambda}^t(s) \right) - \\ - \frac{1}{\varrho\vartheta} \overset{\infty}{\underset{s=0}{\boldsymbol{q}}} (\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ _1\boldsymbol{g},\ \ldots,\ _m\boldsymbol{g}) \cdot {_1\boldsymbol{g}} - \sum_{j=1}^{m} \partial_{jg} \overset{\infty}{\underset{s=0}{p}} (\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ _1\boldsymbol{g},\ \ldots,\ _m\boldsymbol{g}) \cdot {_j\dot{\boldsymbol{g}}}\,. \end{aligned}$$

It is evident from this equation that the quantity $\vartheta\gamma$ depends only on the history $\mathbf{\Lambda}^t(s)$ and on the vectors $_j\boldsymbol{g}$ and $_j\dot{\boldsymbol{g}}$ $(j = 1, 2, \ldots, m)$. It follows from Lemma 3 that for fixed $\boldsymbol{X}$ and t these quantities may be chosen arbitrarily and there always exists an admissible thermodynamic process which is in accordance with them. The Second Law of Thermodynamics may be transformed to the form

$$\vartheta\gamma = \theta - \sum_{j=1}^{m} \partial_{jg} p \cdot {_j\dot{\boldsymbol{g}}} \geqq 0\,,$$

where the quantity θ does not depend on $_j\dot{\boldsymbol{g}}$ $(j = 1, 2, \ldots, m)$. In order that this inequality be fulfilled for any values of $_j\dot{\boldsymbol{g}}$, it is necessary that $\partial_{jg} p = 0$, $j = 1, 2, \ldots, m$. Hence the free energy does not depend on the gradients $_j\boldsymbol{g}$.

Consequently, equation (24) assumes the form

$$(25)\qquad \begin{aligned} \vartheta\gamma = \Big[\overset{\infty}{\underset{s=0}{\mathscr{S}}} (\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ _1\boldsymbol{g},\ \ldots,\ _m\boldsymbol{g}) - \partial_\Lambda \overset{\infty}{\underset{s=0}{p}} (\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda}) + \\ + \nabla \overset{\infty}{\underset{s=0}{p}} (\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda})\Big] \cdot \dot{\mathbf{\Lambda}} + \delta \overset{\infty}{\underset{s=0}{p}} \left(\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda} \,\middle|\, \frac{d}{ds}\mathbf{\Lambda}^t(s) \right) - \frac{1}{\varrho\vartheta} \overset{\infty}{\underset{s=0}{\boldsymbol{q}}} (\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ _1\boldsymbol{g},\ \ldots,\ _m\boldsymbol{g}) \cdot {_1\boldsymbol{g}}\,. \end{aligned}$$

Lemma 4. *Let $\mathbf{\Lambda}^t(s)$ be any differentiable history of the particle $\boldsymbol{X}$. Put $\mathbf{\Lambda} = \mathbf{\Lambda}^t(0)$, $\mathbf{\Lambda}_d^t(s) = \mathbf{\Lambda}^t(s) - \mathbf{\Lambda}^+(s)$, $(d/dt)\,\mathbf{\Lambda}^t(s)$, $\dot{\mathbf{\Lambda}} = -(d/ds)\,\mathbf{\Lambda}^t(s)\,|_{s=0}$. Suppose that $\mathbf{\Lambda}_d^t(s)$ and $(d/ds)\,\mathbf{\Lambda}^t(s)$ are in $\mathscr{H}_h$. Then for any $\delta > 0$ there is a history $\hat{\mathbf{\Lambda}}^t(s)$ which is near to $\mathbf{\Lambda}^t(s)$ in the sense that $\hat{\mathbf{\Lambda}} = \hat{\mathbf{\Lambda}}^t(0)$, $\hat{\mathbf{\Lambda}}_d^t(s) = \hat{\mathbf{\Lambda}}^t(s) - \hat{\mathbf{\Lambda}}^+(s)$ and $(d/ds)\,\hat{\mathbf{\Lambda}}^t(s)$ have the following properties*:

$$(26)\qquad \hat{\mathbf{\Lambda}} = \mathbf{\Lambda}\,,$$

$$(27)\qquad \hat{\mathbf{\Lambda}}_d^t(s) - \mathbf{\Lambda}_d^t(s)\|_h < \delta\,,$$

$$(28)\qquad \left\| \frac{d}{ds}\,\hat{\mathbf{\Lambda}}^t(s) - \frac{d}{ds}\,\Lambda^t(s) \right\|_h < \delta\,,$$

while $\dot{\mathbf{\Lambda}} = -(d/ds)\,\hat{\mathbf{\Lambda}}^t(s)\,|_{s=0}$ *is arbitrary, i.e.*

$$\dot{\mathbf{\Lambda}} = \Omega \tag{29}$$

$\mathbf{\Omega}$ *being an arbitrary element of the space* $\mathscr{A}$.

Proof. Let $\mathbf{\Lambda}^t(s)$, $\delta > 0$ and $\mathbf{\Omega}$ be fixed. Choose $\hat{\mathbf{\Lambda}}^t(s)$ in the form

$$\hat{\mathbf{\Lambda}}^t(s) = \mathbf{\Lambda}^t(s) - f(s)\,(\mathbf{\Omega} - \dot{\mathbf{\Lambda}}) \tag{30}$$

$f(s)$ being a smooth scalar function, $f(0) = 0$. Hence equation (26) is satisfied. Denoting $f' = (d/ds)\,f$ we obtain

$$-\frac{d}{ds}\,\hat{\mathbf{\Lambda}}^t(s) = -\frac{d}{ds}\,\mathbf{\Lambda}^t(s) + f'(s)\,(\mathbf{\Omega} - \dot{\mathbf{\Lambda}})\,. \tag{31}$$

If we choose $f'(0) = 1$, then equation (29) is fulfilled. By means of relations (30) and (31) we obtain

$$\|\hat{\mathbf{\Lambda}}_d^t(s) - \mathbf{\Lambda}_d^t(s)\|_h = \|\mathbf{\Omega} - \dot{\mathbf{\Lambda}}\|^2 \int_0^\infty f(s)^2\, h(s)^2\, ds\,,$$

$$\left\|\frac{d}{ds}\,\hat{\mathbf{\Lambda}}_d^t(s) - \frac{d}{ds}\,\mathbf{\Lambda}_d^t(s)\right\|_h = \|\mathbf{\Omega} - \dot{\mathbf{\Lambda}}\|^2 \int_0^\infty f'(s)^2\, h(s)^2\, ds\,.$$

The function $f(s)$ can be chosen so as to satisfy inequalities (27) and (28).

The assumption of continuity of all functionals $\partial_{\mathbf{\Lambda}} p$, ∇p, $\mathscr{S}$, δp and $\boldsymbol{q}$ with respect to $\mathbf{\Lambda}_d^t(s)$ and of δp also with respect to $(d/ds)\,\mathbf{\Lambda}^t(s)$ in the sense of the h-norm enables us with regard to Lemma 4 to rewrite equation (25) in the form

$$\begin{aligned}\vartheta\gamma = \Big[\mathop{\mathscr{S}}_{s=0}^{\infty}(\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g}) - \partial_{\Lambda} \mathop{p}_{s=0}^{\infty}(\mathbf{\Lambda}_d^t(s); \mathbf{\Lambda}) &+ \\ + \nabla \mathop{p}_{s=0}^{\infty}(\mathbf{\Lambda}_d^t(s); \mathbf{\Lambda})\Big]\,.\,\mathbf{\Omega} + \delta \mathop{p}_{s=0}^{\infty}\left(\mathbf{\Lambda}_d^t(s); \mathbf{\Lambda}\,\middle|\,\frac{d}{ds}\,\mathbf{\Lambda}^t(s)\right) &- \\ - \frac{1}{\varrho\vartheta} \mathop{\boldsymbol{q}}_{s=0}^{\infty}(\mathbf{\Lambda}_d^t(s);\ \mathbf{\Lambda},\ {}_1\boldsymbol{g}, \ldots, {}_m\boldsymbol{g})\,.\,{}_1\boldsymbol{g} + 0(\|\Omega\|) &\geqq 0\,.\end{aligned}$$

Since the vector $\Omega \in \mathscr{A}$ is arbitrary and the error $0(\|\mathbf{\Omega}\|)$ may be arbitrarily small, the coefficient at Ω in the preceding inequality must be zero, i.e.

$$\mathbf{\Sigma} = \partial_{\mathbf{\Lambda}} \mathop{p}_{s=0}^{\infty}(\mathbf{\Lambda}_d^t(s); \mathbf{\Lambda}) - \nabla \mathop{p}_{s=0}^{\infty}(\mathbf{\Lambda}_d^t(s); \mathbf{\Lambda})\,. \tag{32}$$

This equation implies that the vector $\mathbf{\Sigma} = ((1/\varrho)\,{}_1\boldsymbol{\Pi}, \ldots, (1/\varrho)\,{}_n\boldsymbol{\Pi}, -\eta)$ as well as the internal energy e is independent of the temperature gradients ${}_j\boldsymbol{g}$ $(j = 1, 2, \ldots, m)$.

It is evident from equations (21) and (22) that the expressions $\partial_\Lambda f$ and ∇p are vectors of the space $\mathscr{A}$. Denote their components by $\partial_{{}_1G} f, \ldots, \partial_{{}_nG} f, \partial_\vartheta f$ and $\nabla_{{}_1G^t{}_d} p, \ldots, \nabla_{{}_nG^t{}_d} p, \nabla_{\vartheta^t{}_d} p$.

Hence the constitutive equations (32) may be specified

$$\text{(33)} \qquad \begin{aligned} \frac{1}{\varrho}\, {}_k\boldsymbol{\Pi} &= \partial_{{}_kG} p - \nabla_{{}_kG^t{}_d} p\,, \quad k = 1, 2, \ldots, n \\ -\eta &= \partial_\vartheta p - \nabla_{\vartheta_d{}^t} p\,. \end{aligned}$$

The right hand sides of the equations may be expressed more briefly by the operators

$$\begin{aligned} D_{{}_kG} p &= \partial_{{}_kG} p - \nabla_{{}_kG_d{}^t} p\,, \\ D_\vartheta p &= \partial_\vartheta p - \nabla_{\vartheta_d{}^t} p\,. \end{aligned}$$

Considering the fact that the left hand sides of equations (33) are defined by relations (12), we can arrange the system of constitutive equations in the following form:

$$\begin{aligned} {}_{n-1}\boldsymbol{T} &= \varrho D_{{}_nG} p\,, \\ {}_{n-2}\boldsymbol{T} &= \varrho D_{{}_{n-1}G} p - \operatorname{div} {}_{n-1}\boldsymbol{T} - \varrho\, {}_{n-1}\boldsymbol{F}\,, \\ \vdots\ &= \varrho D_{{}_{n-1}G} p - \operatorname{div} (\varrho D_{{}_nG} p) - \varrho\, {}_{n-1}\boldsymbol{F}\,, \\ {}_k\boldsymbol{T} &= \varrho D_{{}_{k+1}G} p - \operatorname{div} {}_{k+1}\boldsymbol{T} - \varrho\, {}_{k+1}\boldsymbol{F}\,, \\ &\ \vdots \\ {}_0\boldsymbol{T} &= \varrho D_{{}_1G} p - \operatorname{div} {}_1\boldsymbol{T} - \varrho\, {}_1\boldsymbol{F}\,, \\ \eta &= -D_\vartheta p\,. \end{aligned}$$

If we know the functional of the free energy, it is possible to find from these equations all the stress tensors and the entropy as functionals dependent on the deformation and temperature history of the material.

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Souhrn

TERMODYNAMICKÁ TEORIE MONOPOLÁRNÍHO KONTINUA *n*-TÉHO ŘÁDU S PAMĚTÍ

KAREL BUCHÁČEK

Oproti teorii prostého materiálu se v článku předpokládá, že hodnoty fysikálních veličin v bodě jsou ovlivněny deformační historií *konečného* okolí bodu. V případě monopolárního kontinua n-tého řádu fysikální veličiny závisejí funkcionálně kromě teploty na n deformačních gradientech, vypočtených od *jediné* funkce posunu. Na základě prvého zákona termodynamiky jsou odvozeny rovnice rovnováhy a okrajové podmínky pro všech n tensorů napětí. Zavedením Hilbertova prostoru, v kterém v normě prvku je vyjádřen útlum paměti, lze z druhého zákona termodynamiky odvodit systém konstitutivních rovnic. Tyto rovnice umožňují výpočet entropie a všech tensorů napětí, je-li dána funkcionální závislost svobodné energie na historii n deformačních gradientů a na historii teploty.

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