Aplikace matematiky

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Aplikace matematiky, Vol. 16 (1971), No. 2, 109–124

Persistent URL: `http://dml.cz/dmlcz/103335`

ON REISSNER'S VARIATIONAL THEOREM FOR BOUNDARY VALUES IN LINEAR ELASTICITY

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(Received February 18, 1970)

1. INTRODUCTION

E. Reissner suggested in [1] a variational theorem for the theory of elasticity, related closely to the well-known Trefftz method. The theorem says, that if the equations of equilibrium in terms of displacements are satisfied a priori by the admissible functions, then all boundary conditions follow from the stationarity of a certain functional as natural conditions. Similar method of approximate solution in elasticity were presented by D. Rüdiger [2]. In the shell theory, a variational principle, analogous to that of Reissner, was established by K. Ф. Черных [3].

In the present paper, we discuss the Reissner's theorem within the range of linear anisotropic and non-homogeneous elasticity. For the traction boundary-value problem the minimal property of the functional and the convergence of any minimizing sequence are proved, which is an extension of a result of [4]. For the displacement boundary-value problem, however, some modification is needed, enabling the Reisnner's theorem to remain in force. Then the maximal property of the functional on a modified class of admissible functions and the convergence of any maximizing sequence can be proved. For mixed problems with separate conditions in the normal and tangential directions to the boundary (see [5], [6]) some particular cases are shown, in which the kinematic boundary conditions do not follow as natural conditions, unless a modification similar to that of displacement boundary-value problem is accomplished. A general condition is established, that is necessary for the original assertion on the natural conditions without modification.

2. DEFINITION OF THE GENERAL BOUNDARY-VALUE PROBLEM

First let us introduce some preliminary definitions and notations. E_3 denotes the Euclidean three-dimensional space with Cartesian coordinates $X \equiv (x_1, x_2, x_3)$. We call a region $\Omega \subset E_3$ Lipschitz region, if it is bounded and its boundary Γ has the following properties: a) to each point $X \in \Gamma$ an open sphere S_X about X exists, such that the intersection $S_X \cap \Gamma$ may be described by means of a Lipschitz function, and b) $S_X \cap \Gamma$ divides S_X into exterior and interior parts with respect to Ω.

Let a Lipschitz region $\Omega \subset E_3$ be given. $L_2(\Omega)$ will denote the space of real functions which are square-integrable on Ω (in the Lebesgue-sense). $W_2^{(1)}(\Omega)$ denotes the subspace of $L_2(\Omega)$ consisting of functions, whose derivatives of the first order, in the sense of distributions, are in $L_2(\Omega)$. Let us introduce the norm on $W_2^{(1)}(\Omega)$ by means of

$$\|u\|^2_{W_2^{(1)}(\Omega)} = \int_\Omega (u^2 + u_{,i}u_{,i})\,\mathrm{d}X\,,$$

where $u_{,i} = \partial u/\partial x_i$ and a repeated suffix (excepting t or n) implies always summation over the range 1, 2, 3.

$L_2(\Gamma)$ denotes the space of real functions which are square-integrable on Γ. $[W_2^{(1)}(\Omega)]^3$ or $[L_2(\Gamma)]^3$ denotes the space of vector-functions each component of which belongs to $W_2^{(1)}(\Omega)$ or $L_2(\Gamma)$, respectively. The norm on $[W_2^{(1)}(\Omega)]^3$ is defined by means of

$$\|\boldsymbol{u}\|^2_{[W_2^{(1)}(\Omega)]^3} = \|u_j\|_{W_2^{(1)}(\Omega)}\,\|u_j\|_{W_2^{(1)}(\Omega)}\,.$$

The norm on $[L_2(\Gamma)]^3$ is analogous. Similarly, the space $[L_2(\Omega)]^3$ is defined. Let the body forces $\boldsymbol{K} \in [L_2(\Omega)]^3$ and the surface tractions $\boldsymbol{P} \in [L_2(\Gamma)]^3$ be prescribed and assume, that the surface displacements are given by means of a function $\boldsymbol{u}_0 \in$ $\in [W_2^{(1)}(\Omega)]^3$.

Suppose that the strain-displacement relations

$$\varepsilon_{ik} = \tfrac{1}{2}(u_{i,k} + u_{k,i})\,, \tag{1}$$

the stress-strain relations

$$\tau_{ik} = c_{iklm}\varepsilon_{lm} \tag{2}$$

and the stress equations of equilibrium

$$\tau_{ik,k} + K_i = 0 \tag{3}$$

hold on Ω. Here u_i, ε_{ik} and τ_{ik} designate respectively the rectangular Cartesian components of the displacement vector $\boldsymbol{u}$, the strain tensor ε and the stress tensor τ. The elastic coefficients $c_{iklm}(X)$ are assumed to be measurable and bounded on $\Omega \cap \Gamma$ and to satisfy the symmetry relations

$$c_{iklm} = c_{kilm} = c_{lmik}\,. \tag{4}$$

Moreover, we suppose that

$$c_{iklm}(X)\,\varepsilon_{ik}\varepsilon_{lm} \geqq \mu_0\varepsilon_{ik}\varepsilon_{ik} \tag{5}$$

for every symmetric tensor ε_{ik} at each point $X \in \Omega$ where a positive constant μ_0 is independent of X.

On the boundary Γ of the region Ω the boundary conditions are prescribed in the form of linear combinations of the displacement and surface traction components, thus

$$(6) \qquad A_n u_n + B_n T_n = C_n$$
$$A_t \boldsymbol{u}_t + B_t \boldsymbol{T}_t = \boldsymbol{C}_t ,$$

where the suffices n or t denote the components of vectors $\boldsymbol{u}$ and $\boldsymbol{T}$ with components $T_i = \tau_{ik} n_k$ in the direction of the unit outward normal n or of the tangential plane to Γ, respectively, that is

$$(7) \qquad u_n = u_k n_k , \qquad T_n = \tau_{ik} n_i n_k ,$$
$$u_{tj} = u_j - u_i n_i n_j , \quad T_{tj} = \tau_{jk} n_k - \tau_{ik} n_i n_k n_j .$$

The coefficients A_n, A_t are piecewise constant functions on Γ whose values are either 0 or 1, while B_n and B_t are bounded measurable functions defined almost everywhere on Γ, such that

$$(8) \qquad B_n \geqq \beta_n > 0 \quad \text{or} \quad B_n = 0 ,$$
$$B_t \geqq \beta_t > 0 \quad \text{or} \quad B_t = 0$$

with β_n, β_t constant. We suppose that

$$(9) \qquad A_n + B_n > 0 , \quad A_t + B_t > 0$$

holds for almost all[1]) $X \in \Gamma$. Let us introduce the following point sets:

$$\mathscr{A}_n = \{X \in \Gamma,\ B_n = 0\} , \quad \mathscr{B}_n = \{X \in \Gamma,\ B_n > 0\} , \quad \Gamma_n = \{X \in \Gamma,\ A_n B_n > 0\} ,$$
$$\mathscr{A}_t = \{X \in \Gamma,\ B_t = 0\} , \quad \mathscr{B}_t = \{X \in \Gamma,\ B_t > 0\} , \quad \Gamma_t = \{X \in \Gamma,\ A_t B_t > 0\} .$$

A set $G \subset \Gamma$ will be called open in Γ if for any point $X_0 \in G$ there exists an $\eta > 0$ such that each $X \in \Gamma$, for which $\operatorname{dist}(X, X_0) < \eta$, belongs to G. Here we denoted

$$\operatorname{dist}(X, X_0) = [(x_i - x_{0i})(x_i - x_{0i})]^{1/2} .$$

Suppose that the sets $\mathscr{A}_n, \mathscr{A}_t, \mathscr{B}_n, \mathscr{B}_t, \Gamma_n, \Gamma_t$ are either empty, or open in Γ. Furthermore, let the given vector-function $\boldsymbol{u}_0 \in [W_2^{(1)}(\Omega)]^3$ define functions $C_n \in L_2(\mathscr{A}_n)$ and $\boldsymbol{C}_t \in [L_2(\mathscr{A}_t)]^3$ on $\mathscr{A}_n$ and $\mathscr{A}_t$ by means of (6), that is, by means of the relations

$$u_{0n} = C_n \quad \text{on} \quad \mathscr{A}_n , \qquad \boldsymbol{u}_{0t} = \boldsymbol{C}_t \quad \text{on} \quad \mathscr{A}_t .$$

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[1]) That is, for $X \in \Gamma \dot{-} \mathscr{N}$, where $\mathscr{N}$ is a set of surface measure zero.

C_n is defined on $\mathscr{B}_n$ by means of the given surface traction $\mathbf{P}$, namely

$$C_n = B_n P_n \in L_2(\mathscr{B}_n)$$

and $\mathbf{C}_t$ is defined on $\mathscr{B}_t$ by means of

$$\mathbf{C}_t = B_t \mathbf{P}_t \in [L_2(\mathscr{B}_t)]^3 .$$

Inserting (1) and (2) into (3) and making use of the symmetry (4), we obtain the system of equilibrium equations in terms of displacements

$$(c_{iklm} u_{l,m})_{,k} + K_i = 0 . \tag{10}$$

3. REISSNER'S THEOREM "FOR BOUNDARY VALUES"

We shall introduce a set of vector-functions satisfying the homogeneous equation (10) (with $\mathbf{K} = \mathbf{O}$) in a sense which is suitable also for cases of discontinuous elasticity coefficients.

Definition 1. *Assume that the region Ω can be subdivided into a finite number of disjoint Lipschitz subregions Ω_j, such that $c_{iklm}(X)$ are continuous in every subregion and*

$$\Omega \cap \Gamma = \bar{\Omega} = \bigcup_{j=1}^{J} \bar{\Omega}_j , \quad \Omega_j \cap \Omega_h = \emptyset \text{ for } j \neq h .$$

Let $\mathbf{M}_0$ be the linear manifold of vector-functions $\mathbf{w}$, whose components are continuous in $\bar{\Omega}$, continuously differentiable in every $\bar{\Omega}_j = \Omega_j \cup \Gamma_j$, i.e., $w_i \in C^{(1)}(\bar{\Omega}_j)$ for $j = 1, 2, \ldots, J$, $w_i \in C^{(0)}(\bar{\Omega})$ and for which

$$\int_\Omega c_{iklm} w_{i,k} v_{l,m} \, \mathrm{d}X = \int_\Gamma c_{iklm} w_{l,m} n_k v_i \, \mathrm{d}\Gamma \tag{11}$$

holds for every $\mathbf{v} \in [W_2^{(1)}(\Omega)]^3$.

Remark 1. Obviously, (11) holds, if a function $\mathbf{w}$ satisfies (10) (with $\mathbf{K} = \mathbf{O}$) in every Ω_j and if

$$n_k[(c_{iklm} w_{l,m})(\Omega_j) - (c_{iklm} w_{l,m})(\Omega_h)] = 0$$

holds for the limits on the interregion boundaries of any two adjacent subregions, i.e., for almost all points

$$X \in \Gamma_{jh} = \bar{\Omega}_j \cap \bar{\Omega}_h , \quad j \neq h .$$

In fact, rewriting the left-hand side of (11) as a sum of integrals over all the subregions Ω_j and integrating by parts, the integrals on Γ_{jh} cancel out.

Theorem 1 (Reissner). *Let the traces of functions from* $\mathbf{M}_0$ *be dense in* $[L_2(\Gamma)]^3$. *Define the functional*

$$(12)\qquad \mathscr{R}(\mathbf{u}) = -\frac{1}{2}\int_\Omega K_i u_i \,\mathrm{d}X + \int_{\mathscr{A}_n} T_n(\mathbf{u})\,(u_{0n} - \tfrac{1}{2}u_n)\,\mathrm{d}\Gamma +$$

$$+ \int_{\mathscr{A}_t} \mathbf{T}_t(\mathbf{u})\,(\mathbf{u}_{0t} - \tfrac{1}{2}\mathbf{u}_t)\,\mathrm{d}\Gamma + \int_{\mathscr{B}_n \div \Gamma_n} u_n(\tfrac{1}{2}T_n(\mathbf{u}) - P_n)\,\mathrm{d}\Gamma + \int_{\mathscr{B}_t \div \Gamma_t} \mathbf{u}_t(\tfrac{1}{2}\mathbf{T}_t(\mathbf{u}) - \mathbf{P}_t)\,\mathrm{d}\Gamma +$$

$$+ \int_{\Gamma_n} \frac{u_n}{B_n}(\tfrac{1}{2}u_n + \tfrac{1}{2}B_n\,T_n(\mathbf{u}) - C_n)\,\mathrm{d}\Gamma + \int_{\Gamma_t} \frac{\mathbf{u}_t}{B_t}(\tfrac{1}{2}\mathbf{u}_t + \tfrac{1}{2}B_t\,\mathbf{T}_t(\mathbf{u}) - \mathbf{C}_t)\,\mathrm{d}\Gamma\,,$$

where $T_n(\mathbf{u})$, $\mathbf{T}_t(\mathbf{u})$ *are defined by means of* (1), (2) *and* (7), *the components of displacements* $u_i \in C^{(1)}(\bar{\Omega}_j)$, $j = 1, 2, \ldots, J$, $u_i \in C^{(0)}(\bar{\Omega})$ *and they satisfy the equations* (10) *in the following sense*:

$$(13)\qquad \int_\Omega c_{iklm} u_{l,m} v_{i,k}\,\mathrm{d}X = \int_\Gamma c_{iklm} u_{l,m} n_k v_i\,\mathrm{d}\Gamma + \int_\Omega K_i v_i\,\mathrm{d}X$$

holds for every $\mathbf{v} \in [W_2^{(1)}(\Omega)]^3$.

Then from the condition $\delta\mathscr{R}(\mathbf{u}) = 0$ *the boundary conditions on* $\mathscr{B}_n$ *and* $\mathscr{B}_t$ *follow as natural conditions.*

The boundary conditions on $\mathscr{A}_n$ *and* $\mathscr{A}_t$ *follow from there only if*

$$(14)\qquad \left\{\int_{\mathscr{A}_n} p_n\,T_n(\delta\mathbf{u})\,\mathrm{d}\Gamma + \int_{\mathscr{A}_t} \mathbf{p}_t\,\mathbf{T}_t(\delta\mathbf{u})\,\mathrm{d}\Gamma = 0 \quad \text{for every} \quad \delta\mathbf{u} \in \mathbf{M}_0\right\} \Rightarrow$$

$$\Rightarrow \begin{Bmatrix} p_n = 0 \text{ on } \mathscr{A}_n \text{ and} \\ \mathbf{p}_t = 0 \text{ on } \mathscr{A}_t \end{Bmatrix},$$

where $\mathbf{p} = \mathbf{a} + \mathbf{b} \times \mathbf{r}$, $\mathbf{a}$, $\mathbf{b}$ *denote constant vectors*, $\mathbf{r}$ *the radius vector and* $\times$ *the vector product.*

Proof. It is easy to derive

$$\delta\mathscr{R}(\mathbf{u}) = -\frac{1}{2}\int_{\mathscr{A}_n} K_i v_i\,\mathrm{d}X + \frac{1}{2}\int_\Gamma [u_n\,T_n(\mathbf{v}) + u_t\,\mathbf{T}_t(\mathbf{v}) - v_n\,T_n(\mathbf{u}) - \mathbf{v}_t\,\mathbf{T}_t(\mathbf{u})]\,\mathrm{d}\Gamma +$$

$$+ \int_{\mathscr{A}_n} (u_{0n} - u_n)\,T_n(\mathbf{v})\,\mathrm{d}\Gamma + \int_{\mathscr{B}_n \div \Gamma_n} (T_n(\mathbf{u}) - P_n)\,v_n\,\mathrm{d}\Gamma +$$

$$+ \int_{\Gamma_n} \left(T_n(\mathbf{u}) + \frac{u_n}{B_n} - \frac{C_n}{B_n}\right) v_n\,\mathrm{d}\Gamma + \int_{\mathscr{A}_t} (\mathbf{u}_{0t} - \mathbf{u}_t)\,\mathbf{T}_t(\mathbf{v})\,\mathrm{d}\Gamma +$$

$$+ \int_{\mathscr{B}_t \div \Gamma_t} (\mathbf{T}_t(\mathbf{u}) - \mathbf{P}_t)\,\mathbf{v}_t\,\mathrm{d}\Gamma + \int_{\Gamma_t} \left(\mathbf{T}_t(\mathbf{u}) + \frac{\mathbf{u}_t}{B_t} - \frac{\mathbf{C}_t}{B_t}\right) \mathbf{v}_t\,\mathrm{d}\Gamma\,.$$

Using the properties of $\boldsymbol{u}$ and $\boldsymbol{v}$, we obtain $(\boldsymbol{u}, \boldsymbol{v} \in [W_2^{(1)}(\Omega)]^3)$:

$$-\int_\Omega K_i v_i \,\mathrm{d}X + \int_\Gamma [u_i\, T_i(\boldsymbol{v}) - v_i T_i(\boldsymbol{u})] \,\mathrm{d}\Gamma = \int_\Omega [u_{i,k} c_{iklm} v_{l,m} - v_{i,k} c_{iklm} u_{l,m}] \,\mathrm{d}X = 0 .$$

From the assumption on the density of $\boldsymbol{M}_0$ in $[L_2(\Gamma)]^3$, we deduce the boundary conditions on $\mathscr{B}_n$ and $\mathscr{B}_t$ as natural ones.

Next let us satisfy the boundary conditions on $\mathscr{B}_n$ and $\mathscr{B}_t$. Then If (14) is not satisfied, there exists a vector $\boldsymbol{p}$ such that it holds at least one of the following two relations

$$p_n \neq 0 \text{ on } \mathscr{A}_n, \qquad \boldsymbol{p}_t \neq 0 \text{ on } \mathscr{A}_t$$

and simultaneously

$$\delta\mathscr{R}(\boldsymbol{u}) = \int_{\mathscr{A}_n} p_n\, T_n(\delta\boldsymbol{u}) \,\mathrm{d}\Gamma + \int_{\mathscr{A}_t} \boldsymbol{p}_t\, \boldsymbol{T}_t(\delta\boldsymbol{u}) \,\mathrm{d}\Gamma = 0$$

holds for every $\delta\boldsymbol{u} \in \boldsymbol{M}_0$. Hence the boundary conditions on $\mathscr{A}_n$ and $\mathscr{A}_t$ are satisfied except for a polynomial p_n and $\boldsymbol{p}_t$, respectively.

Remark 2. We can show several cases, for which the necessary condition (14) is not satisfied. The most important is the case of

α) the displacement boundary-value problem, when $\mathscr{A}_n = \mathscr{A}_t = \Gamma$. Then we have

$$\int_{\mathscr{A}_n} p_n\, T_n(\delta\boldsymbol{u}) \,\mathrm{d}\Gamma + \int_{\mathscr{A}_t} \mathbf{p}_t\, \boldsymbol{T}_t(\delta\boldsymbol{u}) \,\mathrm{d}\Gamma = \int_\Gamma p_i c_{iklm} \delta u_{l,m} n_k \,\mathrm{d}\Gamma =$$

$$= \int_\Omega p_{i,k} c_{iklm} \delta u_{l,m} \,\mathrm{d}X = \int_\Omega \tfrac{1}{2} c_{iklm} (p_{i,k} + p_{k,i})\, \delta u_{l,m} \,\mathrm{d}X = 0$$

for every $\delta\boldsymbol{u} \in \boldsymbol{M}_0$ and any vector $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{r}$ with arbitrary constant coefficients $\boldsymbol{a}$, $\boldsymbol{b}$. Hence (14) is violated.

β) Let Ω be a circular cylinder whose axis is identical with x_3 – axis, bounded by two planes $x_3 = c_1$, $x_3 = c_2$. Let

$$\boldsymbol{u}_t = \boldsymbol{u}_{0t}, \quad T_n = P_n$$

be prescribed almost everywhere on its boundary Γ. Consequently, $\mathscr{A}_n = \emptyset$, $\mathscr{A}_t = \Gamma$. Consider a vector $\boldsymbol{p} = b_3 \boldsymbol{k} \times \boldsymbol{r}$, where $\boldsymbol{k}$ denotes the unit vector of the positive x_3 – axis and b_3 is an arbitrary constant. We have

$$\int_{\mathscr{A}_t} \boldsymbol{p}_t\, \boldsymbol{T}_t(\delta\boldsymbol{u}) \,\mathrm{d}\Gamma = \int_\Gamma [p_i\, T_i(\delta\boldsymbol{u}) - p_n\, T_n(\delta\boldsymbol{u})] \,\mathrm{d}\Gamma =$$

$$= \int_\Omega p_{i,k} c_{iklm} \delta u_{l,m} \,\mathrm{d}X - \int_\Gamma p_n\, T_n(\delta\boldsymbol{u}) \,\mathrm{d}\Gamma = 0$$

for every $\delta \boldsymbol{u} \in \boldsymbol{M}_0$, because p_n vanishes almost everywhere on Γ. As $\boldsymbol{p}_t = \boldsymbol{p} = b_3 \boldsymbol{k} \times \boldsymbol{r}$ on $\mathscr{A}_t$, (14) is violated.

γ) Let Ω be the same cylinder as in the previous case. Denote the two plane bases by Γ_1 and the cylindrical surface by Γ_2. Let the following boundary conditions be prescribed:

$$u_n = u_{0n}, \quad \boldsymbol{T}_t = 0 \quad \text{on } \Gamma_1,$$
$$\boldsymbol{u}_t = \boldsymbol{u}_{0t}, \quad T_n = P_n \quad \text{on } \Gamma_2.$$

Consequently, $\mathscr{A}_n = \Gamma_1$, $\mathscr{A}_t = \Gamma_2$, $\mathscr{A}_n \cup \mathscr{A}_t = \Gamma$ (except for a set of surface measure zero). Let $\boldsymbol{p} = a_3 \boldsymbol{k}$, where a_3 is an arbitrary constant. Then we can write

$$\int_{\Gamma_1} p_n T_n(\delta \boldsymbol{u}) \, d\Gamma + \int_{\Gamma_2} \boldsymbol{p}_t \boldsymbol{T}_t(\delta \boldsymbol{u}) \, d\Gamma =$$
$$= \int_{\Gamma} p_i T_i(\delta \boldsymbol{u}) \, d\Gamma - \int_{\Gamma_1} \boldsymbol{p}_t \boldsymbol{T}_t(\delta \boldsymbol{u}) \, d\Gamma - \int_{\Gamma_2} p_n T_n(\delta \boldsymbol{u}) \, d\Gamma = 0$$

for every $\delta \boldsymbol{u} \in \boldsymbol{M}_0$, because $\boldsymbol{p}_t = \boldsymbol{O}$ on Γ_1 and $p_n = 0$ on Γ_2. At the same time, however, $p_n = \pm a_3$ on $\mathscr{A}_n$ and $\boldsymbol{p}_t = a_3 \boldsymbol{k}$ on $\mathscr{A}_t$, hence (14) is violated.

Remark 3. In Section 5, we shall suggest a modification of the Theorem 1 for the case (α), such that the boundary conditions are natural. A similar approach could be applied to the cases (β), (γ).

4. TRACTION BOUNDARY-VALUE PROBLEM

First let us analyse the important problem with tractions assigned on the whole boundary. Consequently, we have

$$\Gamma = \mathscr{B}_n = \mathscr{B}_t, \quad \mathscr{A}_n = \mathscr{A}_t = \emptyset.$$

Let the conditions of the total equilibrium of the body

$$\int_{\Omega} \boldsymbol{K} \, dX + \int_{\Gamma} \boldsymbol{P} \, d\Gamma = 0, \quad \int_{\Omega} \boldsymbol{r} \times \boldsymbol{K} \, dX + \int_{\Gamma} \boldsymbol{r} \times \boldsymbol{P} \, d\Gamma = 0$$

hold. Assume there exists a particular solution $\hat{\boldsymbol{u}}$ of the equations of equilibrium (10) in the sense of (13), with $\hat{u}_i \in C^{(0)}(\bar{\Omega}) \cap C^{(1)}(\bar{\Omega}_j)$. Let $\boldsymbol{M}_1 \subset \boldsymbol{M}_0$ be a linear manifold of vector-functions $\boldsymbol{w}$, for which

$$\int_{\Omega} \boldsymbol{w} \, dX = 0, \quad \int_{\Omega} \boldsymbol{r} \times \boldsymbol{w} \, dX = 0 \tag{15}$$

(or any equivalent conditions – see [6], Part II., Theorem II.1 and Lemma II.2) hold.

Let us introduce the scalar product on $[L_2(\Gamma)]^3$

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma u_i v_i \, \mathrm{d}\Gamma$$

and define the operator $\boldsymbol{A}$, mapping $\boldsymbol{M}_1$ into $[L_2(\Gamma)]^3$ by means of

$$(\boldsymbol{Au})_i = c_{iklm} u_{l,m} n_k \, .$$

Using (4) and (11), we can write

$$(\boldsymbol{Au}, \boldsymbol{v}) = \int_\Gamma c_{iklm} u_{l,m} n_k v_i \, \mathrm{d}\Gamma = \int_\Omega c_{iklm} u_{l,m} v_{i,k} \, \mathrm{d}X = (\boldsymbol{u}, \boldsymbol{Av}) \, ,$$

so that $\boldsymbol{A}$ is symmetric. Moreover, the following inequalities hold in $\boldsymbol{M}_1$

$$(\boldsymbol{Au}, \boldsymbol{u}) = \int_\Omega c_{iklm} \varepsilon_{ik}(\boldsymbol{u}) \, \varepsilon_{lm}(\boldsymbol{u}) \, \mathrm{d}X \geqq \mu_0 \int_\Omega \varepsilon_{ik}(\boldsymbol{u}) \, \varepsilon_{ik}(\boldsymbol{u}) \, \mathrm{d}X \geqq$$

$$\geqq C_1 \int_\Omega u_{i,k} u_{i,k} \, \mathrm{d}X \geqq C_2 \|\boldsymbol{u}\|^2_{[W_2^{(1)}(\Omega)]^3} \geqq C_3 |\boldsymbol{u}|^2_{[L_2(\Gamma)]^3} \, .$$

This is a consequence of the Korn's and Poincaré's inequalities and of the continuity of embedding of $W_2^{(1)}(\Omega)$ into $L_2(\Gamma)$ (cf. also [6], Theorem II.1). Hence the operator $\boldsymbol{A}$ is positive definite in $\boldsymbol{H} = [L_2(\Gamma)]^3$. Completing $\boldsymbol{M}_1$ by means of the associated norm

$$(\boldsymbol{Au}, \boldsymbol{u}) = |\boldsymbol{u}|^2_A = [\boldsymbol{u}, \boldsymbol{u}] \, ,$$

a new Hilbert space $\boldsymbol{H}_A$ with the scalar product $[\boldsymbol{u}, \boldsymbol{v}]$ arises, such that $\boldsymbol{H}_A \subset [W_2^{(1)}(\Omega)]^3$ and

(16) $$|\boldsymbol{u}|_A \geqq C_4 \|\boldsymbol{u}\|_{[W_2^{(1)}(\Omega)]^3} \geqq C_5 |\boldsymbol{u}|_{[L_2(\Gamma)]^3}$$

holds for every $\boldsymbol{u} \in \boldsymbol{H}_A$ (see [7]).

Let us seek a function $\mathring{\boldsymbol{w}} \in \boldsymbol{H}_A$ such that $\mathring{\boldsymbol{u}} = \hat{\boldsymbol{u}} + \mathring{\boldsymbol{w}}$ satisfies the equations of equilibrium and the conditions on the boundary in the following sense:

(17) $$\boldsymbol{v} \in \boldsymbol{M}_1 \Rightarrow \int_\Omega c_{iklm} \mathring{u}_{i,k} v_{l,m} \, \mathrm{d}X = \int_\Omega K_i v_i \, \mathrm{d}X + \int_\Gamma P_i v_i \, \mathrm{d}\Gamma \, .$$

The relation (17) may be rewritten as follows

(18) $$\boldsymbol{v} \in \boldsymbol{M}_1 \Rightarrow \int_\Omega c_{iklm} \mathring{w}_{i,k} v_{l,m} \, \mathrm{d}X = \int_\Gamma (P_i - c_{iklm} \hat{u}_{l,m} n_k) \, v_i \, \mathrm{d}\Gamma \, .$$

By virtue of

$$P_i - c_{iklm} \hat{u}_{l,m} n_k \in L_2(\Gamma) \, ,$$

on the right-hand side of (18), there is a functional continuous on $\boldsymbol{H}_A$, because of (16). On the left-hand side of (18), we have the scalar product $[\mathring{\boldsymbol{w}}, \boldsymbol{v}]$. From the Riesz theorem and the density of $\boldsymbol{M}_1$ in $\boldsymbol{H}_A$, we deduce that there exists one and only one element $\mathring{\boldsymbol{w}} \in \boldsymbol{H}_A$, satisfying (17).

Lemma 1. *The functional*

$$\mathscr{L}(\boldsymbol{u}) = \frac{1}{2}\int_\Omega c_{iklm} u_{i,k} u_{l,m} \, \mathrm{d}X - \int_\Omega K_i u_i \, \mathrm{d}X - \int_\Gamma P_i u_i \, \mathrm{d}\Gamma \tag{19}$$

attains its minimum on the set $\hat{\boldsymbol{u}} \oplus \boldsymbol{H}_A$, if and only if

$$\|\boldsymbol{u} - \mathring{\boldsymbol{u}}\|_{[W_2^{(1)}(\Omega)]^3} = 0 \,,$$

where $\mathring{\boldsymbol{u}}$ is the solution defined by means of (17).

Proof. Define on $\boldsymbol{H}_A$ the functional

$$\mathscr{F}(\boldsymbol{w}) = [\boldsymbol{w}, \boldsymbol{w}] - 2[\mathring{\boldsymbol{w}}, \boldsymbol{w}] = |\boldsymbol{w} - \mathring{\boldsymbol{w}}|_A^2 - |\mathring{\boldsymbol{w}}|_A^2 \geqq - |\mathring{\boldsymbol{w}}|_A^2 \,. \tag{20}$$

Inserting with respect to (18)

$$[\mathring{\boldsymbol{w}}, \boldsymbol{w}] = \int_\Gamma (P_i - c_{iklm}\hat{u}_{l,m} n_k) \, w_i \, \mathrm{d}\Gamma$$

and $w_i = u_i - \hat{u}_i$, we obtain

$$\mathscr{F}(\boldsymbol{w}) = \mathscr{F}(\boldsymbol{u} - \hat{\boldsymbol{u}}) =$$

$$= \int_\Omega c_{iklm}(u - \hat{u})_{i,k} \, (u - \hat{u})_{l,m} \, \mathrm{d}X - 2\int_\Gamma (P_i - c_{iklm}\hat{u}_{l,m} n_k) \, (u_i - \hat{u}_i) \, \mathrm{d}\Gamma =$$

$$= \int_\Omega c_{iklm} u_{i,k} u_{l,m} \, \mathrm{d}X - 2\int_\Omega c_{iklm} \hat{u}_{i,k} u_{l,m} \, \mathrm{d}X - 2\int_\Gamma (P_i - c_{iklm}\hat{u}_{l,m} n_k) \, u_i \, \mathrm{d}\Gamma + F_1(\hat{\boldsymbol{u}}) \,.$$

Using (13), we can write

$$\mathscr{F}(\boldsymbol{u} - \hat{\boldsymbol{u}}) = \int_\Omega c_{iklm} u_{i,k} u_{l,m} \, \mathrm{d}X - 2\int_\Omega K_i u_i \, \mathrm{d}X - 2\int_\Gamma P_i u_i \, \mathrm{d}\Gamma + F_1(\hat{\boldsymbol{u}}) \,.$$

Defining

$$\mathscr{L}(\boldsymbol{u}) = \tfrac{1}{2}[\mathscr{F}(\boldsymbol{u} - \hat{\boldsymbol{u}}) - F_1(\hat{\boldsymbol{u}})]$$

and making use of (20), we obtain the assertion to be proved.

Remark 4. The functional (19) coincides with that of potential energy, so that the Lemma expresses a restriction of the principle of minimum potential energy.

Next let us consider the functional $\mathscr{F}(\boldsymbol{w})$ only on the set $\boldsymbol{M}_1$. Then we may write

$$[\boldsymbol{w}, \boldsymbol{w}] = \int_\Gamma c_{iklm} w_{l,m} n_k w_i \, \mathrm{d}\Gamma$$

and consequently

$$\mathscr{F}(\boldsymbol{w}) = \mathscr{F}(\boldsymbol{u} - \hat{\boldsymbol{u}}) =$$

$$= \int_\Gamma c_{iklm}(u - \hat{u})_{l,m} \, n_k(u - \hat{u})_i \, \mathrm{d}\Gamma - 2\int_\Gamma (P_i - c_{iklm}\hat{u}_{l,m}n_k)(u - \hat{u})_i \, \mathrm{d}\Gamma =$$

$$= \int_\Gamma [c_{iklm}n_k(u_{l,m}u_i - u_{l,m}\hat{u}_i + \hat{u}_{l,m}u_i) - 2P_iu_i] \, \mathrm{d}\Gamma + \mathscr{F}_2(\hat{\boldsymbol{u}}) .$$

Using (11) for $\boldsymbol{w}$ and (13) for $\hat{\boldsymbol{u}}$, we obtain

$$\int_\Gamma c_{iklm}n_k(\hat{u}_{l,m}u_i - u_{l,m}\hat{u}_i) \, \mathrm{d}\Gamma = \int_\Gamma c_{iklm}n_k[\hat{u}_{l,m}(\hat{u}_i + w_i) - (\hat{u} + w)_{l,m} \, \hat{u}_i] \, \mathrm{d}\Gamma =$$

$$= \int_\Gamma c_{iklm}n_k[\hat{u}_{l,m}w_i - w_{l,m}\hat{u}_i] \, \mathrm{d}\Gamma = -\int_\Omega K_iw_i \, \mathrm{d}X = -\int_\Omega K_i(u_i - \hat{u}_i) \, \mathrm{d}X$$

and

$$\mathscr{F}(\boldsymbol{u} - \hat{\boldsymbol{u}}) = \int_\Gamma (c_{iklm}n_ku_{l,m}u_i - 2P_iu_i) \, \mathrm{d}\Gamma - \int_\Omega K_iu_i \, \mathrm{d}X + \mathscr{F}_3(\hat{\boldsymbol{u}}) .$$

Comparison with (12) leads to the relation

$$\mathscr{R}(\boldsymbol{u}) = \tfrac{1}{2}[\mathscr{F}(\boldsymbol{u} - \hat{\boldsymbol{u}}) - \mathscr{F}_3(\hat{\boldsymbol{u}})] .$$

From (20) and (16) we obtain the following

Theorem 2. *Let a sequence* $\{\boldsymbol{u}_n\}_{n=1}^{\infty}$, $\boldsymbol{u}_n \in \hat{\boldsymbol{u}} \oplus \boldsymbol{M}_1$ *be such that*

$$\lim_{n \to \infty} \mathscr{R}(\boldsymbol{u}_n) = \min_{\hat{u} \oplus H_A} \mathscr{R}(\boldsymbol{u})$$

Then

$$\lim_{n \to \infty} \|\boldsymbol{u}_n - \mathring{\boldsymbol{u}}\|_{[W_2^{(1)}(\Omega)]^3} = 0$$

where $\mathring{\boldsymbol{u}} \in \hat{\boldsymbol{u}} \oplus \boldsymbol{H}_A$ *is defined by means of* (17).

In other words, every sequence from $\hat{\boldsymbol{u}} \oplus \boldsymbol{M}_1$, minimizing the functional $\mathscr{R}(\boldsymbol{u})$ on $\hat{\boldsymbol{u}} \oplus \boldsymbol{H}_A$, converges to the solution (defined in the sense of (17)) in $[W_2^{(1)}(\Omega)]^3$.

Remark 5. Theorem 2 is an extension of an earlier result [4]. The proof is based on the "method of minimal surface integrals" as was presented in [8], § 47.

5. DISPLACEMENT BOUNDARY-VALUE PROBLEM

Let us consider the second particular boundary-value problem, when the displacements are prescribed on the whole boundary Γ. Then we have $\Gamma = \mathscr{A}_n = \mathscr{A}_t$ (except of a set of surface measure zero). Let $[\mathring{W}_2^{(1)}(\Omega)]^3$ denote the subspace of $[W_2^{(1)}(\Omega)]^3$ of vector-functions, whose components vanish on the boundary (in the sense of traces).

The weak solution of the problem (see [6] II.) is defined as a vector-function $\mathring{\boldsymbol{u}}$ such that

a) $\mathring{\boldsymbol{u}} - \boldsymbol{u}_0 \in [\mathring{W}_2^{(1)}(\Omega)]^3$,

b) $\boldsymbol{v} \in [\mathring{W}_2^{(1)}(\Omega)]^3 \Rightarrow \int_\Omega c_{iklm} \mathring{u}_{i,k} v_{l,m} \, \mathrm{d}X = \int_\Omega K_i v_i \, \mathrm{d}X$.

Denote by $T_0 = T(\mathring{\boldsymbol{u}})$ the stress tensor with components

$$\tau_{ik}(\mathring{\boldsymbol{u}}) = c_{iklm} \tfrac{1}{2}(\mathring{u}_{l,m} + \mathring{u}_{m,l}) = c_{iklm} \mathring{u}_{l,m} .$$

Let us recall also the principle of minimum complementary energy [5]: The quadratic functional

$$\tilde{\mathscr{S}}(T) = \int_\Omega (\tfrac{1}{2} a_{iklm} \tau_{ik} \tau_{lm} - \tau_{ik} u_{0i,k}) \, \mathrm{d}X$$

attains its minimum on the set of tensor-functions with all components $\tau_{ik} \in L_2(\Omega)$, which satisfy the equations of equilibrium in the following sense

$$\boldsymbol{v} \in [\mathring{W}_2^{(1)}(\Omega)]^3 \Rightarrow \int_\Omega \tau_{ik} v_{i,k} \, \mathrm{d}X = \int_\Omega K_i v_i \, \mathrm{d}X , \tag{21}$$

if and only if

$$\|T - T_0\|^2 = \int_\Omega [\tau_{ik} - \tau_{ik}(\mathring{\boldsymbol{u}})] [\tau_{ik} - \tau_{ik}(\mathring{\boldsymbol{u}})] \, \mathrm{d}X = 0 .$$

Consider $T = T(\boldsymbol{u})$, $\boldsymbol{u} = \hat{\boldsymbol{u}} + \boldsymbol{w}$, $\boldsymbol{w} \in \boldsymbol{M}_0$, where $\hat{\boldsymbol{u}}$ is the particular solution introduced in Section 4 and $\boldsymbol{M}_0$ the linear manifold according to the Definition 1. It is easy to deduce that $T(\boldsymbol{u})$ satisfy (21), and therefore $T(\boldsymbol{u})$ are admissible fields in the principle of minimum complementary energy.

Lemma 2. *Let* $\{\boldsymbol{w}_n\}_{n=1}^\infty$, $\boldsymbol{w}_n \in \boldsymbol{M}_0$ *be a sequence such that*

$$\lim_{n\to\infty} \tilde{\mathscr{S}}(T(\boldsymbol{u}_n)) = \tilde{\mathscr{S}}(T_0)$$

holds for $\boldsymbol{u}_n = \hat{\boldsymbol{u}} + \boldsymbol{w}_n$. *Then*

$$\lim_{n\to\infty} \|T(\boldsymbol{u}_n) - T_0\| = 0 .$$

Proof. Denoting $\mathring{\boldsymbol{w}} = \mathring{\boldsymbol{u}} - \boldsymbol{u}_0$ and

$$|T|^2 = \int_\Omega a_{iklm}\tau_{ik}\tau_{lm}\,\mathrm{d}X\,,$$

we may write (see [5])

$$(22) \qquad {}^{\sim}\mathscr{S}(T) = \tfrac{1}{2}(|T - T(\boldsymbol{u}_0)|^2 - |T(\boldsymbol{u}_0)|^2) = \tfrac{1}{2}(|T - T_0|^2 + |T(\mathring{\boldsymbol{w}})|^2 - |T(\boldsymbol{u}_0)|^2)$$

and

$$|T(\boldsymbol{u}_n) - T_0|^2 \geqq C\|T(\boldsymbol{u}_n) - T_0\|^2\,,$$

which is a consequence of (5).

Hence the Lemma follows immediately.

If we restrict the functional ${}^{\sim}\mathscr{S}$ to the set $\hat{\boldsymbol{u}} \oplus \boldsymbol{M}_0$, it may be rewritten as follows

$$(23) \qquad -{}^{\sim}\mathscr{S}(T(\boldsymbol{u})) = \int_\Omega (-\tfrac{1}{2}c_{iklm}u_{i,k}u_{l,m} + c_{iklm}u_{l,m}u_{0i,k})\,\mathrm{d}X =$$

$$= \int_\Omega K_i(u_0 - \tfrac{1}{2}u)_i\,\mathrm{d}X + \int_\Gamma c_{iklm}u_{l,m}n_k(u_0 - \tfrac{1}{2}u)_i\,\mathrm{d}\Gamma = \mathscr{R}(\boldsymbol{u}) + \int_\Omega K_i u_{0i}\,\mathrm{d}X\,,$$

where $\mathscr{R}(\boldsymbol{u})$ is the appropriate Reissner's functional "for boundary values". Making use of Lemma 2, we derive

Theorem 3. *Let* $\{\boldsymbol{w}_n\}_{n=1}^{\infty}$, $\boldsymbol{w}_n \in \boldsymbol{M}_0$ *be a sequence such that*

$$\lim_{n\to\infty} \mathscr{R}(\boldsymbol{u}_n) = -{}^{\sim}\mathscr{S}(T_0) - \int_\Omega K_i u_{0i}\,\mathrm{d}X$$

holds for $\boldsymbol{u}_n = \hat{\boldsymbol{u}} + \boldsymbol{w}_n$. *Then*

$$\lim_{n\to\infty} \|T(\boldsymbol{u}_n) - T_0\| = \boldsymbol{O}\,.$$

In other words, for every sequence from $\hat{\boldsymbol{u}} \oplus \boldsymbol{M}_0$, maximizing the functional $\mathscr{R}(\boldsymbol{u})$' the corresponding stress components converge in the mean to the components of stress of the weak solution.

Let us recall the fact, that the boundary condition does not follow from the stationary value of the functional $\mathscr{R}(\boldsymbol{u})$, as was shown in Remark 2 and Theorem 1. In order to remove this defect, let us restrict the linear manifold $\boldsymbol{M}_0$ to $\boldsymbol{M}_2$, by a requirement, that

$$\boldsymbol{M}_2 \subset \boldsymbol{M}_0\,, \quad \boldsymbol{w} \in \boldsymbol{M}_2 \Rightarrow \int_{\Gamma^*} \boldsymbol{w}\,\mathrm{d}\Gamma = \boldsymbol{O}\,, \quad \int_{\Gamma^*} \boldsymbol{r} \times \boldsymbol{w}\,\mathrm{d}\Gamma = \boldsymbol{O}\,,$$

where $\Gamma^* \subset \Gamma$ is an arbitrary open part of the boundary Γ (in particular $\Gamma^* = \Gamma$).

Denote

$$\mathscr{P} = \{\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{r};\ \text{with } \boldsymbol{a}, \boldsymbol{b} \text{ arbitrary constant vectors}\}$$

and

$$\mathbf{V}_p = \left\{ \mathbf{u} \in [W_2^{(1)}(\Omega)]^3 \,, \int_{\Gamma^*} \mathbf{u} \, \mathrm{d}\Gamma = \mathbf{O}, \int_{\Gamma^*} \mathbf{r} \times \mathbf{u} \, \mathrm{d}\Gamma = \mathbf{O} \right\}. \tag{24}$$

$\mathbf{V}_p$ is a subspace of $[W_2^{(1)}(\Omega)]^3$. Let us introduce in $\mathbf{V}_p$ the scalar product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{V}_p} = \int_\Omega c_{iklm} u_{i,k} v_{l,m} \, \mathrm{d}X \,.$$

This definition can be justified by means of (1), (4), (5) and the implication (see [6], Lemma II.2 and eq. (18))

$$|\mathbf{u}|_{\mathbf{V}_p} = \mathbf{O} \Rightarrow \mathbf{u} \in \mathscr{P} \cap \mathbf{V}_p \Rightarrow \mathbf{u} = \mathbf{O} \,.$$

$[\mathring{W}_2^{(1)}(\Omega)]^3$ is a subspace of $\mathbf{V}_p$. In fact, the Korn's inequality ([6], Theorem II.1) yields

$$|\mathbf{u}_n - \mathbf{u}|_{\mathbf{V}_p}^2 \geqq C \|\mathbf{u}_n - \mathbf{u}\|_{[W_2^{(1)}(\Omega)]^3}^2$$

for $\mathbf{u}_n \in [\mathring{W}_2^{(1)}(\Omega)]^3$, consequently the limit of any sequence $\{\mathbf{u}_n\}_{n=1}^\infty$ belongs to $[\mathring{W}_2^{(1)}(\Omega)]^3$.

Remark 6. We may always suppose that $\hat{\mathbf{u}} - \mathbf{u}_0 \in \mathbf{V}_p$, because to any particular solution $\hat{\mathbf{u}}'$ there exists a vector-polynomial $\mathbf{p} = \mathbf{a} + \mathbf{b} \times \mathbf{r}$ such that $\hat{\mathbf{u}} = \hat{\mathbf{u}}' + \mathbf{p}$ is also a particular solution (in the sense of (13))and $\hat{\mathbf{u}} - \mathbf{u}_0 \in \mathbf{V}_p$. In fact, denote concisely the system

$$\int_{\Gamma^*} \mathbf{p} \, \mathrm{d}\Gamma = \mathbf{O} \,, \quad \int_{\Gamma^*} \mathbf{r} \times \mathbf{p} \, \mathrm{d}\Gamma = \mathbf{O} \tag{25}$$

by $p_i(\mathbf{p}) = 0$ $(i = 1, \ldots, 6)$ or $A\alpha = \mathbf{O}$, respectively, where $\alpha = (a_1, a_2, a_3, b_1, b_2, b_3)^T$. As from (25) $\alpha = \mathbf{O}$ follows, $\det |A| \neq 0$ and consequently, the system $A\alpha = \mathbf{c}$ has a solution $\alpha(\mathbf{c})$ for $c_i = -p_i(\hat{\mathbf{u}}' - \mathbf{u}_0)$. Then

$$p_i(\mathbf{p}) + p_i(\hat{\mathbf{u}}' - \mathbf{u}_0) = p_i(\hat{\mathbf{u}}' + \mathbf{p} - \mathbf{u}_0) = 0$$

holds for $\mathbf{p}$ with coefficients $\alpha(\mathbf{c})$.

Theorem 4. *Let $\hat{\mathbf{u}} - \mathbf{u}_0 \in \mathbf{V}_p$. Let $\{\mathbf{w}_n\}_{n=1}^\infty$, $\mathbf{w}_n \in \mathbf{M}_2$ be a sequence such that*

$$\lim_{n \to \infty} \mathscr{R}(\mathbf{u}_n) = -\tilde{\ }\mathscr{S}(T_0) - \int_\Omega K_i u_{0i} \, \mathrm{d}X$$

holds for $\mathbf{u}_n = \hat{\mathbf{u}} + \mathbf{w}_n$. Then

$$\lim_{n \to \infty} \|\mathbf{u}_n - \mathring{\mathbf{u}}\|_{[W_2^{(1)}(\Omega)]^3} = 0 \,.$$

Proof. We have $\boldsymbol{u}_n - \mathring{\boldsymbol{u}} \in \boldsymbol{V}_p$ and

(26) $$|\boldsymbol{u}|_{\boldsymbol{V}_p} = |T(\boldsymbol{u})| .$$

Using (22), (23) and the Korn's inequality for $\boldsymbol{u}_n - \mathring{\boldsymbol{u}}$, the assertion of the Theorem follows.

Denote

$$\boldsymbol{R} = \boldsymbol{V}_p \ominus [\mathring{W}_2^{(1)}(\Omega)]^3$$

the orthogonal complement of $[\mathring{W}_2^{(1)}(\Omega)]^3$ by means of the scalar product in $\boldsymbol{V}_p$. Then $\boldsymbol{M}_2 \subset \boldsymbol{R}$, because $\boldsymbol{M}_2 \subset \boldsymbol{V}_p$ and (11) yields

$$\boldsymbol{w} \in \boldsymbol{M}_2 , \quad \boldsymbol{v} \in [\mathring{W}_2^{(1)}(\Omega)]^3 \Rightarrow (\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{V}_p} = 0 .$$

Lemma 3. *The closure of* $\boldsymbol{M}_2$ *in* $\boldsymbol{V}_p$ *is equal to* $\boldsymbol{R}$, *i.e.*,

$$\overline{\boldsymbol{M}}_2^{\boldsymbol{V}_p} = \boldsymbol{R} ,$$

if and only if for every $\boldsymbol{u}_0 \in [W_2^{(1)}(\Omega)]^3$ *a particular solution* $\hat{\boldsymbol{u}}$ *and a sequence* $\{\boldsymbol{w}_n\}_{n=1}^{\infty}$, $\boldsymbol{w}_n \in \boldsymbol{M}_2$ *exists, such that the sequence* $\{\boldsymbol{u}_n\}_{n=1}^{\infty}$, $\boldsymbol{u}_n = \hat{\boldsymbol{u}} + \boldsymbol{w}_n$, *maximizes the corresponding functional* $\mathscr{R}(\boldsymbol{u}, \boldsymbol{u}_0)$.

Proof. Let us choose an arbitrary element $\varrho \in \boldsymbol{R}$. The sum $\hat{\boldsymbol{u}} + \varrho$ represents a weak solution of the elasticity problem with the boundary condition $\boldsymbol{u} = \boldsymbol{u}_0 = \hat{\boldsymbol{u}} + \varrho$ on Γ. Let the sequence $\{\hat{\boldsymbol{u}} + \boldsymbol{w}_n\}$, $\boldsymbol{w}_n \in \boldsymbol{M}_2$, maximize the functional $\mathscr{R}(\boldsymbol{u}, \boldsymbol{u}_0)$, where $\boldsymbol{u}_0 = \hat{\boldsymbol{u}} + \varrho$. Denote

$$T_0 = T(\hat{\boldsymbol{u}} + \varrho) = T(\hat{\boldsymbol{u}}) + T(\varrho)$$

the stress tensor corresponding with the displacement field $\hat{\boldsymbol{u}} + \varrho$. The tensors $T(\hat{\boldsymbol{u}} + \boldsymbol{w})$, with $\boldsymbol{w} \in \boldsymbol{M}_2$, are admissible in the principle of minimum complementary energy. Using (22), (23), (26) with $\mathring{\boldsymbol{u}} = \hat{\boldsymbol{u}} + \varrho$, $\boldsymbol{u}_n = \hat{\boldsymbol{u}} + \boldsymbol{w}_n$, we obtain

$$|T(\boldsymbol{u}_n) - T_0|^2 = |T(\boldsymbol{w}_n) - T(\varrho)|^2 = |T(\boldsymbol{w}_n - \varrho)|^2 = |\boldsymbol{w}_n - \varrho|_{\boldsymbol{V}_p}^2 \to 0 .$$

Hence $\boldsymbol{R} = \overline{\boldsymbol{M}}_2^{\boldsymbol{V}_p}$.

Conversely, let $\boldsymbol{R} = \overline{\boldsymbol{M}}_2^{\boldsymbol{V}_p}$ hold and an arbitrary $\boldsymbol{u}_0 \in [W_2^{(1)}(\Omega)]^3$ be given. According to Remark 6, we choose $\hat{\boldsymbol{u}}$ such that $\boldsymbol{u}_0 - \hat{\boldsymbol{u}} \in \boldsymbol{V}_p$. Put $\boldsymbol{u}_0 = \hat{\boldsymbol{u}} + \varrho + \boldsymbol{v}$, where $\varrho \in \boldsymbol{R}$, $\boldsymbol{v} \in [\mathring{W}_2^{(1)}(\Omega)]^3$. Denote again

$$\mathring{\boldsymbol{u}} = \hat{\boldsymbol{u}} + \varrho , \quad T_0 = T(\hat{\boldsymbol{u}}) + T(\varrho) .$$

For the element ϱ a sequence $\{\boldsymbol{w}_n\} \in \boldsymbol{M}_2$ exists such that

$$|\boldsymbol{w}_n - \varrho|_{\boldsymbol{V}_p} \to 0 .$$

By virtue of (22), (23) and (26), we have

$$\mathscr{R}(\boldsymbol{u}_n, \boldsymbol{u}_0) = -\tilde{\mathscr{S}}(T(\boldsymbol{u}_n)) - \int_\Omega K_i u_{0i} \, dX =$$

$$= -\tfrac{1}{2}(|T(\boldsymbol{u}_n) - T_0|^2 + |T(\boldsymbol{v})|^2 - |T(\boldsymbol{u}_0)|^2) - \int_\Omega K_i u_{0i} \, dX ,$$

$$|T(\boldsymbol{u}_n) - T_0| = |T(\boldsymbol{w}_n - \varrho)| = |\boldsymbol{w}_n - \varrho|_{\boldsymbol{V}_p} .$$

The sequence $\{\hat{\boldsymbol{u}} + \boldsymbol{w}_n\}$ maximizes the functional $\mathscr{R}(\boldsymbol{u}, \boldsymbol{u}_0)$ and the proof is complete.

Theorem 5. *Let $\hat{\boldsymbol{u}}$ be such that $\boldsymbol{u}_0 - \hat{\boldsymbol{u}} \in \boldsymbol{V}_p$. Then from the condition*

$$\delta\mathscr{R}(\boldsymbol{u}, \boldsymbol{u}_0) = 0$$

on the set $\hat{\boldsymbol{u}} \oplus \boldsymbol{M}_2$, the boundary condition $\boldsymbol{u} = \boldsymbol{u}_0$ on Γ follows as natural condition, if and only if to any function $\boldsymbol{u}_0 \in [W_2^{(1)}(\Omega)]^3$ a particular solution $\hat{\boldsymbol{u}}$ and a sequence $\{\boldsymbol{w}_n\}_{n=1}^\infty$, $\boldsymbol{w}_n \in \boldsymbol{M}_2$ exists such that the sequence $\{\hat{\boldsymbol{u}} + \boldsymbol{w}_n\}_{n=1}^\infty$ maximizes the corresponding functional $\mathscr{R}(\boldsymbol{u}, \boldsymbol{u}_0)$.

Proof. On the set $\hat{\boldsymbol{u}} \oplus \boldsymbol{M}_2$ we have

$$\delta\mathscr{R}(\boldsymbol{u}, \boldsymbol{u}_0) = \int_\Gamma c_{iklm} \delta u_{l,m} n_k (u_{0i} - u_i) \, d\Gamma .$$

Denote $\delta \boldsymbol{u} = \boldsymbol{w} \in \boldsymbol{M}_2$. It holds $\boldsymbol{u} - \boldsymbol{u}_0 \in \boldsymbol{V}_p$ and

$$\delta\mathscr{R}(\boldsymbol{u}, \boldsymbol{u}_0) = \int_\Omega c_{iklm} w_{l,m} (u_0 - u)_{i,k} \, dX = (\boldsymbol{w}, \boldsymbol{u}_0 - \boldsymbol{u})_{\boldsymbol{V}_p} .$$

Let $\delta\mathscr{R}(\boldsymbol{u}, \boldsymbol{u}_0) = 0$, consequently $(\boldsymbol{u}_0 - \boldsymbol{u}, \boldsymbol{w})_{\boldsymbol{V}_p} = 0$ for every $\boldsymbol{w} \in \boldsymbol{M}_2$. Using Lemma 3 and the assumption on the existence of maximizing sequence, we obtain $\overline{\boldsymbol{M}}_2^{\boldsymbol{V}_p} = \boldsymbol{R}$, consequently $(\boldsymbol{u}_0 - \boldsymbol{u}, \varrho)_{\boldsymbol{V}_p} = 0$ for all $\varrho \in \boldsymbol{R}$. Hence $\boldsymbol{u} - \boldsymbol{u}_0 \in [\mathring{W}_2^{(1)}(\Omega)]^3$ follows, which is equivalent to the relation $\boldsymbol{u} = \boldsymbol{u}_0$ on Γ in the sense of traces.

Conversely, let from the zero variation the boundary condition follow, i.e., let

$$(27) \qquad (\boldsymbol{u}_0 - \boldsymbol{u}, \boldsymbol{w})_{\boldsymbol{V}_p} = 0 \quad \text{for all} \quad \boldsymbol{w} \in \boldsymbol{M}_2 \Rightarrow \boldsymbol{u} - \boldsymbol{u}_0 = 0 \quad \text{on} \quad \Gamma .$$

Suppose that $\boldsymbol{M}_2$ is not dense in $\boldsymbol{R}$. Then a nonzero subspace $\boldsymbol{N}_0 \subset \boldsymbol{R}$ exists such that

$$\boldsymbol{R} = \overline{\boldsymbol{M}}_2 \oplus \boldsymbol{N}_0 , \quad \boldsymbol{V}_p = \overline{\boldsymbol{M}}_2 \oplus \boldsymbol{N}_0 \oplus [\mathring{W}_2^{(1)}(\Omega)]^3 .$$

From the condition

$$(\boldsymbol{u}_0 - \boldsymbol{u}, \boldsymbol{w})_{\boldsymbol{V}_p} = 0 \quad \text{for all} \quad \boldsymbol{w} \in \boldsymbol{M}_2$$

only $\boldsymbol{u} - \boldsymbol{u}_0 \in \boldsymbol{N}_0 \oplus [\mathring{W}_2^{(1)}(\Omega)]^3$ follows. Hence it may hold $\boldsymbol{u} - \boldsymbol{u}_0 \in \boldsymbol{N}_0$, $\boldsymbol{u} - \boldsymbol{u}_0 \neq 0$, i.e., $\boldsymbol{u} - \boldsymbol{u}_0 \notin [\mathring{W}_2^{(1)}(\Omega)]^3$, consequently $\boldsymbol{u} - \boldsymbol{u}_0 \neq \boldsymbol{O}$ on Γ, which is a contradiction to (27). Hence $\boldsymbol{M}_2$ is dense in $\boldsymbol{R}$ and Lemma 3 yields the existence of a maximizing sequence for any $\boldsymbol{u}_0$, which completes the proof.

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Souhrn

O REISSNEROVĚ VARIAČNÍ VĚTĚ PRO OKRAJOVÉ PODMÍNKY V LINEÁRNÍ TEORII PRUŽNOSTI

IVAN HLAVÁČCK

E. Reissner navrhl v práci [1] variační větu v teorii pružnosti, odpovídající známé Trefftzově metodě. Věta tvrdí, že jsou-li rovnice rovnováhy v posunutích splněny a priori přípustnými funkcemi, pak všechny okrajové podmínky vyplývají ze stacionární hodnoty jistého funkcionálu jako přirozené podmínky. V tomto článku je dán rozbor Reissnerovy věty v oblasti lineární anisotropní a nehomogenní pružnosti. V případě povrchového zatížení na celém povrchu tělesa se dokazuje minimální vlastnost funkcionálu a konvergence každé minimizující posloupnosti. V případě posunutí daných na celém povrchu je však k zachování platnosti Reissnerovy věty zapotřebí jisté modifikace třídy přípustných funkcí. Pak lze dokázat maximální vlastnost funkcionálu a konvergenci každé maximizující posloupnosti. Pro smíšené okrajové úlohy, s oddělenými podmínkami ve směru normály a tečné roviny k povrchu tělesa, jsou ukázány některé případy, pro které Reissnerova věta rovněž neplatí, není-li příslušným způsobem modifikována. Článek obsahuje též jistou obecnou podmínku, která je nutná k tomu, aby Reissnerova věta platila v původním tvaru bez modifikace.

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