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THE BAYES APPROACH IN MULTIPLE AUTOREGRESSIVE SERIES

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The Bayes approach is applied to the problem of estimating of autoregressive parameters in multiple autoregressive series. The point estimates coincide with the least squares estimates; the posterior distribution of these parameters is given in a simple form.

1. INTRODUCTION

The problem of estimating of autoregressive parameters is a very important one. The autoregressive models are often used in the theory and applications of time series, but the values of their parameters seldom are exactly known. If the length of the series is sufficient, then the least squares method may be used. This method was considered in the well-known paper [4]. The least squares method may be interpreted as a special Bayesian method in this case. Thus Bayesian look at these estimates makes possible an easy derivation of their statistical properties.

The p-dimensional autoregressive series $\{\mathbf{X}_t\}_{t=1}^N$ (with zero mean values) may be defined by the formula

$$\mathbf{A}_0\mathbf{X}_t + \mathbf{A}_1\mathbf{X}_{t-1} + \ldots + \mathbf{A}_n\mathbf{X}_{t-n} = \xi_t\,, \quad \text{for} \quad n < t \leqq N\,,$$

where $|\mathbf{A}_0| \neq 0$ and $\{\xi_t\}_{t=n+1}^N$ are uncorrelated random vectors with unit covariance matrix, uncorrelated with given random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$. The elements of all the matrices $\mathbf{A}_j$ should be estimated. These matrices are of the order $p \times p$. The present paper deals with this problem under the assumption that vectors $\mathbf{X}_1, \ldots, \mathbf{X}_N$ have simultaneous normal distribution with vanishing mean values and that matrix $\mathbf{A}_0$ is diagonal. Then the Bayes approach suggested in [2] for a one-dimensional autoregressive series may be generalized. This method is valid without the assumption of stationarity. The Bayes approach concerning stationary autoregressive series (see [1], [2]) is not generalized here because of a rather complicated structure of the inverse covariance matrix of multiple autoregressive series.

Let $X_1, Y_1, \ldots, X_n, Y_n$ be random variables with vanishing expectations and finite variances. Let ζ_t, η_t $(n < t \leqq N)$ be orthonormal random variables uncorrelated with $X_1, Y_1, \ldots, X_n, Y_n$. Define X_t, Y_t for $n < t \leqq N$ by

$$(1)\qquad \begin{aligned} &\sum_{j=0}^{n} a_j X_{t-j} + \sum_{j=1}^{n} c_j Y_{t-j} = \zeta_t\,, \\ &\sum_{j=0}^{n} b_j Y_{t-j} + \sum_{j=1}^{n} d_j X_{t-j} = \eta_t\,, \quad n < t \leqq N\,, \end{aligned}$$

where a_j, b_j, c_j, d_j are real numbers, $a_0 > 0$, $b_0 > 0$.

Put $c_0 = d_0 = 0$ and define vectors

$$\boldsymbol{a} = (a_0, \ldots, a_n)'\,, \quad \boldsymbol{b} = (b_0, \ldots, b_n)'\,, \quad \boldsymbol{c} = (c_0, \ldots, c_n)'\,, \quad \boldsymbol{d} = (d_0, \ldots, d_n)'\,.$$

Lemma 1. *If $X_1, Y_1, \ldots, X_N, Y_N$ have simultaneous normal distribution, then the conditional density of $X_{n+1}, Y_{n+1}, \ldots, X_N, Y_n$ given $X_1 = x_1, Y_1 = y_1, \ldots, X_n = x_n$, $Y_n = y_n$ equals to*

$$(2)\qquad \begin{aligned} &p(x_{n+1}, y_{n+1}, \ldots, x_N, y_N \mid x_1, y_1, \ldots, x_n, y_n, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = \\ &= (2\pi)^{-N+n}(a_0 b_0)^{N-n} \exp\{-\tfrac{1}{2} \sum_{t=n+1}^{N} (\sum_{j=0}^{n} a_j x_{t-j} + \sum_{j=1}^{n} c_j y_{t-j})^2 - \\ &\qquad - \tfrac{1}{2} \sum_{t=n+1}^{N} (\sum_{j=0}^{n} b_j y_{t-j} + \sum_{j=1}^{n} d_j x_{t-j})^2\}\,. \end{aligned}$$

Proof. It follows from the assumption of normality that the simultaneous density of random variables $\zeta_{n+1}, \ldots, \zeta_N, \eta_{n+1}, \ldots, \eta_N$ is

$$f(u_{n+1}, \ldots, u_N, v_{n+1}, \ldots, v_N) = (2\pi)^{-N+n} \exp\{-\tfrac{1}{2} \sum_{t=n+1}^{N} u_t^2 - \tfrac{1}{2} \sum_{t=n+1}^{N} v_t^2\}\,.$$

By means of the substitution according to (1) we get the density (2).

Further we introduce following vectors and matrices:

$$\begin{aligned} \boldsymbol{A} &= (-a_1/a_0, \ldots, -a_n/a_0, \; -c_1/a_0, \ldots, -c_n/a_0)'\,, \\ \boldsymbol{B} &= (-b_1/b_0, \ldots, -b_n/b_0, \; -d_1/b_0, \ldots, -d_n/b_0)'\,, \\ \boldsymbol{P}^* &= \|p_{ij}\|_{i,j=0}^{n}\,, \quad \boldsymbol{Q}^* = \|q_{ij}\|_{i,j=0}^{n}\,, \quad \boldsymbol{R}^* = \|r_{ij}\|_{i,j=0}^{n}\,, \end{aligned}$$

where

$$p_{ij} = \sum_{t=n+1}^{N} x_{t-i} x_{t-j}\,, \quad q_{ij} = \sum_{t=n+1}^{N} y_{t-i} y_{t-j}\,, \quad r_{ij} = \sum_{t=n+1}^{N} x_{t-i} y_{t-j}$$
$$\text{for} \quad 0 \leqq i\,, \; j \leqq n\,,$$

$$\mathbf{P} = \|p_{ij}\|_{i,j=1}^{n}, \quad \mathbf{Q} = \|q_{ij}\|_{i,j=1}^{n}, \quad \mathbf{R} = \|r_{ij}\|_{i,j=1}^{n},$$

$$\mathbf{p}_0 = (p_{01}, \ldots, p_{0n})', \quad \mathbf{q}_0 = (q_{01}, \ldots, q_{0n})'.$$

$$\mathbf{r}_0 = (r_{01}, \ldots, r_{0n})', \quad \mathbf{r}_1 = (r_{10}, \ldots, r_{n0})',$$

$$\mathbf{N}_1 = \begin{Vmatrix} \mathbf{P} & \mathbf{R} \\ \mathbf{R}' & \mathbf{Q} \end{Vmatrix}, \quad \mathbf{K}_1 = \begin{Vmatrix} \mathbf{p}_0 & \mathbf{r}_1 \\ \mathbf{r}_0 & \mathbf{q}_0 \end{Vmatrix}, \quad \mathbf{M}_1 = \begin{Vmatrix} p_{00} & r_{00} \\ r_{00} & q_{00} \end{Vmatrix},$$

$$\mathbf{N}_2 = \begin{Vmatrix} \mathbf{Q} & \mathbf{R}' \\ \mathbf{R} & \mathbf{P} \end{Vmatrix}, \quad \mathbf{K}_2 = \begin{Vmatrix} \mathbf{q}_0 & \mathbf{r}_0 \\ \mathbf{r}_1 & \mathbf{p}_0 \end{Vmatrix}, \quad \mathbf{M}_2 = \begin{Vmatrix} q_{00} & r_{00} \\ r_{00} & p_{00} \end{Vmatrix},$$

$$\mathbf{W} = \begin{Vmatrix} \mathbf{P}^* & \mathbf{R}^* \\ \mathbf{R}^{*\prime} & \mathbf{Q}^* \end{Vmatrix}, \quad \mathbf{v} = (-1, 0)'.$$

Theorem 2. *Let the prior probability density of $a_0, b_0, \mathbf{A}, \mathbf{B}$ be proportional to $(a_0 b_0)^{-1}$ for $a_0 > 0$, $b_0 > 0$ and to zero in the other cases independently on $x_1, y_1, \ldots, x_n, y_n$. Let matrix $\mathbf{W}$ be positive definite. If the density p is given by formula (2) then the posterior density of $a_0, b_0, \mathbf{A}, \mathbf{B}$ is*

$$(3) \qquad \pi(a_0, b_0, \mathbf{A}, \mathbf{B} \mid \mathbf{x}, \mathbf{y}) = c(a_0 b_0)^{N-n-1} \exp\{-\tfrac{1}{2}a_0^2[(\mathbf{A} - {}^\circ\mathbf{A})' \mathbf{N}_1(\mathbf{A} - {}^\circ\mathbf{A}) + p_{00} - {}^\circ\mathbf{A}'\mathbf{N}_1{}^\circ\mathbf{A}]\} \exp\{-\tfrac{1}{2}b_0^2[(\mathbf{B} - {}^\circ\mathbf{B})' \mathbf{N}_2(\mathbf{B} - {}^\circ\mathbf{B}) + q_{00} - {}^\circ\mathbf{B}'\mathbf{N}_2{}^\circ\mathbf{B}]\}$$

$$\text{for} \quad a_0 > 0, \quad b_0 > 0,$$

where

$${}^\circ\mathbf{A} = -\mathbf{N}_1^{-1}\mathbf{K}_1\mathbf{v}, \quad {}^\circ\mathbf{B} = -\mathbf{N}_2^{-1}\mathbf{K}_2\mathbf{v},$$

$\mathbf{x}, \mathbf{y}$ is written instead of $x_1, y_1, \ldots, x_N, y_N$ and c may depend on $\mathbf{x}, \mathbf{y}$, but not on $a_0, b_0, \mathbf{A}, \mathbf{B}$. If $N > n + 1$, then the modus of posterior density π is $\tilde{a}_0, \tilde{b}_0, {}^\circ\mathbf{A}, {}^\circ\mathbf{B}$, where

$$\tilde{a}_0^{-2} = \frac{1}{N - n - 1}(p_{00} - {}^\circ\mathbf{A}'\mathbf{N}_1{}^\circ\mathbf{A}), \quad \tilde{b}_0^{-2} = \frac{1}{N - n - 1}(q_{00} - {}^\circ\mathbf{B}'\mathbf{N}_2{}^\circ\mathbf{B}).$$

The estimates ${}^\circ\mathbf{A}, {}^\circ\mathbf{B}$ coincide with the least squares estimates of autoregressive parameters.

Proof. We have

$$\sum_{t=n+1}^{N} \Big(\sum_{j=0}^{n} a_j x_{t-j} + \sum_{j=1}^{n} c_j y_{t-j}\Big)^2 = \mathbf{a}'\mathbf{P}^*\mathbf{a} + \mathbf{c}'\mathbf{Q}^*\mathbf{c} + 2\mathbf{a}'\mathbf{R}^*\mathbf{c} =$$

$$= (\mathbf{a}', \mathbf{c}') \begin{Vmatrix} \mathbf{P}^* & \mathbf{R}^* \\ \mathbf{R}^{*\prime} & \mathbf{Q}^* \end{Vmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix} = (\mathbf{a}', \mathbf{c}')\, \mathbf{W} \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix}.$$

Put $\boldsymbol{\alpha} = (-a_1/a_0, \ldots, -a_n/a_0)'$, $\gamma = (-c_1/c_0, \ldots, -c_n/c_0)'$,

$$\boldsymbol{T} = \begin{Vmatrix} 1 & 0 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ 0 & 1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{Vmatrix},$$

where $\boldsymbol{I}$ is the unit matrix of the order $n \times n$. Thus

$$(\boldsymbol{a}', \boldsymbol{c}')\, \boldsymbol{W}(\boldsymbol{a}', \boldsymbol{c}')' = a_0^2(-1, \boldsymbol{\alpha}', 0, \gamma')\, \boldsymbol{TT}' \begin{Vmatrix} p_{00} & \boldsymbol{p}_0' & r_{00} & \boldsymbol{r}_0' \\ \boldsymbol{p}_0 & \boldsymbol{P} & \boldsymbol{r}_1 & \boldsymbol{R} \\ r_{00} & \boldsymbol{r}_1' & q_{00} & \boldsymbol{q}_0' \\ \boldsymbol{r}_0 & \boldsymbol{R}' & \boldsymbol{q}_0 & \boldsymbol{Q} \end{Vmatrix} \boldsymbol{TT}' \begin{pmatrix} -1 \\ \boldsymbol{\alpha} \\ 0 \\ \gamma \end{pmatrix} =$$

$$= a_0^2(\boldsymbol{v}', \boldsymbol{A}') \begin{Vmatrix} \boldsymbol{M}_1 & \boldsymbol{K}_1' \\ \boldsymbol{K}_1 & \boldsymbol{N}_1 \end{Vmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{A} \end{pmatrix} =$$

$$= a_0^2[(\boldsymbol{A} + \boldsymbol{N}_1^{-1}\boldsymbol{K}_1\boldsymbol{v})'\, \boldsymbol{N}_1(\boldsymbol{A} + \boldsymbol{N}^{-1}\boldsymbol{K}_1\boldsymbol{v}) + \boldsymbol{v}'\boldsymbol{M}_1\boldsymbol{v} - \boldsymbol{v}'\boldsymbol{K}_1'\boldsymbol{N}_1^{-1}\boldsymbol{K}_1\boldsymbol{v}] =$$

$$= a_0^2[(\boldsymbol{A} - {}^\circ\boldsymbol{A})'\, \boldsymbol{N}_1(\boldsymbol{A} - {}^\circ\boldsymbol{A}) + p_{00} - {}^\circ\boldsymbol{A}'\boldsymbol{N}_1{}^\circ\boldsymbol{A}]\,.$$

The sum

$$\sum_{t=n+1}^{N} \left(\sum_{j=0}^{n} b_j y_{t-j} + \sum_{j=1}^{n} d_j x_{t-j}\right)^2$$

is analogous. Now, applying the Bayes theorem we get density π. Its modus may be easily derived and we get the values $\tilde{a}_0$, $\tilde{b}_0$, ${}^\circ\boldsymbol{A}$, ${}^\circ\boldsymbol{B}$. Matrix $\boldsymbol{W}$ is positive definite and therefore $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ are positive definite, too. Obviously ${}^\circ\boldsymbol{A}$ and ${}^\circ\boldsymbol{B}$ are the least squares estimates. Further we see that $p_{00} - {}^\circ\boldsymbol{A}'\boldsymbol{N}_1{}^\circ\boldsymbol{A} > 0$, $q_{00} - {}^\circ\boldsymbol{B}'\boldsymbol{N}_2{}^\circ\boldsymbol{B} > 0$.

Theorem 3. *Let $\boldsymbol{W}$ be positive definite. If $N > 3n$ and if density π is given by (3), then the following assertions hold for the posterior distributions:*

(I) *The marginal posterior density of $\boldsymbol{A}$ and $\boldsymbol{B}$ is*

$$\pi_1(\boldsymbol{A}, \boldsymbol{B} \mid \boldsymbol{x}, \boldsymbol{y}) = c\left[1 + \frac{(\boldsymbol{A} - {}^\circ\boldsymbol{A})'\, \boldsymbol{N}_1(\boldsymbol{A} - {}^\circ\boldsymbol{A})}{p_{00} - {}^\circ\boldsymbol{A}'\boldsymbol{N}_1{}^\circ\boldsymbol{A}}\right]^{-(N-n)/2} \left[1 + \right.$$
$$\left. + \frac{(\boldsymbol{B} - {}^\circ\boldsymbol{B})'\, \boldsymbol{N}_2(\boldsymbol{B} - {}^\circ\boldsymbol{B})}{q_{00} - {}^\circ\boldsymbol{B}'\boldsymbol{N}_2{}^\circ\boldsymbol{B}}\right]^{-(N-n)/2}.$$

(II) *The marginal posterior density of a_0 and b_0 is*

$$\pi_2(a_0, b_0 \mid \boldsymbol{x}, \boldsymbol{y}) = c a_0^{N-3n-1} \exp\{-\tfrac{1}{2}a_0^2(p_{00} - {}^\circ\boldsymbol{A}'\boldsymbol{N}_1{}^\circ\boldsymbol{A})\}\, b_0^{N-3n-1}$$
$$\exp\{-\tfrac{1}{2}b_0^2(q_{00} - {}^\circ\boldsymbol{B}'\boldsymbol{N}_2{}^\circ\boldsymbol{B})\}\,.$$

Thus random variables $a_0^2(p_{00} - {}^\circ\boldsymbol{A}'\boldsymbol{N}_1{}^\circ\boldsymbol{A})$ and $b_0^2(q_{00} - {}^\circ\boldsymbol{B}'\boldsymbol{N}_2{}^\circ\boldsymbol{B})$ are independent and both of them have χ^2-distribution with $N - 3n$ degrees of freedom.

(III) *Random variables*

$$F_1 = \frac{N-3n}{2n} \frac{(\mathbf{A} - {}^\circ\mathbf{A})' \mathbf{N}_1 (\mathbf{A} - {}^\circ\mathbf{A})}{p_{00} - {}^\circ\mathbf{A}' \mathbf{N}_1 {}^\circ\mathbf{A}}, \quad F_2 = \frac{N-3n}{2n} \frac{(\mathbf{B} - {}^\circ\mathbf{B})' \mathbf{N}_2 (\mathbf{B} - {}^\circ\mathbf{B})}{q_{00} - {}^\circ\mathbf{B}' \mathbf{N}_2 {}^\circ\mathbf{B}}$$

are independent and both of them have F-distribution with $2n$ *and* $N - 3n$ *degrees of freedom.*

(IV) *Denote by* $\mathbf{N}_{1(i,i)}$ *and* $\mathbf{N}_{2(i,i)}$ *the matrices obtained from* $\mathbf{N}_1$ *and* $\mathbf{N}_2$, *respectively, when i-th column and i-th row are omitted* $(1 \leqq i \leqq 2n)$. *Then each of the variables*

$$t_i = (A_i - {}^\circ A_i) \left[\frac{|\mathbf{N}_1| (N - 3n)}{|\mathbf{N}_{1(i,i)}| (p_{00} - {}^\circ\mathbf{A}' \mathbf{N}_1 {}^\circ\mathbf{A})} \right]^{1/2}, \quad (1 \leqq i \leqq 2n),$$

and

$$t_i' = (B_i - {}^\circ B_i) \left[\frac{|\mathbf{N}_2| (N - 3n)}{|\mathbf{N}_{2(i,i)}| (q_{00} - {}^\circ\mathbf{B}' \mathbf{N}_2 {}^\circ\mathbf{B})} \right]^{1/2}, \quad (1 \leqq i \leqq 2n)$$

has Student distribution with $N - 3n$ *degrees of freedom, where* A_i, ${}^\circ A_i$, B_i, ${}^\circ B_i$ *are the i-th components of vectors* $\mathbf{A}$, ${}^\circ\mathbf{A}$, $\mathbf{B}$, ${}^\circ\mathbf{B}$, *respectively. The variable* t_i *is independent on* t_j' *for* $1 \leqq i, j \leqq 2n$.

Proof is analogous to that given in [2] for the one-dimensional stationary autoregressive series, and therefore it will be sketched only. Also see [3].

(I) We get the density π_1 by integrating π on the set $a_0 > 0$, $b_0 > 0$.

(II) The density π_2 follows from π by integration.

(III) Introduce vectors

$$\mathbf{u}_1 = (p_{00} - {}^\circ\mathbf{A}' \mathbf{N}_1 {}^\circ\mathbf{A})^{-1/2} \mathbf{N}_1^{1/2} (\mathbf{A} - {}^\circ\mathbf{A}),$$
$$\mathbf{u}_2 = (q_{00} - {}^\circ\mathbf{B}' \mathbf{N}_2 {}^\circ\mathbf{B})^{-1/2} \mathbf{N}_2^{1/2} (\mathbf{B} - {}^\circ\mathbf{B}).$$

Their simultaneous density derived from π_1 contains the sum of squares of components $\mathbf{u}_1$ and $\mathbf{u}_2$. Further introduce polar coordinates and compute marginal distributions.

(IV) As for the variable t_1, introduce vector $\mathbf{u} = (u_1, \ldots, u_{2n})' = \mathbf{G}(\mathbf{A} - {}^\circ\mathbf{A})$, where $\mathbf{G} = \|g_{ij}\|_{i,j=1}^{2n}$ is a matrix such that $\mathbf{G}'\mathbf{G} = \mathbf{N}_1$, $g_{ij} = 0$ for $1 \leqq i < j \leqq 2n$. Integrating we obtain the marginal distribution of u_1 and further transformation leads to the density of t_1. The distributions of $t_2, \ldots, t_{2n}$ and $t_1', \ldots, t_{2n}'$ may be reduced to this case.

Theorem 2 gives point estimates of autoregressive parameters which are the same as least squares estimates. Theorem 3 enables us to construct confidence regions and to test the hypotheses about these parameters. The usual confidence approach uses the limiting normal approximation only (see [4]).

3. THE p-DIMENSIONAL AUTOREGRESSIVE SERIES

Let $\{X_t^k\}$, $1 \leqq t \leqq n$; $1 \leqq k \leqq p$ be random variables with vanishing mean values and finite variances. Let $\{\xi_t^k\}$, $n < t \leqq N$, $1 \leqq k \leqq p$, be a system of orthonormal random variables such that $\operatorname{cov}(X_t^k, \xi_s^j) = 0$ for $1 \leqq j, k \leqq p$; $1 \leqq t \leqq n$; $n < s \leqq N$. Define random variables $\{X_t^k\}$, $1 \leqq k \leqq p$; $n < t \leqq N$, by

$$(4) \qquad \sum_{k=1}^{p} \sum_{j=0}^{n} a_{ikj} X_{t-j}^k = \xi_t^i \,, \quad 1 \leqq i \leqq p \,; \quad n < t \leqq N \,,$$

where a_{ikj} are real numbers such that $a_{ii0} > 0$, $a_{ik0} = 0$ for $1 \leqq i \neq k \leqq p$.

Introduce vectors $\boldsymbol{a}_{ik} = (a_{ik0}, \ldots, a_{ikn})'$, $1 \leqq i$, $k \leqq p$; $\boldsymbol{a}_i = (\boldsymbol{a}'_{i1}, \ldots, \boldsymbol{a}'_{ip})'$, $1 \leqq i \leqq p$, $\boldsymbol{X}_t = (X_t^1, \ldots, X_t^p)'$, $\boldsymbol{x}_t = (x_t^1, \ldots, x_t^p)'$, $1 \leqq t \leqq N$. Further put

$$r_{\beta\alpha}^{qu} = \sum_{t=n+1}^{N} x_{t-\alpha}^q x_{t-\beta}^u \,, \quad 1 \leqq q,\ u \leqq p \,; \quad 0 \leqq \alpha,\ \beta \leqq n \,,$$

and define matrices

$$\boldsymbol{R}_{qu} = \left\| r_{\alpha\beta}^{qu} \right\|_{\alpha,\beta=0}^{n} \,, \quad \boldsymbol{R} = \left\| \begin{matrix} \boldsymbol{R}_{11} & \ldots & \boldsymbol{R}_{1p} \\ \ldots & \ldots & \ldots \\ \boldsymbol{R}_{p1} & \ldots & \boldsymbol{R}_{pp} \end{matrix} \right\| \,.$$

Lemma 4. *If X_t^k $(1 \leqq t \leqq N; 1 \leqq k \leqq p)$ are normal, then the conditional density of $\boldsymbol{X}_{n+1}, \ldots, \boldsymbol{X}_N$ given $\boldsymbol{X}_1 = \boldsymbol{x}_1, \ldots, \boldsymbol{X}_n = \boldsymbol{x}_n$ is*

$$(5) \quad p(\boldsymbol{x}_{n+1}, \ldots, \boldsymbol{x}_N \mid \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p) = (2\pi)^{-(N-n)p/2} \prod_{i=1}^{p} a_{ii0}^{N-n} \exp\left\{-\tfrac{1}{2} \boldsymbol{a}'_i \boldsymbol{R} \boldsymbol{a}_i\right\} .$$

Proof. The conditional density p is

$$(2\pi)^{-(N-n)p/2} \prod_{i=1}^{p} a_{ii0}^{N-n} \exp\left\{-\tfrac{1}{2} \sum_{t=n+1}^{N} \left(\sum_{k=1}^{p} \sum_{j=1}^{n} a_{ikj} x_{t-j}^k\right)^2\right\}$$

and it may be obviously written in the form (5).

Put $\alpha_{ijk} = -a_{ijk}/a_{ii0}$ for $1 \leqq i$, $j \leqq p$; $0 \leqq k \leqq n$, $\boldsymbol{\alpha}_{ij} = (\alpha_{ij1}, \ldots, \alpha_{ijn})'$, $\boldsymbol{\alpha}_i = (\boldsymbol{\alpha}'_{i1}, \ldots, \boldsymbol{\alpha}'_{ip})'$, $\boldsymbol{A}_i = (\boldsymbol{\alpha}'_{ii}, \boldsymbol{\alpha}'_{i2}, \boldsymbol{\alpha}'_{i3}, \ldots, \boldsymbol{\alpha}'_{i,i-1}, \boldsymbol{\alpha}'_{i1}, \boldsymbol{\alpha}'_{i,i+1}, \ldots, \boldsymbol{\alpha}'_{ip})'$. Let $\boldsymbol{v} = (-1, 0, 0, \ldots, 0)'$ be a p-dimensional vector. Denote by $\boldsymbol{I}$ the unit matrix of the order $(n + 1) \times (n + 1)$ and define matrices

$$\boldsymbol{S}_i = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & \ldots & i-1 & i & i+1 & \ldots & p \end{array} \\ \left\| \begin{array}{ccccccccc} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \ldots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \ldots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \ldots & \boldsymbol{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{I} \end{array} \right\| \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \ldots \\ i-1 \\ i \\ i+1 \\ \ldots \\ p \end{array}$$

for $1 \leqq i \leqq p$. Further let

$$\mathbf{T} = \left\| \begin{matrix} 1\,0\,0\,0 & \ldots & 0\,\mathbf{0}\,\mathbf{0}\,\mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0}\,\mathbf{0}\,\mathbf{0}\,\mathbf{0} & \ldots & \mathbf{0}\,\mathbf{I}\,\mathbf{0}\,\mathbf{0} & \ldots & \mathbf{0} \\ 0\,1\,0\,0 & \ldots & 0\,\mathbf{0}\,\mathbf{0}\,\mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0}\,\mathbf{0}\,\mathbf{0}\,\mathbf{0} & \ldots & \mathbf{0}\,\mathbf{0}\,\mathbf{I}\,\mathbf{0} & \ldots & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0\,0\,0\,0 & \ldots & 1\,\mathbf{0}\,\mathbf{0}\,\mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0}\,\mathbf{0}\,\mathbf{0}\,\mathbf{0} & \ldots & \mathbf{0}\,\mathbf{0}\,\mathbf{0}\,\mathbf{0} & \ldots & \mathbf{I} \end{matrix} \right\|$$

be matrix of the order $p(n+1) \times p(n+1)$, where $\mathbf{I}$ is the unit matrix of the order $n \times n$ only.

Theorem 5. *Let the prior probability density of $a_{110}, \ldots, a_{pp0}, \boldsymbol{\alpha}'_1, \ldots, \boldsymbol{\alpha}'_p$ be proportional to $\sum_{i=1}^{p} a_{ii0}^{-1}$ independently on $\mathbf{X}_1, \ldots, \mathbf{X}_n$. If the matrix $\mathbf{R}$ is positive definite and density p is given by (5), then the posterior density of parameters $a_{110}, \ldots, a_{pp0}$, $\mathbf{A}_1, \ldots, \mathbf{A}_p$ is*

$$\pi(a_{110}, \ldots, a_{pp0}, \mathbf{A}_1, \ldots, \mathbf{A}_p \mid \mathbf{x}_1, \ldots, \mathbf{x}_N) = \tag{6}$$

$$= c \prod_{i=1}^{p} a_{ii0}^{N-1} \exp\left\{-\tfrac{1}{2} a_{ii0}^2 [(\mathbf{A}_i - {}^\circ\mathbf{A}_i)' \, \mathbf{N}_i (\mathbf{A}_i - {}^\circ\mathbf{A}_i) + \mathbf{v}'\mathbf{M}_i\mathbf{v} - {}^\circ\mathbf{A}'_i \mathbf{N}_i {}^\circ\mathbf{A}_i]\right\},$$

where

$${}^\circ\mathbf{A}_i = -\,\mathbf{N}_i^{-1}\mathbf{K}_i\mathbf{v}\,,$$

$$\mathbf{R}_i = \mathbf{T}'\mathbf{S}'_i\mathbf{R}\mathbf{S}_i\mathbf{T} = \left\| \begin{matrix} \mathbf{M}_i & \mathbf{K}'_i \\ \mathbf{K}_i & \mathbf{N}_i \end{matrix} \right\|, \quad 1 \leqq i \leqq p\,.$$

Matrices $\mathbf{M}_i$ are of the order $p \times p$.

If $N > n + 1$, then the modus of posterior density π is $\tilde{a}_{110}, \ldots, \tilde{a}_{pp0}, {}^\circ\mathbf{A}_1, \ldots, {}^\circ\mathbf{A}_p$, where

$$\tilde{a}_{ii0}^{-2} = \frac{1}{N-n-1}\left(\mathbf{v}'\mathbf{M}_i\mathbf{v} - {}^\circ\mathbf{A}'_i\mathbf{N}_i{}^\circ\mathbf{A}_i\right), \quad 1 \leqq i \leqq p\,.$$

The estimates ${}^\circ\mathbf{A}_1, \ldots, {}^\circ\mathbf{A}_p$ are the same as the least squares estimates.

Proof. We have

$$\mathbf{a}'_i\mathbf{R}\mathbf{a}_i = \mathbf{a}'_i\mathbf{S}_i\mathbf{T}\mathbf{T}'\mathbf{S}'_i\mathbf{R}\mathbf{S}_i\mathbf{T}\mathbf{T}'\mathbf{S}'_i\mathbf{a}_i = a_{ii0}^2(\mathbf{v}', \mathbf{A}'_i)\,\mathbf{R}_i \begin{pmatrix} \mathbf{v} \\ \mathbf{A}_i \end{pmatrix} =$$

$$= a_{ii0}^2[(\mathbf{A}_i - {}^\circ\mathbf{A}_i)' \, \mathbf{N}_i(\mathbf{A}_i - {}^\circ\mathbf{A}_i) + \mathbf{v}'\mathbf{M}_i\mathbf{v} - {}^\circ\mathbf{A}'_i\mathbf{N}_i{}^\circ\mathbf{A}'_i]\,.$$

The rest of the proof is obvious.

Theorem 6. *Let* $\mathbf{R}$ *be positive definite and* $N > n + np$. *If* π *is given by formula* (6), *then the following assertions hold*:

(I) *The marginal posterior density of* $\mathbf{A}_1, \ldots, \mathbf{A}_p$ *is*

$$\pi_1(\mathbf{A}_1, \ldots, \mathbf{A}_p \mid \mathbf{x}_1, \ldots, \mathbf{x}_N) = c \prod_{i=1}^{p} \left[1 + \frac{(\mathbf{A}_i - {}^\circ\mathbf{A}_i)' \, \mathbf{N}_i (\mathbf{A}_i - {}^\circ\mathbf{A}_i)}{\mathbf{v}'\mathbf{M}_i\mathbf{v} - {}^\circ\mathbf{A}_i'\mathbf{N}_i{}^\circ\mathbf{A}_i} \right]^{-(N-n)/2} .$$

(II) *The marginal density of* $a_{110}, \ldots, a_{pp0}$ *is*

$$\pi_2(a_{110}, \ldots, a_{pp0} \mid \mathbf{x}_1, \ldots, \mathbf{x}_N) = c \prod_{i=1}^{p} a_{ii0}^{N-n-np-1} \exp \left\{ -\tfrac{1}{2} a_{ii0}^2 (\mathbf{v}'\mathbf{M}_i\mathbf{v} - {}^\circ\mathbf{A}_i'\mathbf{N}_i{}^\circ\mathbf{A}_i) \right\} .$$

Random variables $a_{ii0}^2(\mathbf{v}'\mathbf{M}_i\mathbf{v} - {}^\circ\mathbf{A}_i'\mathbf{N}_i{}^\circ\mathbf{A}_i)$ *are independent and each of them has* χ^2-*distribution with* $N - n - np$ *degrees of freedom.*

(III) *Random variables*

$$F_i = \frac{N - n - np}{np} \, \frac{(\mathbf{A}_i - {}^\circ\mathbf{A}_i)' \, \mathbf{N}_i (\mathbf{A}_i - {}^\circ\mathbf{A}_i)}{\mathbf{v}'\mathbf{M}_i\mathbf{v} - {}^\circ\mathbf{A}_i'\mathbf{N}_i{}^\circ\mathbf{A}_i} \quad (1 \leqq i \leqq p)$$

are independent and each of them has F-distribution with np *and* $N - n - np$ *degrees of freedom.*

(IV) *Let* A_i^k *and* ${}^\circ A_i^k$ *be the k-th components of vectors* $\mathbf{A}_i$ *and* ${}^\circ\mathbf{A}_i$, *respectively, and* $\mathbf{N}_{i(k,k)}$ *be the matrix obtained from* $\mathbf{N}_i$ *by omitting the k-th row and k-th column* $(1 \leqq i \leqq p,\ 1 \leqq k \leqq np)$. *Then each of the random variables*

$$t_i^{(k)} = (A_i^k - {}^\circ A_i^k) \left[\frac{|\mathbf{N}_i| \, (N - n - np)}{|\mathbf{N}_{i(k,k)}| \, (\mathbf{v}'\mathbf{M}_i\mathbf{v} - {}^\circ\mathbf{A}_i'\mathbf{N}_i{}^\circ\mathbf{A}_i)} \right]^{1/2}$$

has Student distribution with $N - n - np$ *degrees of freedom*; *moreover,* $t_i^{(k)}$ *and* $t_j^{(s)}$ *are independent for* $i \neq j$.

Proof. Theorem 6 may be proved similarly as Theorem 3.

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Souhrn

BAYESOVSKÝ PŘÍSTUP V MNOHOROZMĚRNÝCH AUTOREGRESNÍCH POSLOUPNOSTECH

Jiří Anděl

Budiž $\mathbf{X}_1, \ldots, \mathbf{X}_N$ konečná část p-rozměrné normální autoregresní posloupnosti vytvářené vztahem

$$\sum_{k=0}^{n} \mathbf{A}_k \mathbf{X}_{t-k} = \xi_t ,$$

kde ξ_t jsou nekorelované náhodné vektory, z nichž každý má nulovou střední hodnotu a jednotkovou kovarianční matici. Předpokládejme, že matice $\mathbf{A}_0$ je diagonální. Pro řešení problému odhadu autoregresních parametrů byla zvolena Bayesova metoda, při níž je v tomto případě použita nevlastní „vágní" apriorní hustota autoregresních parametrů. Vychází se z podmíněné hustoty vektorů $\mathbf{X}_{n+1}, \ldots, \mathbf{X}_N$ při daném počátku $\mathbf{X}_1 = \mathbf{x}_1, \ldots, \mathbf{X}_n = \mathbf{x}_n$ a daných hodnotách autoregresních parametrů. V článku je dokázáno, že bodové odhady těchto parametrů získané Bayesovou metodou jako modus apostriorního rozdělení jsou totožné s odhady metodou nejmenších čtverců. Kromě toho jsou odvozena aposteriorní rozdělení parametrů, která pro svou jednoduchost mohou být použita při testování statistických hypotéz a při konstrukci konfidenčních oblastí.

Výhodou uvedeného postupu je to, že nemusí být nikde činěn předpoklad stacionarity a že výsledná aposteriorní rozdělení jsou podstatně jednodušší než rozdělení, ke kterým se dospívá v obvyklém konfidenčním přístupu.

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