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ON THE EXISTENCE AND UNIQUENESS OF SOLUTION OF THE CAUCHY PROBLEM FOR A CLASS OF LINEAR INTEGRO-DIFFERENCIAL EQUATIONS

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INTRODUCTION

Some problems in the theory of viscoelasticity [1], [2] may be described by means of integro-differential equations. In the present paper a class of problems is considered, which includes these physical examples. The weak solution is defined on the variational basis [3] and its existence, uniqueness and continuous dependence on the given data proved, using the theory of integral equations of Volterra's type in Banach spaces.

Sections 1 and 4 deal with the equations of the first order in time coordinate, Sections 2 and 5 with equations of the second order. The theory is restricted to the equations only, possessing the highest spatial derivative by the term with the highest time derivative. In Section 3 the existence and uniqueness theorem for integral equations of Volterra's type in a Banach space is proved.

1. THE PROBLEM OF THE FIRST ORDER AND THE CORRESPONDING INTEGRAL EQUATION

Let a bounded interval $I = \langle 0, T\rangle$ and a basic Hilbert space H be given, with the scalar product (u, v) and the norm $|u| = (u, v)^{1/2}$.

V will denote a Hilbert space with the scalar product $((u, v))$ and the norm $\|u\| = ((u, u))^{1/2}$.

$L_2(I, X_0)$ will denote the space of functions $u(t)$, mapping the interval I into a part X_0 of a Banach space X, and such that

$$\int_0^T |u|_X^2 \, dt < \infty .$$

Similarly, $L_2(I \times I, X_0)$ is defined.

$\mathscr{L}(X, Y)$ denotes the space of linear continuous mappings of a Hilbert space X into a Hilbert space Y. Let us denote $u'(t) = \mathrm{d}u/\mathrm{d}t$ and the domain of the operator A by D_A.

Consider the following equation in H

$$\frac{\mathrm{d}}{\mathrm{d}t}(B(t)\,u(t)) + A_0(t)\,u(t) + \int_0^t A_1(t, \tau)\,u(\tau)\,\mathrm{d}\tau = f(t) \tag{1}$$

and the initial condition

$$u(0) = u_0\,. \tag{2}$$

Here $B(t)$, $A_0(t)$, $A_1(t, \tau)$ are linear (in general unbounded) operators in H. Assume that there exists a Hilbert space V, positive constants c, β and operators $\mathscr{B}(t) \in \mathscr{L}(V, V)$ for all $t \in I$, such that

$$V \subset H\,, \quad \|u\| \geqq c|u| \quad \text{for every} \quad u \in V\,, \tag{3}$$

$$D_{B(t)} \quad \text{is dense in } V \text{ for all} \quad t \in I\,, \tag{3'}$$

$$((\mathscr{B}(t)\,u, v)) = (B(t)\,u, v) \quad \text{for} \quad u \in D_{B(t)}\,, \quad v \in V\,, \quad t \in I\,, \tag{4}$$

$$((\mathscr{B}(t)\,u, u)) \geqq \beta\|u\|^2 \qquad \text{for} \quad u \in V\,, \quad t \in I \quad \text{and} \tag{5}$$

$$((\mathscr{B}(t)\,u, v)) \quad \text{is bounded on } I \text{ for any fixed} \quad u, v \in V\,. \tag{5'}$$

Furthermore, assume that the operators $\mathscr{A}_0(t) \in \mathscr{L}(V, V)$ and $\mathscr{A}_1(t, \tau) \in \mathscr{L}(V, V)$ exist for almost all $t \in I$ and $t, \tau \in I$, respectively, such that

$$((\mathscr{A}_0(t)\,u, v)) = (A_0(t)\,u, v)\,, \tag{6}$$

$$((\mathscr{A}_1(t, \tau)\,u, v)) = (A_1(t, \tau)\,u, v)\,, \tag{7}$$

$$\mathscr{A}_1(t, \tau) = \Theta \quad \text{for} \quad t < \tau$$

hold for almost all $t \in I$, $v \in V$, $u \in D_{A_0(t)} \cap V$ and almost all $t, \tau \in I$, $u \in D_{A_1(t,\tau)} \cap V$, respectively.

Finally, let

$$\mathscr{A}_0(t) \in L_2(I; \mathscr{L}(V, V))\,, \tag{8}$$

$$\mathscr{A}_1(t, \tau) \in L_2(I \times I; \mathscr{L}(V, V)) \tag{9}$$

and $u_0 \in V$, $f \in L_2(I; H)$.

Remark 1. In case of differential operators, the above-mentioned conditions imply that the operators A_0 and A_1 may involve only spatial derivatives, the order of which is bounded from above by the order of the spatial derivatives of the operator B.

Define

$$(10)\qquad \mathscr{A}_2(t,\tau) = \int_\tau^t \mathscr{A}_1(z,\tau)\,\mathrm{d}z\,,$$

$$(11)\qquad \mathscr{K}_0(t,\tau) = \mathscr{A}_0(\tau) + \mathscr{A}_2(t,\tau)\,.$$

It is obvious that $\mathscr{K}_0(t,\tau) \in L_2(I \times I;\ \mathscr{L}(V, V))$. In fact,

$$(11')\qquad \|\mathscr{A}_2(t,\tau)\| = 0 \quad \text{for} \quad \tau > t\,,$$

$$\|\mathscr{A}_2(t,\tau)\|^2 \leqq \int_\tau^t \mathrm{d}z \int_\tau^t \|\mathscr{A}_1(z,\tau)\|^2\,\mathrm{d}z \leqq T\int_0^T \|\mathscr{A}_1(z,\tau)\|^2\,\mathrm{d}z \quad \text{for} \quad \tau \leqq t\,.$$

Definition 1. (Weak solution of the Cauchy problem.) *We say that a function u is a solution of the Cauchy problem $\mathscr{P}(u_0, f)$, if $u \in L_2(I; V)$ and*

$$(12)\qquad \int_0^T \left(\left(\mathscr{B}(t)\,u(t) + \int_0^t \mathscr{K}_0(t,\tau)\,u(\tau)\,\mathrm{d}\tau,\ \varphi(t)\right)\right)\mathrm{d}t =$$

$$= \int_0^T \left[\left(\int_0^t f(\tau)\,\mathrm{d}\tau,\ \varphi(t)\right) + ((\mathscr{B}(0)\,u_0,\ \varphi(t)))\right]\mathrm{d}t$$

holds for every $\varphi \in L_2(I;\ D_{B(t)})$.

Remark 2. Let us suppose the "convolution symmetry" of the operators, occuring in (1), i.e., let

$$\int_0^T (B(t)\,u(t),\, v(T-t))\,\mathrm{d}t = \int_0^T (B(t)\,v(t),\, u(T-t))\,\mathrm{d}t\,,$$

$$\int_0^T \left(A_0(t)\,u(t) + \int_0^t A_1(t,\tau)\,u(\tau)\,\mathrm{d}\tau,\ v(T-t)\right)\mathrm{d}t =$$

$$= \int_0^T \left(A_0(t)\,v(t) + \int_0^t A_1(t,\tau)\,v(\tau)\,\mathrm{d}\tau,\ u(T-t)\right)\mathrm{d}t\,.$$

Then (12) means the condition of the stationary value for the functional (see [3])

$$(13)\qquad \mathscr{F}(u) = \int_0^T \left\{\left(\left(\mathscr{B}(t)\,u(t) + \int_0^t \left[\mathscr{A}_0(\tau)\,u(\tau) + \int_0^\tau \mathscr{A}_1(\tau, z)\,u(z)\,\mathrm{d}z\right]\mathrm{d}\tau\, - \right.\right.\right.$$

$$\left.\left.\left. - 2\,\mathscr{B}(0)\,u_0,\ u(T-t)\right)\right) - 2\left(\int_0^t f(\tau)\,\mathrm{d}\tau,\ u(T-t)\right)\right\}\mathrm{d}t\,,$$

if we set $\delta u(T-t) = \varphi(t)$.

Remark 3. The relation (12) follows from (1) formally, if we integrate it with respect to t, insert (2), multiply by $\varphi(t)$ in H, extend the result with the use of (4), (6), (7) and integrate over I.

Example. Some three-dimensional problems of linear viscoelasticity for ageing isotropic and homogeneous materials ([1], [2]) may be described in terms of displacements $u_i(X, t)$, $(i = 1, 2, 3)$, $X \in \Omega \subset E_3$, $t \in I$, by an integro-differential equation

$$Lu(X, t) - \int_0^t K_0(t, \tau)\, Lu(X, \tau)\, \mathrm{d}\tau = F(t)\,, \tag{14}$$

where $K_0(t, \tau)$ is a real continuous function on $I \times I$ with continuous $\partial K_0(t, \tau)/\partial t$, $\boldsymbol{Lu}$ is a vector-function with the components

$$(\boldsymbol{Lu})_i = \nabla^2 u_i + (c_0 + \tfrac{1}{3}) \sum_{k=1}^{3} \frac{\partial^2 u_k}{\partial x_k\, \partial x_i}$$

(where c_0 is a positive constant, ∇^2 Laplace operator). For simplicity, we shall consider the conditions

$$u_i(t) = 0$$

on the boundary of the (bounded) region Ω, for all $t \in I$. Let us set

$$H = [L_2(\Omega)]^3\,, \quad B(t) = L, \quad V = [\mathring{W}_2^{(1)}(\Omega)]^3\,, \quad D_{B(t)} = [\mathscr{D}(\Omega)]^3\,.$$

Here $\mathscr{D}(\Omega)$ denotes the set of functions having continuous derivatives of all orders and a compact support in Ω. $\mathring{W}_2^{(1)}(\Omega)$ denotes the closure of $\mathscr{D}(\Omega)$ in the sense of the norm of $W_2^{(1)}\Omega)$, i.e.,

$$|u|^2_{W_2^{(1)}(\Omega)} = \int_\Omega \left[u^2 + \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x_i} \right)^2 \right] \mathrm{d}X\,.$$

Extending the product $(\boldsymbol{Lu}, \boldsymbol{v})$ according to (4), we derive

$$((\mathscr{B}\boldsymbol{u}, \boldsymbol{v})) = \int_\Omega \sum_{i,k=1}^{3} \left[\frac{\partial u_k}{\partial x_i} + (c_0 + \tfrac{1}{3}) \frac{\partial u_i}{\partial x_k} \right] \frac{\partial v_k}{\partial x_i}\, \mathrm{d}X\,,$$

where

$$((\boldsymbol{u}, \boldsymbol{v})) = \int_\Omega \sum_{i,k=1}^{3} \frac{\partial u_k}{\partial x_i} \frac{\partial v_k}{\partial x_i}\, \mathrm{d}X\,.$$

Consequently, $\mathscr{B}$ is defined by means of

$$\frac{\partial}{\partial x_i} (\mathscr{B}\boldsymbol{u})_k = \frac{\partial u_k}{\partial x_i} + (c_0 + \tfrac{1}{3}) \frac{\partial u_i}{\partial x_k}\,.$$

Evidently, $\mathscr{B} \in \mathscr{L}(V, V)$, because

$$\|\mathscr{B}\|^2 \leqq 2 \max\, [1, (c_0 + \tfrac{1}{3})^2]$$

and the inequality (5) holds. In fact, the operator $\mathscr{B}$ corresponds with that of isotropic elasticity with the Poisson's ratio

$$\nu = \tfrac{1}{2} - (2c_0 + \tfrac{2}{3})^{-1} .$$

Consequently, $-1 < \nu < \frac{1}{2}$ for $c_0 > 0$ and the quadratic form of the strain energy in terms of strain tensor components is positive definite. Then (5) follows from so called KORN's inequality (see [4], [5]). The operators A_0 and A_1 are defined by means of

$$A_0(t) = -K_0(t, t)\, L , \quad A_1(t, \tau) = -\frac{\partial}{\partial t} K_0(t, \tau)\, L ,$$

as follows from (14) by differentiation with respect to t and by comparison with (1). Then (8) and (9) can be verified for

$$\mathscr{A}_0(t) = -K_0(t, t)\, \mathscr{B} , \quad \mathscr{A}_1(t, \tau) = -\frac{\partial}{\partial t} K_0(t, \tau)\, \mathscr{B} .$$

Suppose $\boldsymbol{u}_0 \in [\mathring{W}_2^{(1)}(\Omega)]^3$, $\boldsymbol{f} \in L_2(I; [L_2(\Omega)]^3)$ are given and

$$\boldsymbol{F}(t) = \int_0^t \boldsymbol{f}(\tau)\, \mathrm{d}\tau .$$

The solution of the Cauchy problem $\mathscr{P}(u_0, f)$ is any function $\boldsymbol{u} \in L_2(I; [\mathring{W}_2^{(1)}(\Omega)]^3)$, satisfying (according to (12)) the relation

$$\int_0^T \int_\Omega \left\{ \sum_{i,k=1}^{3} \left[\frac{\partial u_k(t)}{\partial x_i} + (c_0 + \tfrac{1}{3}) \frac{\partial u_i(t)}{\partial x_k} - \int_0^t K_0(t, \tau) \left[\frac{\partial u_k(\tau)}{\partial x_i} + \right.\right.\right.$$
$$\left.\left.\left. + (c_0 + \tfrac{1}{3}) \frac{\partial u_i(\tau)}{\partial x_k} \right] \mathrm{d}\tau \right\} \frac{\partial \varphi_k(t)}{\partial x_i}\, \mathrm{d}X\, \mathrm{d}t =$$
$$= \int_0^T \int_\Omega \left\{ F_i(t)\, \varphi_i(t) + \sum_{i,k=1}^{3} \left(\frac{\partial u_{0k}}{\partial x_i} + (c_0 + \tfrac{1}{3}) \frac{\partial u_{0i}}{\partial x_k} \right) \frac{\partial \varphi_k(t)}{\partial x_i} \right\} \mathrm{d}X\, \mathrm{d}t$$

for every $\varphi \in L_2(I; [\mathscr{D}(\Omega)]^3)$.

In accordance with Definition 1, we shall consider the integral equation

$$\mathscr{B}(t)\, u(t) + \int_0^t \mathscr{K}_0(t, \tau)\, u(\tau)\, \mathrm{d}\tau = G(t) , \tag{15}$$

in $L_2(I; V)$, where $G(t) \in V$ is defined by means of the relation

$$((G(t), \varphi)) = ((\mathscr{B}(0)\, u_0, \varphi)) + \left(\int_0^t f(\tau)\, \mathrm{d}\tau, \varphi \right) \tag{15'}$$

for every $\varphi \in V$, $t \in I$.

Note that the norms $\|\mathscr{B}(t)\|$ are bounded on I. This may be concluded on the base of (5') (see Lemma 1 in [6]). Hence we can prove easily, that $G(t) \in L_2(I; V)$. In fact,

$$\|G(t)\| \leqq \mathscr{B}_1 \|u_0\| + T^{1/2} c^{-1} \left(\int_0^T |f(t)|^2 \, \mathrm{d}t \right)^{1/2},$$

where $\mathscr{B}_1$ is the upper bound of $\|\mathscr{B}(t)\|$, consequently

$$\int_0^T \|G(t)\|^2 \, \mathrm{d}t \leqq 2T \left(\mathscr{B}_1 \|u_0\|^2 + T c^{-2} \int_0^T |f(t)|^2 \, \mathrm{d}t \right). \tag{16}$$

2. THE PROBLEM OF THE SECOND ORDER AND THE CORRESPONDING INTEGRAL EQUATION

Consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} (C(t) \, u'(t)) + B_0(t) \, u'(t) + \int_0^t B_1(t, \tau) \, u'(\tau) \, \mathrm{d}\tau + + A_0(t) \, u(t) + \int_0^t A_1(t, \tau) \, u(\tau) \, \mathrm{d}\tau = f(t) \tag{17}$$

in H and the initial conditions

$$u(0) = u_0 \, , \quad u'(0) = v_0 \, . \tag{18}$$

Here $A_0(t)$, $A_1(t, \tau)$, $B_0(t)$, $B_1(t, \tau)$ and $C(t)$ are linear operators in H. Assume that three Hilbert spaces $V_A, V_B, V_C \subset H$, positive constants $\alpha, \beta, \gamma, c_1$ and operators $\mathscr{A}_0(t)$, $\mathscr{B}_0(t)$, $\mathscr{C}(t)$ exist such that

$$\|u\|_A \geqq \tfrac{1}{2}|u| \, , \quad \|u\|_B \geqq \beta|u| \, , \quad \|u\|_C \geqq \gamma|u| \, , \tag{19}$$

$$\begin{aligned} ((\mathscr{A}_0(t) \, u, v))_A &= (A_0(t) \, u, v) \quad \text{for} \quad u \in D_{A_0(t)} \subset V_A \, , \quad v \in V_A \, , \quad t \in I \, , \\ ((\mathscr{B}_0(t) \, u, v))_B &= (B_0(t) \, u, v) \quad \text{for} \quad u \in D_{B_0(t)} \subset V_B \, , \quad v \in V_B \, , \quad t \in I \, , \\ ((\mathscr{C}(t) \, u, v))_C &= (C(t) \, u, v) \quad \text{for} \quad u \in D_{C(t)} \subset V_C \, , \quad v \in V_C \, , \quad t \in I \, , \end{aligned} \tag{20}$$

$$\mathscr{A}_0(t) \in \mathscr{L}(V_A, V_A) \, , \quad \mathscr{B}_0(t) \in \mathscr{L}(V_B, V_B) \, , \quad \mathscr{C}(t) \in \mathscr{L}(V_C, V_C) \, , \tag{21}$$

$$((\mathscr{C}(t) \, u, u))_C \geqq c_1 \|u\|_C^2 \quad \text{for} \quad t \in I \, , \quad u \in V_C \quad \text{and} \tag{22}$$

$((\mathscr{C}(t) \, u, v))_C$ is bounded on I for every fixed $u, v \in V_C$.

Suppose that

$$V = V_A \cap V_B \cap V_C \tag{23}$$

is not empty nor restricted to zero element only. Let us define the norm in V by

$$\|u\|^2 = \|u\|_A^2 + \|u\|_B^2 + \|u\|_C^2$$

and suppose that

$$\|u\| \leqq c_2 \|u\|_C . \tag{24}$$

The latter inequality implies, in case of differential operators, that the operator C contains the spatial derivatives of the maximal order in the equation (17).

Choosing a fixed $u \in V$,

$$g(v) = ((\mathscr{A}_0(t)\, u, v))_A$$

is a continuous functional on V, because

$$|((\mathscr{A}_0(t)\, u, v))_A| \leqq \|\mathscr{A}_0(t)\|\ \|u\|_A\ \|v\| .$$

Therefore an operator $\mathscr{A}_{00}(t) \in \mathscr{L}(V, V)$ exists, such that

$$((\mathscr{A}_0(t)\, u, v))_A = ((\mathscr{A}_{00}(t)\, u, v)) \quad \text{for} \quad u, v \in V ,$$

where

$$\|\mathscr{A}_{00}(t)\, u\| = \sup_{\|v\|=1} |((\mathscr{A}_0(t)\, u, v))_A| \leqq \|\mathscr{A}_0(t)\|\ \|u\| ,$$

consequently

$$\|\mathscr{A}_{00}(t)\| \leqq \|\mathscr{A}_0(t)\| . \tag{25}$$

In the same way, we define the operators $\mathscr{B}_{00}(t) \in \mathscr{L}(V, V)$ and $\mathscr{C}_{00}(t) \in \mathscr{L}(V, V)$ on the base of operators $\mathscr{B}_0(t)$ and $\mathscr{C}(t)$, respectively.

Furthermore, let us assume that operators $\mathscr{A}_1(t, \tau) \in \mathscr{L}(V, V)$ and $\mathscr{B}_1(t, \tau) \in \in \mathscr{L}(V, V)$ exist for almost all $t, \tau \in I$, such that

$$\begin{aligned} ((\mathscr{A}_1(t, \tau)\, u, v)) &= (A_1(t, \tau)\, u, v) , \\ ((\mathscr{B}_1(t, \tau)\, u, v)) &= (B_1(t, \tau)\, u, v) \end{aligned} \tag{26}$$

holds for almost all $t, \tau \in I$, $t \geqq \tau$, $v \in V$, $u \in D_{A_1(t,\tau)} \cap V$ and $u \in D_{B_1(t,\tau)} \cap V$, respectively, $\mathscr{A}_1(t, \tau) = \mathscr{B}_1(t, \tau) = \Theta$ for $t < \tau$.

Let

$$\mathscr{A}_0(t) \in L_2(I; \mathscr{L}(V_A, V_A)) , \quad \mathscr{B}_0(t) \in L_2(I; \mathscr{L}(V_B, V_B)) . \tag{27}$$

Then by virtue of (25) and an analogous inequality

$$\mathscr{A}_{00}(t), \mathscr{B}_{00}(t) \in L_2(I; \mathscr{L}(V, V)) .$$

The norms $\|\mathscr{C}_{00}(t)\|$ are bounded on I. This follows from (21), (22) and an analogue of (25) (see Lemma 1 in [6]).

Moreover, let

(28) $$\mathscr{A}_1(t,\tau)\,,\quad \mathscr{B}_1(t,\tau) \in L_2(I\times I;\,\mathscr{L}(V,V))\,,$$
$$u_0,\, v_0 \in V\,,\quad f \in L_2(I;\,H)\,.$$

Define

$$\mathscr{A}_{01}(t,\tau) = \int_\tau^t \mathscr{A}_{00}(z)\,\mathrm{d}z \quad \text{for} \quad t \geqq \tau\,,\quad \mathscr{A}_{01}(t,\tau) = \Theta \quad \text{for} \quad t < \tau\,,$$

$$\mathscr{A}_2(t,\tau) = \int_\tau^t \mathscr{A}_1(z,\tau)\,\mathrm{d}z\,,\quad \mathscr{B}_2(t,\tau) = \int_\tau^t \mathscr{B}_1(z,\tau)\,\mathrm{d}z\,,$$

$$\mathscr{A}_3(t,\tau) = \int_\tau^t\left(\int_\tau^z \mathscr{A}_1(z,s)\,\mathrm{d}s\right)\mathrm{d}z\,.$$

Setting

$$\mathscr{K}_0(t,\tau) = \mathscr{B}_{00}(\tau) + \mathscr{B}_2(t,\tau) + \mathscr{A}_{01}(t,\tau) + \mathscr{A}_3(t,\tau)\,,$$

we obtain

(29) $$\mathscr{K}_0(t,\tau) \in L_2(I\times I;\,\mathscr{L}(V,V))\,.$$

This can be proved, using the following inequalities together with (25), (27) and (28)

$$\int_0^T\int_0^T \|\mathscr{B}_{00}(\tau)\|^2\,\mathrm{d}t\,\mathrm{d}\tau \leqq T\int_0^T \|\mathscr{B}_0(\tau)\|^2\,\mathrm{d}\tau\,,$$

$$\int_0^T\int_0^T \|\mathscr{A}_{01}(t,\tau)\|^2\,\mathrm{d}t\,\mathrm{d}\tau \leqq \int_0^T\int_0^T\left(\int_\tau^t \|\mathscr{A}_{00}(z)\|\,\mathrm{d}z\right)^2 \mathrm{d}t\,\mathrm{d}\tau \leqq T^3\int_0^T \|\mathscr{A}_0(z)\|^2\,\mathrm{d}z\,,$$

$$\|\mathscr{A}_3(t,\tau)\|^2 \leqq \left[\int_\tau^t\left(\int_\tau^z \|\mathscr{A}_1(z,s)\|\,\mathrm{d}s\right)\mathrm{d}z\right]^2 \leqq T^2\int_0^T\int_0^T \|\mathscr{A}_1(z,s)\|^2\,\mathrm{d}z\,\mathrm{d}s\,.$$

$\mathscr{B}_2(t,\tau)$ is quite analogous to $\mathscr{A}_2(t,\tau)$ in (10), therefore we have an analogue to (11′).

Definition 2. (Weak solution of the Cauchy problem.) *Let $D(I)$ denote the linear manifold of functions $\varphi \in L_2(I;V)$, for which $\varphi' \in L_2(I;V)$, $\varphi(T) = \Theta$. We say that a function u is a solution of the Cauchy problem $\mathscr{P}(u_0, v_0, f)$, if $u \in L_2(I;V)$, $u' \in \in L_2(I;V)$, $u(0) = u_0$ and*

(30) $$\int_0^T\left(\left(\mathscr{C}_{00}(t)\,u'(t) + \int_0^t \mathscr{K}_0(t,\tau)\,u'(\tau)\,\mathrm{d}\tau,\, \varphi(t)\right)\right)\mathrm{d}t =$$
$$= \int_0^T\left\{\left(\left(\mathscr{C}_{00}(0)\,v_0 - \mathscr{A}_{01}(t,0)\,u_0 - \int_0^t \mathscr{A}_2(t,\tau)\,u_0\,\mathrm{d}\tau,\, \varphi(t)\right)\right) + \left(\int_0^t f(\tau)\,\mathrm{d}\tau,\, \varphi(t)\right)\right\}\mathrm{d}t$$

holds for every $\varphi \in D(I)$.

Remark 4. Let the operators be "symmetric in convolutions" in the following sense

$$\int_0^T (C(t)\, u(t), v(T-t))\, \mathrm{d}t = \int_0^T (C(t)\, v(t), u(T-t))\, \mathrm{d}t\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (C(\tau)\, u(\tau), v(t-\tau))\, \mathrm{d}\tau\Big|_{t=T} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (C(\tau)\, v(\tau), u(t-\tau))\, \mathrm{d}\tau\Big|_{t=T}\,,$$

$$\int_0^t \left(B_0(\tau)\, u(\tau) + \int_0^\tau B_1(\tau, z)\, u(z)\, \mathrm{d}z\,, v(t-\tau)\right) \mathrm{d}\tau =$$

$$= \int_0^t \left(B_0(\tau)\, v(\tau) + \int_0^\tau B_1(\tau, z)\, v(z)\, \mathrm{d}z\,, \quad u(t-\tau)\right) \mathrm{d}\tau$$

for any $t \in I$ and let an analogous equation hold for $A_0(t)$ and $A_1(t, \tau)$. Then (30) means the condition of the stationary value of the functional ([3])

$$\mathscr{F}(u) = \int_0^T \Big\{\Big(\Big(\mathscr{C}_{00}(t)\, u'(t) + \int_0^t \Big(\mathscr{B}_{00}(\tau)\, u'(\tau) + \int_0^\tau \mathscr{B}_1(\tau, z)\, u'(z)\, \mathrm{d}z\Big) \mathrm{d}\tau +$$

$$+ \int_0^t \Big(\mathscr{A}_{00}(\tau)\, u(\tau) + \int_0^\tau \mathscr{A}_1(\tau, z)\, u(z)\, \mathrm{d}z\Big) \mathrm{d}\tau - 2\, \mathscr{C}_{00}(0)\, v_0 -$$

$$- \mathscr{B}_{00}(t)\, u_0 - \int_0^t \mathscr{B}_1(t, \tau)\, u_0\, \mathrm{d}\tau, u(T-t)\Big)\Big) - 2\Big(\int_0^t f(\tau)\, \mathrm{d}\tau, u(T-t)\Big)\Big\}\, \mathrm{d}t -$$

$$- ((u_0, \mathscr{C}_{00}(T)\, u(T)))\,,$$

if we insert

$$u(t) = u_0 + \int_0^t u'(\tau)\, \mathrm{d}\tau \tag{31}$$

and set $\delta u(T-t) = \varphi(t)$.

Remark 5. The relation (30) follows from (17) formally, if we integrate it on $(0, t)$ with respect to t, insert (31), multiply by φ in H, extend the result with the use of (20), (26) and integrate over I.

In accordance with Definition 2, we shall consider the integral equation

$$\mathscr{C}_{00}(t)\, u'(t) + \int_0^t \mathscr{K}_0(t, \tau)\, u'(\tau)\, \mathrm{d}\tau = G(t) \tag{32}$$

in $L_2(I; V)$, where $G(t) \in V$ is defined by means of the relation

(32')

$$((G(t), \varphi)) = \Big(\Big(\mathscr{C}_{00}(0)\, v_0 - \mathscr{A}_{01}(t, 0)\, u_0 - \int_0^t \mathscr{A}_2(t, \tau)\, u_0\, \mathrm{d}\tau, \varphi\Big)\Big) + \Big(\int_0^t f(\tau)\, \mathrm{d}\tau, \varphi\Big)$$

for every $\varphi \in V$ and $t \in I$. We can prove easily, that $G(t) \in L_2(I; V)$. In fact, the norms $\|\mathscr{C}_{00}(t)\|$ are bounded on I, therefore

$$\|\mathscr{C}_{00}(0)\, v_0\| \leqq C_1 \|v_0\|\,, \quad C_1 = \text{const.}$$

Further, using (25) and (27), we derive

$$\|\mathscr{A}_{01}(t, 0)\, u_0\| \leqq \|u_0\|\, t^{1/2} \left(\int_0^T \|\mathscr{A}_0(\tau)\|^2 \,\mathrm{d}\tau\right)^{1/2}$$

and similarly, with the use of (11') and (28), we obtain

$$\left\|\int_0^t \mathscr{A}_2(t, \tau)\, u_0 \,\mathrm{d}\tau\right\| \leqq \|u_0\|\, T\left(\int_0^T\int_0^T \|\mathscr{A}_1(t, \tau)\|^2 \,\mathrm{d}t\,\mathrm{d}\tau\right)^{1/2}.$$

Altogether, we have

$$\|G(t)\| \leqq C_1\|v_0\| + \|u_0\| \left[T^{1/2} \left(\int_0^T \|\mathscr{A}_0(t)\|^2 \,\mathrm{d}t\right)^{1/2} + \right. \tag{33}$$

$$\left. + T\left(\int_0^T\int_0^T \|\mathscr{A}_1(t, \tau)\|^2 \,\mathrm{d}t\,\mathrm{d}\tau\right)^{1/2}\right] + c^{-1} T^{1/2} \left(\int_0^T |f(t)|^2 \,\mathrm{d}t\right)^{1/2},$$

$$c = (\alpha^2 + \beta^2 + \gamma^2)^{1/2}\,,$$

consequently $G(t)$ is bounded and therefore square-integrable on I.

3. INTEGRAL EQUATION OF VOLTERRA'S TYPE IN $L_2(I, V)$

The main object of this section is the following

Theorem 1. *Let* $K(t, \tau) \in L_2(I \times I; \mathscr{L}(V, V))$, $F(t) \in L_2(I; V)$. *Then the integral equation*

$$U(t) - \int_0^t K(t, \tau)\, U(\tau) \,\mathrm{d}\tau = F(t) \tag{34}$$

has precisely one solution $U \in L_2(I; V)$. *This solution is determined by the formula*

$$U(t) = F(t) - \int_0^t \mathscr{G}(t, \tau)\, F(\tau) \,\mathrm{d}\tau\,, \tag{35}$$

where the resolvent kernel

$$\mathscr{G}(t, \tau) \in L_2(I \times I; \mathscr{L}(V, V))$$

is given by the series of iterated kernels

$$(36) \qquad \mathscr{G}(t, \tau) = - \sum_{n=1}^{\infty} K_n(t, \tau),$$

$$K_1(t, \tau) = K(t, \tau), \quad K_{n+1}(t, \tau) = \int_{\tau}^{t} K(t, z) K_n(z, \tau) \, dz,$$

$$(n = 1, 2, \ldots, \tau \leqq t),$$

which converges almost everywhere on $I \times I$. Moreover, it holds

$$(37) \qquad \int_0^T \|U(t)\|^2 \, dt \leqq 2\left(1 + \int_0^T \int_0^T \|\mathscr{G}(t, \tau)\|^2 \, dt \, d\tau\right) \int_0^T \|F(t)\|^2 \, dt .$$

Proof. The Fubini's theorem yields the existence of functions

$$\alpha^2(x) = \int_0^T \|K(x, z)\|^2 \, dz, \quad \beta^2(y) = \int_0^T \| K(z, y)\|^2 \, dz$$

for almost all $x \in I$ and $y \in I$, respectively, where $\alpha(x) \in L_2(I)$, $\beta(y) \in L_2(I)$. Let us set

$$(38) \qquad \int_0^T \alpha^2(x) \, dx = \int_0^T \beta^2(y) \, dy = \int_0^T \int_0^T \|K(t, \tau)\|^2 \, dt \, d\tau = N^2 .$$

We have the following estimates

$$(39) \qquad \|K_2(x, y)\|^2 \leqq \alpha^2(x) \beta^2(y),$$

$$\|K_{n+2}(x, y)\|^2 \leqq \alpha^2(x) \beta^2(y) h_n(x, y), \quad (n = 1, 2, \ldots)$$

where

$$h_1(x, y) = \int_y^x \alpha^2(z) \, dz, \quad h_{n+1}(x, y) = \int_y^x \alpha^2(z) h_n(z, y) \, dz, \quad (y \leqq x).$$

The formula

$$(40) \qquad h_n(x, y) = \frac{1}{n!} h_1^n(x, y), \quad (n = 1, 2, \ldots)$$

can be derived by induction and from (38)

$$0 \leqq h_1(x, y) \leqq N^2$$

follows. Then

$$h_n(x, y) \leqq \frac{1}{n!} N^{2n}$$

holds and inserting into (39), we obtain

$$\|K_{n+2}(x, y)\| \leqq \alpha(x) \beta(y) N^n/\sqrt{n!}, \quad (n = 0, 1, 2, \ldots).$$

Consequently

$$(41) \qquad \sum_{n=1}^{m} \|K_n(t, \tau)\| \leqq \|K(t, \tau)\| + \alpha(t)\, \beta(\tau) \sum_{n=0}^{\infty} N^n (n!)^{-1/2} = M(t, \tau)$$

holds for every finite m. The latter infinite series converges for any N, so that $M(t, \tau) \in L_2(I \times I)$. Consequently, the series of norms (41) converges for $m \to \infty$ almost everywhere on $I \times I$. Hence the series (36) converges almost everywhere to $\mathscr{G}(t, \tau) \in \mathscr{L}(V, V)$.

From the Lebesgue theorem ([7], III.6.16) it follows that

$$\lim_{m \to \infty} \int_0^T \int_0^T \|\sum_{n=1}^{m} K_n(t, \tau) - \mathscr{G}(t, \tau)\|^2 \, \mathrm{d}t \, \mathrm{d}\tau = 0$$

and

$$\mathscr{G}(t, \tau) \in L_2(I \times I;\ \mathscr{L}(V, V))$$

(if we define simply $\mathscr{M}(t, \tau) \in \mathscr{L}(V, V)$ as the usual multiplication of u by $M(t, \tau)$).

Next we shall derive the relation

$$(42) \qquad K(t, \tau) + \mathscr{G}(t, \tau) = \int_\tau^t K(t, z)\, \mathscr{G}(z, \tau) \, \mathrm{d}z$$

for almost every $t \in I$ and $\tau \in I$. In fact, choosing τ_0 such that $\beta(\tau_0) < \infty$, $M(t, \tau_0)$ is a majorant for the series

$$\sum_{n=1}^{\infty} \| K_n(t, \tau_0)\|$$

and belongs to $L_2(I)$. Consequently,

$$\sum_{n=1}^{\infty} K_n(t, \tau_0) = -\mathscr{G}(t, \tau_0)$$

in $L_2(I; \mathscr{L}(V, V))$ again by virtue of the Lebesgue theorem. Therefore we may write almost everywhere

$$\int_{\tau_0}^t K(t, z)\, \mathscr{G}(z, \tau_0) \, \mathrm{d}z = -\lim_{m \to \infty} \int_{\tau_0}^t \sum_{n=1}^{m} K(t, z)\, K_n(z, \tau_0) \, \mathrm{d}z =$$

$$= -\lim_{m \to \infty} \sum_{n=1}^{m} K_{n+1}(t, \tau_0) = K(t, \tau_0) - \sum_{n=1}^{\infty} K_n(t, \tau_0) = K(t, \tau_0) + \mathscr{G}(t, \tau_0)\,,$$

that is (42). Now inserting (35) into the left-hand side of (34), we obtain

$$F(t) - \int_0^t \mathscr{G}(t, \tau)\, F(\tau) \, \mathrm{d}\tau - \int_0^t K(t, \tau) \left[F(\tau) - \int_0^\tau \mathscr{G}(\tau, z)\, F(z) \, \mathrm{d}z \right] \mathrm{d}\tau =$$

$$= F(t) - \int_0^t [\mathscr{G}(t, \tau) + K(t, \tau)]\, F(\tau) \, \mathrm{d}\tau + \int_0^t \mathrm{d}\tau \int_0^\tau K(t, \tau)\, \mathscr{G}(\tau, z)\, F(z) \, \mathrm{d}z\,.$$

Applying the Fubini theorem ([7] III.11.9) to the last integral, we obtain

$$\int_0^t \mathrm{d}\tau \int_0^\tau K(t, \tau)\, \mathcal{G}(\tau, z)\, F(z)\, \mathrm{d}z = \int_0^t \mathrm{d}z \int_z^t K(t, \tau)\, \mathcal{G}(\tau, z)\, F(z)\, \mathrm{d}\tau =$$
$$= \int_0^t \left(\int_\tau^t K(t, z)\, \mathcal{G}(z, \tau)\, \mathrm{d}z \right) F(\tau)\, \mathrm{d}\tau .$$

Consequently, making use of (42), we can verify that (35) satisfies the equation (34) almost everywhere in I. The inequality (37) follows from (35), if we realize that

$$\|U(t)\|^2 \leqq 2\|F(t)\|^2 + 2 \int_0^T \|\mathcal{G}(t, \tau)\|^2\, \mathrm{d}\tau \int_0^T \|F(t)\|^2\, \mathrm{d}t$$

and use the properties of $F(t)$ and $\mathcal{G}(t, \tau)$.

It remains to prove the uniqueness of the solution of (34) in $L_2(I; V)$. Suppose that $v(t) \in L_2(I, V)$ satisfies the homogeneous equation (34) with $F(t) = \Theta$, and denote

$$\int_0^T \|v(t)\|^2\, \mathrm{d}t = \nu^2 .$$

Using the Schwartz-Cauchy inequality, we obtain successively

$$\|v(t)\|^2 \leqq \nu^2\, \alpha^2(t)\,, \quad \|v(t)\|^2 \leqq \nu^2\, \alpha^2(t) \int_0^t \alpha^2(\tau)\, \mathrm{d}\tau, \ldots$$
$$\|v(t)\|^2 \leqq \nu^2\, \alpha^2(t)\, h_n(t, 0)\,, \quad (n = 1, 2, \ldots)\,.$$

From (38) and (40) it follows that

$$h_n(t, 0) = \frac{1}{n!}\, h_1^n(t, 0) = \frac{1}{n!} \left[\int_0^t \alpha^2(\tau)\, \mathrm{d}\tau \right]^n \leqq N^{2n}/n!\,,$$

consequently

$$\int_0^T \|v(t)\|^2\, \mathrm{d}t \leqq (n!)^{-1}\, \nu^2 N^{2n} \int_0^T \alpha^2(t)\, \mathrm{d}t \leqq \nu^2 N^{2n+2}/n!\,.$$

As the last term converges to zero for $n \to \infty$, $v = \Theta$ in $L_2(I, V)$.

4. EXISTENCE AND UNIQUENESS THEOREM FOR THE PROBLEM OF THE FIRST ORDER

We shall need the auxiliary

Lemma 1. *There exists the inverse operator $\mathcal{B}^{-1}(t) \in \mathcal{L}(V, V)$ for $t \in I$ and the norms $\|\mathcal{B}^{-1}(t)\|$ are bounded on I.*

Proof. Let $w \in V$ be an arbitrary element and $t \in I$. The bilinear form $((\mathscr{B}(t)\, u, v))$ and the linear functional $((w, v))$ satisfy all the assumptions of the Lax-Milgram theorem in V, consequently there exists precisely one element $u \in V$ such that

$$((\mathscr{B}(t)\, u, v)) = ((w, v))$$

for every $v \in V$. Therefore $\mathscr{B}(t)\, u = w$. Making use of (5), we obtain

$$\beta\|u\| \leqq \|\mathscr{B}(t)\, u\|,$$

which yields

$$\|\mathscr{B}^{-1}(t)\, w\| \leqq \beta^{-1}\|w\|,$$

so that

$$\|\mathscr{B}^{-1}(t)\| \leqq \beta^{-1} \quad \text{for all} \quad t \in I. \tag{43}$$

Theorem 2. *Let* (3) *till* (9) *hold. Then there exists one and only one solution* u *of the problem* $\mathscr{P}(u_0, f)$ *and it holds*

$$\int_0^T \|u(t)\|^2 \,\mathrm{d}t \leqq c\left(\|u_0\|^2 + \int_0^T |f(t)|^2 \,\mathrm{d}t\right). \tag{44}$$

Proof. *Existence.* Let us consider the equation (15) and apply the operator $\mathscr{B}^{-1}(t)$ to it. Using Lemma 1, we obtain

$$u(t) - \int_0^t \mathscr{K}(t, \tau)\, u(\tau)\,\mathrm{d}\tau = F(t), \tag{45}$$

where

$$\mathscr{K}(t, \tau) = -\mathscr{B}^{-1}(t)\, \mathscr{K}_0(t, \tau) \in L_2(I \times I;\ \mathscr{L}(V, V)), \tag{46}$$
$$F(t) = \mathscr{B}^{-1}(t)\, G(t) \in L_2(I; V).$$

Consequently, we may apply Theorem 1 to obtain a solution $u \in L_2(I, V)$ of (45). Then u is a solution of the problem $\mathscr{P}(u_0, f)$. In fact, applying $\mathscr{B}(t)$ to (45), multiplying the result by φ in V, inserting (15′) and integrating over I, we derive (12).

Uniqueness. First we shall prove the following

Lemma 2. $L_2(I;\ D_{B(t)})$ *is dense in* $L_2(I; V)$.

Proof. The set $\mathscr{C}_0$ of continuous mappings $w(t)$ of I into V is dense in $L_2(I, V)$ (see e.g. Lemma IV.8.19 in [7]), therefore it suffices to prove the density of $L_2(I;\ D_{B(t)})$ in $\mathscr{C}_0$. Choose an arbitrary $t \in I$ and $w \in \mathscr{C}_0$. As $w(t) \in V$ and $D_{B(t)}$ is dense in V, a sequence $\{v_n(t)\} \subset D_{B(t)}$ exists such that

$$\lim_{n\to\infty} \|v_n(t) - w(t)\| = 0, \quad \|v_n(t)\| \leqq 2\|w(t)\|, \quad n = 1, 2, \ldots$$

Define in this way a sequence of functions $v_n(t) \in L_2(I;\, D_{B(t)})$. As the function $2w \in \in L_2(I, V)$ represents a majorant of the sequence $\{v_n(t)\}$ and $v_n(t)$ converge everywhere in I to $w(t)$, we may apply the Lebesgue theorem to obtain

$$\lim_{n\to\infty} \int_0^T \|v_n(t) - w(t)\|^2 \,\mathrm{d}t = 0$$

and the proof is complete.

Next let $u \in L_2(I; V)$ satisfy the equation

$$\int_0^T \left(\left(\mathscr{B}(t)\, u(t) + \int_0^t \mathscr{K}_0(t, \tau)\, u(\tau)\,\mathrm{d}\tau,\ \varphi(t) \right) \right) \mathrm{d}t = 0$$

for every $\varphi \in L_2(I;\, D_{B(t)})$. By virtue of Lemma 2, we have

$$\mathscr{B}(t)\, u(t) + \int_0^t \mathscr{K}_0(t, \tau)\, u(\tau)\,\mathrm{d}\tau = \Theta\,.$$

Applying also the inverse operator $\mathscr{B}^{-1}(t)$ and Lemma 1, we are led to the equation (45) with $F(t) = \Theta$; hence

$$\int_0^T \|u(t)\|^2 \,\mathrm{d}t = 0$$

according to (37), and the uniqueness of solution is proved.

It remains to prove the inequality (44). From Theorem 1, (37), (46), (43) and (16) it follows that

$$\int_0^T \|u(t)\|^2 \,\mathrm{d}t \leqq 2\left(1 + \int_0^T\int_0^T \|\mathscr{G}(t, \tau)\|^2 \,\mathrm{d}t\,\mathrm{d}\tau\right) \beta^{-2}\, 2T\left(\tilde{\mathscr{B}} \|u_0\|^2 + Tc^{-2} \int_0^T |f(t)|^2 \,\mathrm{d}t\right),$$

which is of the form (44).

5. EXISTENCE AND UNIQUENESS THEOREM FOR THE PROBLEM OF THE SECOND ORDER

In the present section we shall prove the following

Theorem 3. *Let* (19) *till* (24), (26), (27) *and* (28) *hold. Then there exists one and only one solution u of the problem* $\mathscr{P}(u_0, v_0, f)$ *and*

$$\text{(48)} \qquad \int_0^T (\|u(t)\|^2 + \|u'(t)\|^2)\,\mathrm{d}t \leqq c\left(\|u_0\|^2 + \|v_0\|^2 + \int_0^T |f(t)|^2 \,\mathrm{d}t\right)$$

holds.

Proof. *Existence*. There exists the inverse operator $\mathscr{C}_{00}^{-1}(t) \in \mathscr{L}(V, V)$ for any $t \in I$ and

$$\|\mathscr{C}_{00}^{-1}(t)\| \leqq c_1^{-1} \quad \text{for} \quad t \in I . \tag{49}$$

This follows from (21) and (22) in a way similar to the proof of Lemma 1. Let us consider the equation (32) and apply the operator $\mathscr{C}_{00}^{-1}(t)$ to it, We obtain the equation

$$u'(t) - \int_0^t \mathscr{K}(t, \tau)\, u'(\tau)\, \mathrm{d}\tau = F(t) , \tag{50}$$

where

$$\mathscr{K}(t, \tau) = -\mathscr{C}_{00}^{-1}(t)\, \mathscr{K}_0(t, \tau) \in L_2(I \times I; \mathscr{L}(V, V)) , \tag{51}$$

$$F(t) = \mathscr{C}_{00}^{-1}(t)\, G(t) \in L_2(I; V) . \tag{52}$$

By virtue of Theorem 1, a solution $u' \in L_2(I, V)$ of (50) exists. Setting

$$u(t) = u_0 + \int_0^t u'(\tau)\, \mathrm{d}\tau , \tag{53}$$

we obtain $u \in L_2(I; V)$, $u(0) = u_0$. Then u is a solution of the problem $\mathscr{P}(u_0, v_0, f)$. In fact, applying $\mathscr{C}_{00}(t)$ to (50), multiplying the result by $\varphi \in D(I)$ in V, inserting (32') and integrating over I, we derive (30).

Uniqueness. First we shall prove the following

Lemma 3. *The linear manifold $D(I)$ is dense in $L_2(I; V)$.*

Proof. As the set $\mathscr{C}_0$ of continuous mappings $v(t)$ of I into V is dense in $L_2(I; V)$, it will suffice to prove the density of $D(I)$ in $\mathscr{C}_0$. Let $v \in \mathscr{C}_0$ and let the real functions $\vartheta_n(t)$, $(n = 1, 2, \ldots)$, be defined as follows

$$\vartheta_n(t) = 1 \quad \text{for} \quad 0 \leqq t \leqq T - 2/n ,$$

$$\vartheta_n(t) = n(T - 1/n - t) \quad \text{for} \quad T - 2/n \leqq t \leqq T - 1/n ,$$

$$\vartheta_n(t) = 0 \quad \text{for} \quad T - 1/n \leqq t \leqq T .$$

Then the products $v_n(t) = \vartheta_n(t)\, v \in \mathscr{C}_0$ and

$$\int_0^T \|v_n(t) - v(t)\|^2\, \mathrm{d}t \leqq \int_{T-2/n}^T \|v(t)\|^2\, \mathrm{d}t \to 0 \tag{54}$$

for $n \to \infty$. Let us extend $v_n(t)$ continuously on the interval $I_1 = (-\delta, T + \delta)$ for a $\delta > 0$, so that

$$\bar{v}_n(t) = \Theta \quad \text{for} \quad T - 1/n \leqq t \leqq T + \delta\,,$$

denoting the extension of $v_n(t)$ by $\bar{v}_n(t)$. Let us regularize $\bar{v}_n$ by means of the function

$$\omega_h(x) = \begin{cases} \exp\left(\dfrac{x^2}{x^2 - h^2}\right) & \text{for} \quad |x| < h\,, \\ 0 & \text{for} \quad |x| \geqq h\,, \end{cases}$$

that is, we introduce

$$\bar{v}_{nh}(t) = \frac{1}{h\varkappa} \int_{-\delta}^{T+\delta} \omega_h(t - \tau)\, \bar{v}_n(\tau)\, \mathrm{d}\tau \quad \text{for} \quad t \in I\,, \quad h < \delta\,,$$

where

$$\varkappa = \int_{-1}^{1} \omega_1(x)\, \mathrm{d}x\,.$$

From there

(55) $$\int_{t-h}^{t+h} \omega_h(t - \tau)\, \mathrm{d}\tau = h\varkappa$$

follows. Then

$$\lim_{h \to 0} \bar{v}_{nh}(t) = \bar{v}_n(t)$$

uniformly on I. In fact, using (55), we may write (for $t \in I$)

(56) $$\|\bar{v}_{nh}(t) - \bar{v}_n(t)\| = \left\| \frac{1}{h\varkappa} \int_{t-h}^{t+h} \omega_h(t - \tau)\, [\bar{v}_n(\tau) - \bar{v}_n(t)]\, \mathrm{d}\tau \right\| \leqq$$
$$\leqq \frac{1}{h\varkappa} \int_{t-h}^{t+h} \|\bar{v}_n(\tau) - \bar{v}_n(t)\|\, \omega_h(t - \tau)\, \mathrm{d}\tau\,.$$

To any $\varepsilon > 0$, we can find $h > 0$, such that

(57) $$\|\bar{v}_n(\tau) - \bar{v}_n(t)\| < \varepsilon$$

holds for any pair of $t, \tau \in I_1$, satisfying $|t - \tau| < h$. Inserting (57) into (56), we obtain the uniform convergence, which implies

(58) $$\lim_{h \to \infty} \int_0^T \|\bar{v}_{nh}(t) - v_n(t)\|^2\, \mathrm{d}t = 0\,.$$

Obviously $\bar{v}_{nh}(T) = \Theta$ for all $n > 1/\delta$, $h < 1/n$. Further, $\bar{v}_{nh}(t)$ and $\bar{v}'_{nh}(t)$ belong to $\mathscr{C}_0$, as a consequence of the uniform continuity of $\omega_h(t)$ and $\mathrm{d}\omega_h(t)/\mathrm{d}t$ on any compact interval. Hence $\bar{v}_{nh}$, $\bar{v}'_{nh}$ belong to $L_2(I; V)$, if we restrict them again to the interval I.

Hence $\bar{v}_{nh} \in D(I)$. Finally, from (54) and (58), we obtain

$$\int_0^T \|\bar{v}_{nh}(t) - v(t)\|^2 \, dt \to 0$$

for $n \to \infty$, $h < 1/n$, and the Lemma is proved.

Next let $u, u' \in L_2(I; V)$ satisfy the equations

$$u(0) = \Theta\,,$$

$$\int_0^T \left(\left(\mathscr{C}_{00}(t)\, u'(t) + \int_0^t \mathscr{K}_0(t, \tau)\, u'(\tau)\, d\tau, \varphi(t)\right)\right) dt = 0$$

for every $\varphi \in D(I)$. By virtue of Lemma 3, we have

$$\mathscr{C}_{00}(t)\, u'(t) + \int_0^t \mathscr{K}_0(t, \tau)\, u'(\tau)\, d\tau = \Theta$$

and applying the inverse operator $\mathscr{C}_{00}^{-1}(t)$, we are led to the equation (50) with $F(t) = \Theta$, consequently

$$\int_0^T \|u'(t)\|^2 \, dt = 0$$

according to (37). Then we have also

$$\int_0^T \|u(t)\|^2 \, dt = \int_0^T \left\| \int_0^t u'(\tau)\, d\tau \right\|^2 dt \leqq \int_0^T t \, dt \int_0^T \|u'(\tau)\|^2 \, d\tau = 0\,.$$

In order to prove the inequality (48), we use (37), (52) and (33). Thus we obtain

$$\int_0^T \|u'(t)\|^2 \, dt \leqq 2\left(1 + \int_0^T\int_0^T \|\mathscr{G}(t, \tau)\|^2 \, dt\, d\tau\right) Tc_1^{-2} \,. \tag{59}$$

$$\cdot \left\{ C_1 \|v_0\| + \|u_0\| \left[T^{-1/2} \left(\int_0^T \|\mathscr{A}_0(t)\|^2 \, dt \right)^{1/2} + T \left(\int_0^T\int_0^T \|\mathscr{A}_1(t, \tau)\|^2 \, dt\, d\tau \right)^{1/2} \right] + \right.$$

$$\left. + c^{-1} T^{1/2} \left(\int_0^T |f(t)|^2 \, dt \right)^{1/2} \right\}^2 .$$

Finally, a similar inequality follows for $u(t)$ from (53). Adding the latter to (59), we obtain (48).

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Souhrn

EXISTENCE A JEDNOZNAČNOST ŘEŠENÍ CACHYOVY ÚLOHY PRO JEDNU TŘÍDU LINEÁRNÍCH INTEGRO-DIFERENCIÁLNÍCH ROVNIC

IVAN HLAVÁČEK

V teorii vazkopružnosti se vyskytují úlohy, které lze popsat integro-diferenciálními rovnicemi s počátečními podmínkami. Cílem tohoto článku je dokázat korektnost variační formulace jisté třídy úloh, zahrnující zmíněné fyzikální příklady. Teorie se omezuje na rovnice, které mají nejvyšší derivace podle prostorových souřadnic u členu s nejvyšší derivací podle času.

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