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ON MULTIPLE NORMAL PROBABILITIES OF RECTANGLES

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Denote $A = \langle -a_1, a_1 \rangle \times \ldots \times \langle -a_n, a_n \rangle$, where $a_1, \ldots, a_n$ are given positive numbers. Let the n-dimensional random vector $\mathbf{X}$ have normal distribution with zero expectation and regular covariance matrix $\mathbf{G}$. A method for the numerical evaluation of the probability $P(A) = P(\mathbf{X} \in A)$ is suggested in this paper. The probability $P(A)$ is expressed as the sum of an infinite series. The formula for the remainder term of the series is derived, too. The series has a very simple form if $n = 2$. It is proved in this case that it converges more rapidly than a geometrical series with the quotient r^2, where r is the correlation coefficient between the components of the vector $\mathbf{X}$. Two numerical examples are given in order to compare numerically our results with other methods.

1. PRELIMINARIES

The problems concerning the probability $P(A)$ were solved by several authors. Very useful results of Šidák [6] enable to give a lower bound for $P(A)$. If $n = 2$, the Cramér's formula (see [1], Chap. VII, formula 21.12.5) may be used for the derivation of an infinite series, the sum of which gives $P(A)$. This method is commonly known and it is mentioned in our Section 3. Some multivariate normal integrals are solved in [2] and [5] (still further references may be found in these papers).

Denote

$$\varphi(x) = (2\sigma)^{-1/2} \exp\{-x^2/2\}, \quad \Phi(x) = \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t \tag{1}$$

the density and the distribution function of the univariate normal distribution $N(0, 1)$, respectively.

Let $\mathbf{X}$ have an n-dimensional normal distribution with vanishing expectation and regular covariance matrix $\mathbf{G} = \|g_{ij}\|_{i,j=1}^{n}$. Denote

$$\mathbf{Q} = \|q_{ij}\|_{i,j=1}^{n} = \mathbf{G}^{-1}. \tag{2}$$

Then the probability density of $\boldsymbol{X}$ is

(3)
$$f(x_1, \ldots, x_n) = (2\pi)^{-n/2} |\mathbf{Q}|^{1/2} \exp\{-\tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j q_{ij}\} .$$

2. EVALUATION OF THE PROBABILITY $P(A)$

Denote $q = q_{11}q_{22} \ldots q_{nn}$. The matrix $\mathbf{Q}$ is positive definite and, therefore, $q > 0$. Further, all the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix $\mathbf{Q}$ are positive numbers. Put

$$m = \min(\lambda_1, \lambda_2, \ldots, \lambda_n, q_{11}, q_{22}, \ldots, q_{nn}) .$$

Obviously, $m > 0$.

Theorem 1. *Put $A = \langle -a_1, a_1 \rangle \times \ldots \times \langle -a_n, a_n \rangle$, where $a_1, \ldots, a_n$ are finite positive numbers. If P is the measure corresponding to the density* (3), *then*

(4)
$$P(A) = \sum_{k=0}^{\infty} c_k ,$$

where

(5)
$$c_k = |\mathbf{Q}|^{1/2} \frac{1}{k!} (2\pi)^{-n/2} \int_{-a_1}^{a_1} \ldots \int_{-a_n}^{a_n} (-\tfrac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} x_i x_j q_{ij})^k .$$

$$. \exp\{-\tfrac{1}{2} \sum_{i=1}^{n} x_i^2 q_{ii}\} \, dx_1 \ldots dx_n .$$

Especially,

$$c_0 = |\mathbf{Q}|^{1/2} q^{-1/2} 2^n [\Phi(q_{11}^{1/2} a_1) - 0{\cdot}5] \ldots [\Phi(q_{nn}^{1/2} a_n) - 0{\cdot}5] ,$$
$$c_1 = 0 .$$

Further, if t is a natural number, then

(6)
$$\left|P(A) - \sum_{k=0}^{t-1} c_k\right| \leqq Z_t ,$$

where

(7)
$$Z_t = [t!\, 2^t (2\pi)^{n/2}]^{-1} |\mathbf{Q}|^{1/2} \int_{-a_1}^{a_1} \ldots \int_{-a_n}^{a_n} \left| \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} x_i x_j q_{ij} \right|^t .$$

$$. \exp\{-\tfrac{1}{2} m \sum_{i=1}^{n} x_i^2\} \, dx_1 \ldots dx_n .$$

Proof. Let $\Sigma\Sigma'$ mean $\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n}$. Using the Taylor formula we obtain

$$f(x_1, \ldots, x_n) = (2\pi)^{-n/2} \left|\mathbf{Q}\right|^{1/2} \exp\left\{-\tfrac{1}{2}\Sigma\Sigma' x_i x_j q_{ij}\right\} \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{n} x_i^2 q_{ii}\right\} =$$

$$= (2\pi)^{-n/2} \left|\mathbf{Q}\right|^{1/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\tfrac{1}{2}\Sigma\Sigma' x_i x_j q_{ij}\right)^k \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{n} x_i^2 q_{ii}\right\}.$$

Obviously $P(A) = \int_A f(x_1, \ldots, x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$. But

$$\left|\sum_{k=0}^{s} \frac{1}{k!} \left(-\tfrac{1}{2}\Sigma\Sigma' x_i x_j q_{ij}\right)^k \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{n} x_i^2 q_{ii}\right\}\right| \leqq$$

$$\leqq \exp\left\{\tfrac{1}{2}\left|\Sigma\Sigma' x_i x_j q_{ij}\right| - \tfrac{1}{2}\sum_{i=1}^{n} x_i^2 q_{ii}\right\} = g(x_1, \ldots, x_n).$$

The set A is compact and $g(x_1, \ldots, x_n)$ is bounded on A in view of its continuity. Using the Lebesgue theorem we get the formulas (4) and (5). From the symmetry of A with respect to the origin it follows $c_1 = 0$.

Taylor formula gives

$$f(x_1, \ldots, x_n) = (2\pi)^{-n/2} \left|\mathbf{Q}\right|^{1/2} \sum_{k=0}^{t-1} \frac{1}{k!} \left(-\tfrac{1}{2}\Sigma\Sigma' x_i x_j q_{ij}\right)^k .$$

$$. \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{n} x_i^2 q_{ii}\right\} + R_t ,$$

$$R_t = \left[t!\, 2^t (2\pi)^{n/2}\right]^{-1} (-1)^t \left|\mathbf{Q}\right|^{1/2} \left(\Sigma\Sigma' x_i x_j q_{ij}\right)^t .$$

$$. \exp\left\{-\tfrac{1}{2}\Theta\, \Sigma\Sigma' x_i x_j q_{ij} - \tfrac{1}{2}\sum_{i=1}^{n} x_i^2 q_{ii}\right\},$$

where $0 < \Theta < 1$; Θ depends on $x_1, \ldots, x_n$. From the well-known formulas (see, e.g., Rao [4], Chap. 1 f. 2) it follows easily that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j q_{ij} \geqq m \sum_{i=1}^{n} x_i^2 , \quad \sum_{i=1}^{n} x_i^2 q_{ii} \geqq m \sum_{i=1}^{n} x_i^2 .$$

We have

$$-\tfrac{1}{2}\Theta\, \Sigma\Sigma' x_i x_j q_{ij} - \tfrac{1}{2}\sum_{i=1}^{n} x_i^2 q_{ii} =$$

$$= -\tfrac{1}{2}\left[\Theta \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j q_{ij} + (1 - \Theta)\sum_{i=1}^{n} x_i^2 q_{ii}\right] \leqq -\tfrac{1}{2} m \sum_{i=1}^{n} x_i^2 .$$

Finally, we get

$$P(A) = \int_A f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = \sum_{k=0}^{t-1} c_k + \int_A R_t\, dx_1 \ldots dx_n\,,$$

$$\left|\int_A R_t\, dx_1 \ldots dx_n\right| \leqq [t!\, 2^t (2\pi)^{n/2}]^{-1}\, |\mathbf{Q}|^{1/2}\,.$$

$$\cdot \int_A |\Sigma\Sigma' x_i x_j q_{ij}|^t \exp\{-\tfrac{1}{2} m \sum_{i=1}^{n} x_i^2\}\, dx_1 \ldots dx_n = Z_t\,,$$

$$\left|P(A) - \sum_{k=0}^{t-1} c_k\right| \leqq Z_t\,.$$

Note that the use of the formula (7) is simpler in the case when t is even and the absolute value in the integrand need not be considered.

As for the numerical evaluation of the coefficients c_k given by (5), one rewrites the expression

$$(-\tfrac{1}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq 1}}^{n} x_i x_j q_{ij})^k$$

according to the polynomial theorem. Then the integral in (5) equals to a sum of integrals of the type

$$\varrho(\alpha_1, \ldots, \alpha_n) = C \int_{-a_1}^{a_1} \ldots \int_{-a_n}^{a_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \exp\{-\tfrac{1}{2} \sum_{i=1}^{n} x_i^2 q_{ii}\}\, dx_1 \ldots dx_n =$$

$$= C \prod_{i=1}^{n} \int_{-a_i}^{a_i} x_i^{\alpha_i} \exp\{-\tfrac{1}{2} x_i^2 q_{ii}\}\, dx_i\,,$$

where C is a constant, $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 2k, 0 \leqq \alpha_i \leqq k$. If at least one number α_i is odd then $\varrho(\alpha_1, \ldots, \alpha_n) = 0$. Hence, it suffices to calculate the integrals $\varrho(\alpha_1, \ldots, \alpha_n)$ only in the cases when all the numbers $\alpha_1, \ldots, \alpha_n$ are even. Then the formula (14) (see later) is very useful.

If $n = 2$, then the formulas in Theorem 1 have a simpler form.

Theorem 2. *Let* $n = 2$, $A = \langle -a_1, a_1 \rangle \times \langle -a_2, a_2 \rangle$, $0 < a_1 < \infty$, $0 < a_2 < \infty$. *Denote* $r = g_{12}(g_{11} g_{22})^{-1/2} = -q_{12}(q_{11} q_{22})^{-1/2}$ *the correlation coefficient. Then*

$$P(A) = \sum_{j=0}^{\infty} b_j\,, \tag{8}$$

where

$$b_j = \frac{2}{\pi} (1 - r^2)^{1/2} \frac{r^{2j}}{(2j)!} \left(\int_0^{a_1 q_{11}^{1/2}} x^{2j} e^{-x^2/2}\, dx\right) \left(\int_0^{a_2 q_{22}^{1/2}} x^{2j} e^{-x^2/2}\, dx\right). \tag{9}$$

If t is a natural number, then

$$0 \leqq P(A) - \sum_{j=0}^{t-1} b_j \leqq S_t\,,$$

where

$$S_t = \left[(2t)!\,(2\pi)\right]^{-1} (q_{11}q_{22} - q_{12}^2)^{1/2}\, q_{12}^{2t}\,. \tag{10}$$

$$\cdot \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} (x_1x_2)^{2t} \exp\left\{-\frac{m}{2}(x_1^2 + x_2^2)\right\} \mathrm{d}x_1\,\mathrm{d}x_2\,,$$

$$m = \tfrac{1}{2}\{q_{11} + q_{22} - \left[(q_{11} - q_{22})^2 + 4q_{12}^2\right]^{1/2}\}\,. \tag{11}$$

Moreover,

$$S_t < \frac{2}{\pi}(q_{11}q_{22} - q_{12}^2)^{1/2} \frac{q_{12}^{2t}}{(2t)!} \frac{(a_1a_2)^{2t+1}}{(2t+1)^2}\,. \tag{12}$$

The series $\sum_{j=0}^{\infty} b_j$ has non-negative members only and converges more rapidly than a geometrical series with the quotient r^2 in the sense that $b_{j+1}/b_j < r^2$ for $j \geqq 0$.

Proof. We use Theorem 1. If $n = 2$, then

$$c_{2k+1} = (2\pi)^{-1} \frac{1}{(2k+1)!} (q_{11}q_{22} - q_{12}^2)^{1/2} \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} (-x_1x_2q_{12})^{2k+1}\,.$$

$$.\exp\{-\tfrac{1}{2}(x_1^2q_{11} + x_2^2q_{22})\}\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0\,,$$

$$c_{2k} = b_k\,,$$

$$k = 0, 1, 2, \ldots$$

Therefore, $P(A) = \sum_{k=0}^{\infty} c_k = \sum_{j=0}^{\infty} b_j$. Clearly, $b_j \geqq 0$ for $j \geqq 0$. Then $0 \leqq P(A) - \sum_{j=0}^{t-1} b_j$ obviously holds. Further, according to (6) and (7),

$$P(A) - \sum_{j=0}^{t-1} b_j = P(A) - \sum_{k=0}^{2t-1} c_k \leqq Z_{2t}$$

and it implies formula (10).

The roots of the matrix $\mathbf{Q}$ are

$$\lambda_{1,2} = \tfrac{1}{2}\{q_{11} + q_{22} \pm \left[(q_{11} - q_{22})^2 + 4q_{12}^2\right]^{1/2}\}\,.$$

By an elementary way it may be proved that

$$\lambda_2 = \min(\lambda_1, \lambda_2, q_{11}, q_{22}) = m\,.$$

Formula (12) follows from (10) in view of

$$\int_{-a}^{a} x^{2t} \exp\{-\tfrac{1}{2}mx^2\}\,dx < 2\int_0^a x^{2t}\,dx = 2\,\frac{a^{2t+1}}{2t+1}$$

if $a > 0$.

The right-hand side of the formula (12) is simpler than formula (10); moreover, it does not depend on m.

Let D be a fixed positive number. Define

$$J_k(D) = \int_0^D x^k e^{-x^2/2}\,dx\,.$$

Integrating by parts we obtain

$$J_{2k+2}(D) = (2k+1)\,J_{2k}(D) - D^{2k+1}e^{-D^2/2}\,,$$

and thus

$$\frac{J_{2k+2}(D)}{J_{2k}(D)} = 2k + 1 - D^{2k+1}e^{-D^2/2}[J_{2k}(D)]^{-1} < 2k+1\,,$$

$$k = 0, 1, 2, \ldots$$

Using this fact we get

$$\frac{b_{j+1}}{b_j} = \frac{r^{2+2j}}{(2j+2)!}\,\frac{(2j)!}{r^{2j}}\,\frac{J_{2j+2}(a_1 q_{11}^{1/2})}{J_{2j}(a_1 q_{11}^{1/2})}\,\frac{J_{2j+2}(a_2 q_{22}^{1/2})}{J_{2j}(a_2 q_{22}^{1/2})} <$$

$$< \frac{r^2}{(2j+2)(2j+1)}(2j+1)^2 < r^2\,,\quad j = 0, 1, 2, \ldots$$

The proof is finished.

Note that the last assertion of Theorem 2 guarantees the regular behaviour of the series (8). Similar properties need not occur in other formulas commonly used for evaluation of the probability $P(A)$ (see the next Section).

3. EXAMPLES

The following formulas are useful in practical applications of the previous theorems:

$$\int_0^z x^{2r-1}\exp\{-x^2/2\}\,dx = (r-1)!\,2^{r-1} - \tag{13}$$

$$- [z^{2(r-1)} + 2(r-1)\,z^{2(r-2)} + \ldots + 2^{r-2}(r-1)(r-2)\ldots 3.2z^2 +$$

$$+ 2^{r-1}(r-1)(r-2)\ldots 3.2.1]\exp\{-z^2/2\}\,,$$

$$(14)\qquad \int_0^z x^{2r} \exp\{-x^2/2\}\, dx = (2r-1)(2r-3)\dots 3.1.(2\pi)^{1/2}\,.$$

$$.\,[\Phi(z) - 0{\cdot}5] - [z^{2r-1} + (2r-1)\, z^{2r-3} + (2r-1)(2r-3)\, z^{2r-5} + \dots$$

$$\dots + (2r-1)(2r-3)\dots 5.3z]\exp\{-z^2/2\}\,,$$

where r is natural number.

Example 1. Let X_1, X_2 have a simultaneous normal distribution with the parameters

$$EX_1 = EX_2 = 0\,,\quad \operatorname{var} X_1 = \operatorname{var} X_2 = 1\,,\quad \operatorname{cov}(X_1, X_2) = \sqrt{0{\cdot}75}\,.$$

Put $A = \langle -1; 1\rangle \times \langle -1; 1\rangle$.

We have

$$\mathbf{G} = \begin{Vmatrix} 1 & \sqrt{(0{\cdot}75)} \\ \sqrt{(0{\cdot}75)} & 1 \end{Vmatrix}\,,\quad \mathbf{Q} = \mathbf{G}^{-1} = \begin{Vmatrix} 4 & -2\sqrt{3} \\ -2\sqrt{3} & 4 \end{Vmatrix}\,.$$

From the formulas (8) and (9) we get, in view of (14),

$$P(A) = \frac{2}{\pi}\sqrt{(0{\cdot}25)}\sum_{j=0}^{\infty}\frac{0{\cdot}75^j}{(2j)!}\left(\int_0^2 x^{2j} e^{-x^2/2}\, dx\right)^2 =$$

$$= 0{\cdot}455\,535 + 0{\cdot}102\,269 + 0{\cdot}021\,413 + 0{\cdot}003\,197 +$$

$$+ 0{\cdot}000\,340 + 0{\cdot}000\,026 + \dots$$

The sum $\sum_{j=0}^{5} b_j$ gives $0{\cdot}582\,78$. We obtain $m = 0{\cdot}535\,898$. From (12) it follows

$$S_6 < \frac{12^6\,.\,4}{12!\;169\pi} = 0{\cdot}000\,047\,.$$

Thus,

$$(15)\qquad 0{\cdot}582\,78 < P(A) < 0{\cdot}582\,83\,.$$

We compare the above values with other results. Using the Corollary 1 in Šidák's paper [6] we obtain

$$P(A) \geqq [P(|X_1| \leqq 1)]^2 = 0{\cdot}465\,9\,.$$

Still a further method is based on the Cramér's formula (see [1], § 21.12). Let X, Y have a simultaneous normal distribution with the parameters $EX = EY = 0$, $\operatorname{var} X = \sigma_1^2$, $\operatorname{var} Y = \sigma_2^2$ and with the correlation coefficient r. Then their simultaneous density may be expressed in the form

$$(16) \qquad f(x, y) = \frac{1}{\sigma_1 \sigma_2} \sum_{k=0}^{\infty} \frac{r^k}{k!} \Phi^{(k+1)}\left(\frac{x}{\sigma_1}\right) \Phi^{(k+1)}\left(\frac{y}{\sigma_2}\right).$$

Define the Hermite polynomials $H_n(x)$ by

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n e^{-x^2/2} = (-1)^n H_n(x)\, e^{-x^2/2}, \quad n = 0, 1, 2, \ldots$$

(see Cramér [1], § 12.6). The definition slightly differs from the more usual one (see Jarník [3], p. 579).

If $A = \langle -a_1, a_1 \rangle \times \langle -a_2, a_2 \rangle$, then we get from (16)

$$(17) \quad P(A) = \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} f(x, y)\, \mathrm{d}x\, \mathrm{d}y = 4[\Phi(a_1/\sigma_1) - 0{\cdot}5]\,[\Phi(a_2/\sigma_2) - 0{\cdot}5] +$$

$$+ \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{r^{2k}}{(2k)!} H_{2k-1}(a_1/\sigma_1)\, H_{2k-1}(a_2/\sigma_2) \exp\{-\tfrac{1}{2}a_1/\sigma_1^2 - \tfrac{1}{2}a_2/\sigma_2^2\}.$$

In our example $a_1 = a_2 = 1$, $\sigma_1^2 = \sigma_2^2 = 1$, $r^2 = 0{\cdot}75$. Formula (17) gives

$$(18) \qquad P(A) = 0{\cdot}466\,059 + 0{\cdot}087\,825 + 0{\cdot}021\,956 + 0{\cdot}004\,940 +$$
$$+ 0{\cdot}000\,735 + 0{\cdot}000\,012 + 0{\cdot}000\,076 + 0{\cdot}000\,003 +$$
$$+ 0{\cdot}000\,179 + 0{\cdot}000\,203 + 0{\cdot}000\,166 + \ldots$$

The sum of the first 6 numbers gives 0·581 527. It is out from the bounds (15). The further members in (18) do not decrease. I don't know if the remainder term corresponding to (16) or to (17) was stated elsewhere. Our example points out that the use of (17) may be somewhat dangerous. (The sum of the first 11 terms in (18) is 0·582 154.)

Example 2. Let X_1, X_2, X_3 have a simultaneous normal distribution with zero expectations and the covariance matrix

$$\boldsymbol{G} = \frac{1}{0{\cdot}47} \begin{Vmatrix} 0{\cdot}84 & -0{\cdot}40 & -0{\cdot}26 \\ -0{\cdot}40 & 0{\cdot}75 & -0{\cdot}10 \\ -0{\cdot}26 & -0{\cdot}10 & 0{\cdot}64 \end{Vmatrix}.$$

Put $A = \langle -2, 2 \rangle \times \langle -2, 2 \rangle \times \langle -2, 2 \rangle$. In order to obtain $P(A)$ we evaluate

$$\boldsymbol{Q} = \boldsymbol{G}^{-1} = \begin{Vmatrix} 1{\cdot}0 & 0{\cdot}6 & 0{\cdot}5 \\ 0{\cdot}6 & 1{\cdot}0 & 0{\cdot}4 \\ 0{\cdot}5 & 0{\cdot}4 & 1{\cdot}0 \end{Vmatrix}.$$

The roots of the matrix $\mathbf{Q}$ are 0·382 45, 0·613 09, 2·004 45. It gives $m = 0{\cdot}382\,45$. According to (4) and (5) we have

$$P(A) = \sqrt{(0{\cdot}47)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,10^k} \int_A (6x_1x_2 + 5x_1x_3 + 4x_2x_3)^k (2\pi)^{-3/2} \,.$$

$$. \exp\{-\tfrac{1}{2}\sum_{i=1}^{3} x_i^2\}\, dx_1\, dx_2\, dx_3 =$$

$$= 0{\cdot}596\,18 + 0{\cdot}137\,41 - 0{\cdot}033\,14 + 0{\cdot}034\,55 - 0{\cdot}014\,25 + \ldots$$

The sum of the first five numbers written above is 0·720 75. According to (7) we get the remainder term

$$Z_6 = 0{\cdot}086\,81 \,.$$

From [6] we obtain the inequality

$$P(A) \geqq \prod_{i=1}^{3} \left[(2\pi)^{-1/2}\,\sigma_i^{-1} \int_{-2}^{2} \exp\{-\tfrac{1}{2}x^2/\sigma_i^2\}\, dx\right] = 0{\cdot}700\,85 \,.$$

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Souhrn

O NORMÁLNÍCH PRAVDĚPODOBNOSTECH PRAVOÚHELNÍKŮ

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Nechť $a_1, \ldots, a_n$ jsou kladná čísla. Označme

$$A = \langle -a_1, a_1\rangle \times \ldots \times \langle -a_n, a_n\rangle \,.$$

Uvažujme n-rozměrný náhodný vektor $\mathbf{X}$, který má n-rozměrné normální rozdělení s nulovou střední hodnotou a s regulární kovarianční maticí $\boldsymbol{G}$. V práci je odvozena metoda pro výpočet pravděpodobnosti $P(\boldsymbol{X} \in A)$. Její princip spočívá v tom, že se hustota vektoru $\boldsymbol{X}$ napíše ve tvaru

$$f(x_1, \ldots, x_n) = c \exp \left\{ -\tfrac{1}{2} \sum_{i \neq j} \sum x_i x_j q_{ij} \right\} \exp \left\{ -\tfrac{1}{2} \sum x_j^2 q_{jj} \right\},$$

kde c je konstanta. První z těchto exponenciel se rozvine v Taylorovu řadu, takže se pro hledanou pravděpodobnost

$$P(\boldsymbol{X} \in A) = \int_A f(x_1, \ldots, x_n) \, \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

dostane take nekonečná řada. V práci je stanoven odhad pro zbytek této řady. Podrobněji je vyšetřován případ $n = 2$, kdy se pro $P(\boldsymbol{X} \in A)$ dostává mocninná řada se sudými mocninami korelačního koeficientu r. Přitom jde o řadu s nezápornými členy, takže jakýkoliv její částečný součet je současně dolním odhadem pro $P(\boldsymbol{X} \in A)$. V tomto případě jsou také explicitněji uvedeny vzorce pro odhad zbytku a je dokázáno, že řada konverguje rychleji než geometrická řada s kvocientem r^2. Jako ukázka jsou uvedeny dva numerické příklady, jeden pro $n = 2$, druhý pro $n = 3$. Na nich je provedeno porovnání výsledků s výsledky dosaženými jinými metodami, a to se Šidákovým odhadem a s metodou založenou na Cramérově vzorci. Z výpočtů vyplývá, že řada odvozená z Cramérova vzorce obecně nemá ani monotónně klesající členy. Tato okolnost spolu s tím, že pro ni patrně není znám odhad zbytku, zřejmě omezuje její použitelnost. Z tohoto hlediska se jeví metoda navržená v článku jako výhodnější.

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