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THE FORM OF DISCRETE SPECTRUM IN THE CASE OF HIGH SINGULAR POTENTIAL

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The solution of the inverse problem of scattering and the investigation of the set of physically interpretable parameters determining the potential is one of the basic physical problems. Further we shall deal with the form of discrete spectrum in the case of high singular potentials.

Let be given the Schrödinger equation

$$-y'' + V(x)\, y = \lambda^2 y\,. \tag{1}$$

Let us study the problem of eigenfunctions $y(x)$ of this equation in the space $L^2(0, \infty)$ with the boundary value condition $y(0) = 0$. If for the function $V(x)$ the condition

$$\int_0^\infty x|V(x)|\, \mathrm{d}x < \infty \tag{2}$$

is fulfilled then there exists a finite number of eigen-values λ all having the form $\lambda_j = -i\mu_j$, $\mu_j > 0$ as it is demonstrated e.g. in [1]. We shall be interested in the case that the function in the neighbourhood of zero has a singularity of higher order and we shall find under which conditions, even in this case, there exists only a finite number of eigen-values.

Let us suppose that the function $V(x) \to \infty$ for $x \to 0$ and that for every $a > 0$ it holds

$$\int_a^\infty x|V(x)|\, \mathrm{d}x < \infty\,. \tag{3}$$

The equation (1) is investigated under these assumptions in [2], [3], where even the fundamental system of solutions is shown. We shall prove that the following theorem holds:

Theorem. *If $V(x)$ is positive in the neighbourhood of the origin and (3) is valid,* *then the equation* (1) *has a finite number of eigen-values all of them having the form* $\lambda = -i\mu$, $\mu > 0$.

Proof. Since the operator defined by the equation (1) is self-adjoint all eigenvalues are real, i.e. λ^2 is real. For $\lambda \neq 0$ ($Im\lambda \leqq 0$) there exists a fundamental system of solutions

$$\left.\begin{aligned} y_1(x, \lambda) &= e^{-i\lambda x}[1 + o(x^{-1})] \\ y_2(x, \lambda) &= e^{i\lambda x}[1 + o(x^{-1})] \end{aligned}\right\} x \to \infty$$

Hence, the general form of the solution is

$$y(x, \lambda) = C_1\, y_1(x, \lambda) + C_2\, y_2(x, \lambda)$$

where C_1, C_2 are constants. For the real λ the solution of $y(x, \lambda)$ does not fulfil the homogeneous boundary value condition at the infinity, i.e. such a λ is not an eigen-value. Thus all eigen-values have the form

$$\lambda = -i\mu\,, \quad \mu > 0\,.$$

To prove that their number is finite we use the method of operator splitting [4]. If $V(x)$ is positive everywhere then proof of theorem is evident. In other case let us denote by α the first zero of the function $V(x)$ and let us put

$$l[y] = -y'' + V(x)\, y$$

and investigate the self-adjoint operator L defined by the operation $l[y]$ and by the boundary value condition $y(0) = 0$. The domain D_L of the operator L let be the set of functions $y(x) \in L^2(0, \infty)$ fulfilling the conditions:

a) $y'(x)$ exists and is absolutely continuous in each finite interval $(0, k)$;
b) $l[y] \in L^2(0, \infty)$;
c) $y(0) = 0$.

For $y \in D_L$ we put $Ly = l[y]$.

The equation (1) is equivalent with the operator equation $Ly = \lambda^2 y$.

We further introduce two self-adjoint operators:

L_1 – operator defined in the space $L^2(0, \alpha)$ by the operation $l[y]$ and by the boundary value conditions $y(0) = y(\alpha) = 0$.

L_2 – operator defined in the space $L^2(\alpha, +\infty)$ by the operation $l[y]$ and by the boundary value condition $y(\alpha) = 0$.

Domains of these operators D_{L_1} and D_{L_2} are analogous to D_L. We put $L_i y = l[y]$ for all $y \in D_{L_i}$ ($i = 1, 2$). Let us demonstrate that the operator L_1 is positive definite. Namely, for $y \neq 0$ it is

$$(L_1 y, y) = (l[y], y)_{\langle 0, \alpha \rangle} = \int_0^\alpha (-y'' + V(x)\, y)\, y\, \mathrm{d}x = \int_0^\alpha (y'^2 + V(x)\, y^2)\, \mathrm{d}x > 0\,.$$

The operator L_2 has a finite number of negative eigen-values m. This follows from the condition (3). Let us denote by p the number of eigen-values of the operator L. We demonstrate that $p < m + 2$. Let us assume that $p \geqq m + 2$; therefore it exists at least $m + 2$ linear independent eigenfunctions of the operator L, $y_1, y_2, \ldots, y_{m+2}$; from them we take $m + 1$ linearly independent linear combinations $z_1, z_2, \ldots, z_{m+1}$ fulfilling the conditions $z_i(0) = z_i(\alpha) = 0$ $(i = 1, 2, \ldots, m + 1)$. Let us now construct the non-zero function $u(x) = \sum_{i=1}^{m+1} p_i\, z_i(x)$ orthogonal to all eigen-subspaces of the operator L_2 corresponding to its negative eigenvalues. It holds $(L_1 u, u) > 0$ since L_1 is positive definite.

According to the condition (3), the continuous spectrum of operator L_2 is on the positive semiaxis. From that fact and from the way in which the function u was constructed it follows $(L_2 u, u) \geqq 0$, thus $(Lu, u) = (L_1 u, u) + (L_2 u, u) > 0$. Function $u(x)$ is a linear combination of eigenfunctions of the operator L; all eigenvalues of L are negative, hence $(Lu, u) \leqq 0$; that is not possible which completes the proof.

Remark. The theorem need not be valid for $V(x)$ negative in the neighbourhood of the origin; it can be shown as follows:

$$V(x) = \begin{cases} \dfrac{\alpha(\alpha - 1)}{x^2} & 0 \leqq x < 1 \\ 0 & 1 \leqq x < +\infty . \end{cases}$$

The equation (1) assumes the form

$$(4) \qquad \begin{aligned} x^2 y'' - [-\lambda^2 x^2 + \alpha(\alpha - 1)]\, y &= 0 \quad \text{for} \quad 0 \leqq x < 1 \\ y'' + \lambda^2 y &= 0 \quad \text{for} \quad 1 \leqq x < +\infty . \end{aligned}$$

Since only the negative values of λ^2 can be eigen-values, let us put $\lambda^2 = -\mu^2$. Then the equation (4) is

$$(5) \qquad \begin{aligned} x^2 y'' - [\mu^2 x^2 + \alpha(\alpha - 1)]\, y &= 0 \quad \text{for} \quad 0 \leqq x < 1 \\ y'' - \mu^2 y &= 0 \quad \text{for} \quad 1 \leqq x < +\infty . \end{aligned}$$

According to [5] in the interval $0 \leqq x < 1$ the solution has the form

$$y(x) = \sqrt{(x)}\,(C_1 J_{\alpha - 1/2}(i\mu x) + C_2 Y_{\alpha - 1/2}(i\mu x)) \,.$$

From the form of the solution at infinity and from the continuity conditions for the function $y(x)$ and the first derivative $y'(x)$ for $x = 1$ it follows

$$(\mu + \tfrac{1}{2}) - i\mu \frac{J'_{\alpha-1/2}(i\mu) + CY'_{\alpha-1/2}(i\mu)}{J_{\alpha-1/2}(i\mu) + CY_{\alpha-1/2}(i\mu)} = 0 \,, \quad C = \frac{C_2}{C_1} \,.$$

In the case of $\alpha(\alpha - 1) \geqq 0$ it holds $C = 0$ (this follows from the relation $y(0) = 0$ which must hold for the eigen function), in the opposite case it is possibly $C \neq 0$ (since the both solutions fulfil the homogeneous boundary value condition at zero). For $\alpha = \frac{1}{2}$ it holds

$$(\mu + \tfrac{1}{2}) = i\mu \frac{J_1(i\mu) + CY_1(i\mu)}{J_0(i\mu) + CY_0(i\mu)}.$$

Thus there exists an infinite number of μ which are eigen-values of the given differential equation.

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Souhrn

TVAR DISKRÉTNÍHO SPEKTRA PRO VYSOKOSINGULÁRNÍ POTENCIÁLY

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V článku je rozebrán tvar diskrétního spektra pro vysokosingulární potenciály při řešení obrácené úlohy teorie rozptylu. Je dokázána jeho konečnost pro potenciály, které jsou kladné v okolí počátku.

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