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Boris Gruber

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ON THE POSSIBILITY OF APPLYING A COMPUTER WHEN SOLVING THE FOUR COLOUR PROBLEM

BORIS GRUBER

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To my father
on the occasion of his 80th birthday

1. GEOGRAPHICAL NOTIONS

A *country* is a set of points in a euclidean plane which is homeomorphic with a closed circle area (having a positive finite diameter). A finite set M of countries is called a *map*, if the following is fulfilled:

1. M consists of at least two countries;
2. two arbitrary countries of M never have an interior point in common;
3. none point of the plane belongs to four countries of M;
4. the union of all countries of M (the *domain* of M) is homeomorphic with a closed circle area.

As it is well known point 3 does not mean any restriction of the problem. As a matter of fact, the same may be said about point 4.[1])

The number of countries of a map M is called its *degree*; symbol $\deg M$. The countries Z_1, Z_2 of a map are called *adjacent*, if they are neither identical nor disjoint. We say that the set U is a *boundary segment* (in short: *segment* only) of the map M, if

1. it is a subset of the boundary of the domain of M;
2. it is a subset of a country of M;
3. it is connected;
4. $U' \subset U$ is true provided that U' satisfies points 1, 2, 3.

[1]) For a *given n*, however, point 4 *does* mean a restriction. But the above proposition is to be understood in this sense. If we have a map M consisting of n countries which does not satisfy point 4, we consider instead of M a new map M' consisting of n' ($n' > n$) countries which has originated from M by "changing the interior seas into countries". Then M' fulfils point 4 and we apply the algorithm to the number n'.

The number of the boundary segments of a map M is called its *order*; symbol ord M. Obviously

$$1 < \operatorname{ord} M \leqq 2(\deg M - 1) \tag{1.1}$$

for any map M.

Every boundary segment is a subset of one and only one country of the map. We say that it *belongs* to this country. If U_1, U_2 are boundary segments of the same map, then they are either

1. identical or
2. disjoint or
3. have one or two points in common.

In the last case they are called *adjacent*. Such segments always belong to adjacent countries but the opposite is not true. In any map there exists at least one country to which only one boundary segment belongs.

We say that the map M of the order m is *numbered*, if an integer j $(1 \leqq j \leqq m)$ is associated with any boundary segment of M in such a way that this mapping is one-to-one and that to any pair of adjacent segments either the numbers i, $i + 1$ $(1 \leqq i < m)$ or the numbers m, 1 correspond. If the integer j is associated with a particular segment, we speak simply of the segment j.

Two numberings of a map are called *accordant*, of one of them may be obtained from the other by a cyclic permutation

$$\begin{pmatrix} 1 & 2 & \dots & m - p + 1 & m - p + 2 & \dots & m \\ p & p + 1 & \dots & m & 1 & \dots & p - 1 \end{pmatrix} \quad (1 \leqq p \leqq m) \tag{1.2}$$

All numberings of a map the order of which is greater than 2 may be divided into two classes of mutually accordant numberings. We speak following the common convention of *positive* (i.e. counterclockwise) and *negative* numberings. The numbering of a second order map is considered both positive and negative.

We say that the map M' has originated from the map M by *annexation*, if

$$M \subset M', \quad \deg M' = \deg M + 1 .$$

The set $M' - M$ consists of one (the co-called *annexed*) country. If h denotes the number of the boundary segments of the map M which have points in common with the annexed country, then

$$\begin{aligned} h &= \operatorname{ord} M & &\text{for} \quad \operatorname{ord} M' = 2 , \\ h &= \operatorname{ord} M - \operatorname{ord} M' + 3 & &\text{for} \quad \operatorname{ord} M' > 2 . \end{aligned} \tag{1.3}$$

This is immediately seen. Thus we have

$$1 < \operatorname{ord} M' \leqq \operatorname{ord} M + 2$$

provided that the map M' has originated from the map M by annexation. This is obvious for ord $M' = 2$; the other cases follow from (1.3) since $h \geqq 1$. On the other hand if M is an arbitrary map and the integer i fulfils

$$1 < i \leqq \operatorname{ord} M + 2 \tag{1.4}$$

then there exists a map M' of the order i which has originated from M by annexation.

Let the map M' have originated from the map M by annexation. We say that these maps are numbered in a *corresponding* way, if both of them are numbered in a negative way and one of the three following cases occurs:

I. $\operatorname{ord} M' = 2$ or $\operatorname{ord} M' = 3$;

II. a) $3 < \operatorname{ord} M' < \operatorname{ord} M + 2$,

b) the annexed country has points in common with the segments $\operatorname{ord} M' - 1, \ldots$ $\ldots, \operatorname{ord} M, 1$ of the map M,

c) the number $\operatorname{ord} M'$ is associated with that boundary segment of M' which belongs to the annexed country;

III. a) $\operatorname{ord} M' = \operatorname{ord} M + 2$,

b) the annexed country has points in common with the segment 1 of the map M,

c) the number $\operatorname{ord} M'$ is associated with that boundary segment of M' which belongs to the annexed country.

If a map M' has originated from the map M by annexation then they always may be numbered in a corresponding way. On the other hand if M is a negatively numbered map and the integer i fulfils the inequalities (1.4) then a map M' of the order i exists which has originated from M by annexation and may be numbered in a corresponding way with M.

If M is a map of the degree n, then every map which has originated from it by annexation has the degree $n + 1$. The contrary is true as well. If M' is an arbitrary map of the degree $n + 1$ $(n > 1)$, then there exists a map M of the degree n such that M' has originated from M by annexation. It suffices namely to take away from M' that country into which only one boundary segment belongs.

A non-void set of maps is called an *atlas*. If $\mathbf{A}$ is an atlas, then the symbol $[\mathbf{A}]$ denotes the atlas defined like this: $M' \in [\mathbf{A}]$ holds if and only if such a map $M \in \mathbf{A}$ exists that M' has originated from M by annexation. We introduce two particular sequences of atlases

$$\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \ldots \tag{1.5}$$

$$\mathbf{A}'_2, \mathbf{A}'_3, \mathbf{A}'_4, \ldots \tag{1.6}$$

Here $\mathbf{A}_i$ $(i > 1)$ designates the set of all maps of the degree i and $\mathbf{A}'_i$ the set of all maps of the degree smaller or equal to i. Thus we have

$$\mathbf{A}'_i = \mathbf{A}_2 \cup \mathbf{A}_3 \cup \ldots \cup \mathbf{A}_i \tag{1.7}$$

$$\mathbf{A}'_i \subset \mathbf{A}'_{i+1} \tag{1.8}$$

$$\mathbf{A}_{i+1} = [\mathbf{A}_i] \tag{1.9}$$

for any integer $i > 1$.

We say that a map is *regularly stained by at most four colours* (in short: *stained* only) if it is decomposed into at most four classes such that none of them contains two adjacent countries. The countries belonging to the same class are said to be *stained by the same colour.*

Let the map M' have originated from the map M by annexation, let M, M' be stained. We say that they are stained in a *corresponding* way if the following is true: Any two countries Z_1, $Z_2 \in M$ are stained by the same colour in the map M if and only if they are stained by the same colour in the map. M'.

We say that two boundary segments of a stained map are stained by the *same colour*, if the countries to which they belong are stained by the same colour. Otherwise they are said to be stained by *different colours*. Thus two adjacent segments of a stained map are always stained by different colours.

When staining a map we define a decomposition in the set of all its boundary segments. Two segments belong to the same class of this decomposition if and only if they are stained by the same colour. The map is said to be stained in a *reduced* way, if the above decomposition does not contain more than 3 classes.

A map is called *regular*, if it may be stained in a reduced way. It is called *singular*, if it cannot be regularly stained by at most four colours. In all other cases it is called *semisingular*. Such a map can be stained, but not in a reduced way.

We say that an atlas $\boldsymbol{A}$ is *regular*, if every map of it is regular. We say that $\boldsymbol{A}$ is *singular*, if it contains at least one singular map. In the remaining cases $\boldsymbol{A}$ is called *semi-singular*. A semi-singular atlas does not contain any singular map, but it contains at least one semi-singular map.

Let the map M' have originated from the map M by annexation. Then the following obviously holds: If M is singular, then M' is singular; if M is regular, M' is not singular; if M is semi-singular, then a singular map M'' exists which has originated from M by annexation. (It suffices for example if M'' is of the second order.)

This necessitates the following for the atlases (1.5): If $\boldsymbol{A}_i$ is singular, then $\boldsymbol{A}_{i+1}$ is singular; if $\boldsymbol{A}_i$ is regular, $\boldsymbol{A}_{i+1}$ is not singular; if $\boldsymbol{A}_i$ is semi-singular, $\boldsymbol{A}_{i+1}$ is singular. Thus in the sequence (1.5) one of these two cases occurs:

1. Every atlas $\boldsymbol{A}_i$ $(i > 1)$ is regular.
2. There exists such an integer k $(k > 3)$ that the atlases $\boldsymbol{A}_2, \ldots, \boldsymbol{A}_{k-1}$ are regular, the atlas $\boldsymbol{A}_k$ is semi-singular, the atlasses $\boldsymbol{A}_{k+1}, \boldsymbol{A}_{k+2}, \ldots$ are singular.

To ascertain which of these two cases actually occurs means to solve the four colour problem.

The aim of this paper is more modest. We introduce an algorithm which enables one to state for any integer $n > 4$ whether every map consisting of n countries may be regularly stained by at most four colours or not. The negative answer for a particular n would mean, of course, that the solution of the four colour problem was found.

A decomposition K of the set of integers

$$(2.0) \qquad 1, 2, \ldots, m \quad (m > 1)$$

is called a *flower*, if

1. it consists of at most four classes;
2. in any class there is neither a pair of integers of the form i, $i + 1$ $(1 \leqq i < m)$ nor the pair m, 1.

If this decomposition consists of just four classes, we say that the flower K is *gay*. The number m is called the *order* of the flower K; symbol $\operatorname{ord} K$. Performing the cyclic permutation (1.2) on the set (2.0) the flower K changes into a new flower which will be denoted K^p.

Let M be a numbered and stained map of the order m. According to paragraph 1 the staining of M determines a decomposition of the set of all its boundary segments. Since M is numbered, this decomposition is transferred to the set of integers (2.0) forming a flower of the order m which will be denoted $\varkappa(M)$. It is gay if and only if the map M is not stained in a reduced way.

We say that the flower K' has originated from the flower K by *pollination*, if one of the following three cases occurs:

I. a) $\operatorname{ord} K' = 2$ or $\operatorname{ord} K' = 3$,
  b) the flower K is not gay;

II. a) $3 < \operatorname{ord} K' < \operatorname{ord} K + 2$,
  b) there exists a decomposition L of the set of integers $0, 1, 2, \ldots, \operatorname{ord} K$ such that
    α) L consists of at most four classes,
    β) K is a partial decomposition of L,
    γ) none of the integers $\operatorname{ord} K' - 1, \ldots, \operatorname{ord} K, 1$ belongs to the same class of L as the number 0,
    δ) omitting the integers $\operatorname{ord} K', \ldots, \operatorname{ord} K$[2]) in the decomposition L and replacing afterwards the number 0 by the number $\operatorname{ord} K'$ we get the flower K';

III. a) $\operatorname{ord} K' = \operatorname{ord} K + 2$,
  b) K is a partial decomposition of K',
  c) the number $\operatorname{ord} K + 1$ belongs to the same class of K' as the number 1.

If the flower K' has originated from the flower K by pollination, then

$$\operatorname{ord} K' \leqq \operatorname{ord} K + 2 .$$

[2]) Of course this set is void for $\operatorname{ord} K' = \operatorname{ord} K + 1$.

If

$$(2.1) \qquad 1 < s \leqq \operatorname{ord} K + 2$$

holds for a flower K and an integer s, then there exist at most three flowers of the order s which have originated from K by pollination. However, none flower of this property need exist at all. But it does exist if the flower K is not gay.

If the map M' has originated from the map M by annexation and if they are numbered as well as stained in a corresponding way, then the flower $\varkappa(M')$ has originated from the flower $\varkappa(M)$ by pollination. On the other hand if

1. the map M' has originated from the map M by annexation,
2. M, M' are numbered in a corresponding way,
3. M is stained,
4. the flower K' is of the order ord M' and has originated from the flower $\varkappa(M)$ by pollination,

then the map M' may be stained in such a corresponding way with M that $K' = \varkappa(M')$. To see this it is sufficient to consider the definitions of the respective notions.

A finite (void or non-void) set of flowers of the same order is called an *inflorescence.* The *order* of a non-void inflorescence C (symbol ord C) is the order of its flowers. The order of the void inflorescence is any integer greater than 1; but the symbol ord C for $C = \emptyset$ is meaningless.

The void inflorescence is called also *singular.* A non-void inflorescence such that all its flowers are gay is said to be *semi-singular.* All other inflorescences are called *regular.* They contain at least one flower which is not gay.

Let M be a numbered map. Then $\lambda(M)$ designates the set of flowers $\varkappa(M)$ provided that M has been stained in all possible ways. Thus $\lambda(M)$ is an inflorescence of the order ord M. It is regular or semi-singular or singular simultaneously with the map M.

If a non-void inflorescence C consists of the flowers

$$K_1, \ldots, K_t \quad (t \geqq 1)$$

and the integer p fulfils

$$1 \leqq p \leqq \operatorname{ord} C$$

then C^p designates the inflorescence consisting of the flowers

$$K_1^p, \ldots, K_t^p .$$

If $C = \emptyset$ we put $C^p = \emptyset$ for any integer $p \geqq 1$. The inflorescences C, C^p are both either regular, or semi-singular or singular.

Is C a non-void inflorescence and s an integer fulfilling

$$(2.2) \qquad 1 < s \leqq \operatorname{ord} C + 2$$

then the symbol sC denotes the inflorescence defined in the following way: The flower K' belongs to sC if and only if

1. $\operatorname{ord} K' = s$,

2. there exists a flower $K \in C$ such that K' has originated from K by pollination. Further we put $sC = \emptyset$ for $C = \emptyset$ and any integer $s > 1$. It may be, of course, $sC = \emptyset$ also for $C \neq \emptyset$. However, if C is regular and s satisfies the inequalities (2.2), then sC is non-void. The symbol sC – providing it is meaningful – denotes always an inflorescence of the order s.

We say that the inflorescence C' has originated from the inflorescence C by *pollination*, if such an integer s exists that $C' = sC$. Assuming the inflorescence C' has originated from the inflorescence C by pollination, the following propositions are clear: If C is singular, C' is singular. If C is regular, C' is not singular. If C is semi-singular, a singular inflorescence C'' exists which has originated from C by pollination. We may put, for example, $C'' = 2C$.

If the map M' has originated from the map M by annexation and if they are numbered in a corresponding way, then the inflorescence $\lambda(M')$ has originated from the inflorescence $\lambda(M)$ by pollination. It is namely

$$\lambda(M') = \operatorname{ord} M' \lambda(M) .$$

On the other hand if M is a numbered map and the inflorescence C' has originated from the inflorescence $\lambda(M)$ by pollination, then a map M' exists which has originated from M by annexation and fulfils $C' = \lambda(M')$ provided that it has been numbered in a corresponding way with M.

If C is a non-void inflorescence, then $|C|$ designates the set consisting of the elements

$$C^1, C^2, \ldots, C^{\operatorname{ord} C} .$$

Is C void $|C|$ is considered void as well.

The set R is called a *herb*, if such an inflorescence C exists that

$$R = |C| . \tag{2.3}$$

The *order* of a non-void herb R (symbol $\operatorname{ord} R$) is the order of the inflorescences belonging to R. The order of a void herb is any integer greater than 1; but the symbol $\operatorname{ord} R$ for $R = \emptyset$ is meaningless. There exists obviously one and only one herb of the order 2; it will be denoted *2*.

The herb R fulfilling (2.3) is called *regular*, or *semi-singular* or *singular* in accordance with the inflorescence C. The symbol $|\lambda(M)|$ means the same herb may the map M be numbered in any negative way. This herb is denoted $\chi(M)$ and will be referred to as the *characteristic* of the map M. Its order is $\operatorname{ord} M$. It is regular, or semi-singular or singular simultaneously with the map M.

We say that the herb R' has originated from the herb R by *pollination*, if inflorescences C, C' exist in such a way that C' has originated from C by pollination and

$$R = |C|, \quad R' = |C'|$$

hold.

Provided that the herb R' has originated from the herb R by pollination, the following may be easily verified: If R is singular, R' is singular. If R is regular, R' is not singular. If R is semi-singular, a singular herb R'' exists which has originated from R by pollination.

If the map M' has originated from the map M by annexation, then the herb $\chi(M')$ has originated from the herb $\chi(M)$ by pollination. On the other hand if the herb R' has originated from the herb $\chi(M)$ by pollination, then a map M' exists which has originated from M by annexation and fulfils $R' = \chi(M')$.

We say that the herb R_1 is *lower* than the herb R_2 (symbol $R_1 < R_2$), if inflorescences C_1, C_2 exist in such a way that

$$R_1 = |C_1|, \quad R_2 = |C_2|, \quad \emptyset \neq C_1 \subset C_2, \quad C_1 \neq C_2 .$$

When writing $R_1 \leqq R_2$ we mean, of course, that either $R_1 < R_2$ or $R_1 = R_2$ is valid. If the herbs R_1, R_2 fulfil $R_1 < R_2$ then

1. none of them is singular,
2. if R_1 is regular, R_2 is regular,
3. if R_2 is semi-singular, R_1 is semi-singular,
4. $\operatorname{ord} R_1 = \operatorname{ord} R_2$.

We say that the integer j is an *index* of the herb R, if

1. there exists a map M fulfilling

$$\deg M = j, \quad \chi(M) = R ;$$

2. there does not exist a map M such that

$$\deg M < j, \quad \chi(M) = R .$$

If the herb R cannot be considered a characteristic of any map, we put its index equal to 0. Thus every herb R is associated with one and only one non-negative integer which is its index. We denote it ind R.[3])

If $R = \chi(M)$, then

$$\operatorname{ind} R \leqq \deg M .$$

If the herb R' has originated from the herb R by pollination and ind $R > 0$, then

$$\operatorname{ind} R' \leqq \operatorname{ind} R + 1 . \tag{2.4}$$

[3]) To determine the index of a singular herb means to solve the four colour problem.

If ind $R' > 2$, then such a herb R exists that R' has originated from R by pollination and

$$\text{ind } R < \text{ind } R' . \tag{2.5}$$

A non-void set of herbs is called a *herbarium*. The herbarium consisting of the herb 2 is denoted $\mathbf{2}$. If $\boldsymbol{H}$ is a herbarium, then $[\boldsymbol{H}]$ designates the herbarium defined in this way: $R' \in [\boldsymbol{H}]$ holds if and only if such a herb $R \in \boldsymbol{H}$ exists that R' has originated from R by pollination.

We say that the herbarium $\boldsymbol{H}$ is *regular*, if any of its herbs is regular. We say that $\boldsymbol{H}$ is *singular*, if it includes a singular herb. In all other cases $\boldsymbol{H}$ is called *semi-singular*. Such a herbarium does not include singular herbs, but it contains at least one semi-singular herb. The herbarium $\boldsymbol{H}$ is called *simple*, if $R_1 < R_2$ never holds for $R_1, R_2 \in \boldsymbol{H}$.

Is $\boldsymbol{H}$ a herbarium, then $^-\boldsymbol{H}$ designates the set of herbs R fulfilling

1. $R \in \boldsymbol{H}$,
2. if $R' \in \boldsymbol{H}$, then $R' < R$ is false.

The set $^-\boldsymbol{H}$ is a simple herbarium and a subset of $\boldsymbol{H}$. If $R \in \boldsymbol{H}$, then such a herb $R' \in {}^-\boldsymbol{H}$ exists that $R' \leqq R$. The herbarium $^-\boldsymbol{H}$ is regular or semi-singular or singular simultaneously with the herbarium $\boldsymbol{H}$.

Is $\boldsymbol{A}$ an atlas, then $\{\boldsymbol{A}\}$ denotes the set of all the herbs $\chi(M)$ where $M \in \boldsymbol{A}$. The herbarium $\{\boldsymbol{A}\}$ is regular or semi-singular or singular simultaneously with the atlas $\boldsymbol{A}$. For any atlas $\boldsymbol{A}$

$$[\{\boldsymbol{A}\}] = \{[\boldsymbol{A}]\} . \tag{2.6}$$

We introduce two particular sequences of herbaria

$$\boldsymbol{H}_2, \boldsymbol{H}_3, \ldots \tag{2.7}$$

$$\boldsymbol{H}'_2, \boldsymbol{H}'_3, \ldots \tag{2.8}$$

putting

$$\boldsymbol{H}_i = \{\boldsymbol{A}_i\}, \quad \boldsymbol{H}'_i = \{\boldsymbol{A}'_i\} \quad (i > 1) .$$

Thus

$$\boldsymbol{H}'_i = \boldsymbol{H}_2 \cup \boldsymbol{H}_3 \cup \ldots \cup \boldsymbol{H}_i \tag{2.9}$$

$$\boldsymbol{H}'_i \subset \boldsymbol{H}'_{i+1} \tag{2.10}$$

$$\boldsymbol{H}_{i+1} = [\boldsymbol{H}_i] \tag{2.11}$$

for any integer $i > 1$. (See (1.7), (1.8), (1.9), (2.6).) Further

$$\boldsymbol{H}_2 = \boldsymbol{H}'_2 = \mathbf{2} , \tag{2.12}$$

$$\boldsymbol{H}_2 \subset \boldsymbol{H}_3 . \tag{2.13}$$

From (2.9), (2.11), (2.13) it follows

$$\begin{aligned}[\boldsymbol{H}'_i] &= [\boldsymbol{H}_2] \cup [\boldsymbol{H}_3] \cup \ldots \cup [\boldsymbol{H}_i] = \\ &= \boldsymbol{H}_3 \cup \boldsymbol{H}_4 \cup \ldots \cup \boldsymbol{H}_{i+1} = \\ &= \boldsymbol{H}_2 \cup \boldsymbol{H}_3 \cup \ldots \cup \boldsymbol{H}_{i+1}\end{aligned}$$

so that

$$\boldsymbol{H}'_{i+1} = [\boldsymbol{H}'_i] \quad (i = 2, 3, \ldots)\,. \tag{2.14}$$

Using (2.12), (2.11) and (2.12), (2.14) we can construct the sequences (2.7), (2.8) in a recurrent way regardless of the atlases (1.5), (1.6). Owing to (2.12)

$$\boldsymbol{H}_i = \boldsymbol{H}'_i \quad (i = 2, 3, \ldots) \tag{2.15}$$

is valid so that (2.7) and (2.8) denote the same sequence. For this sequence one of the two following cases occurs:

1. Every herbarium $\boldsymbol{H}_i$ $(i > 1)$ is regular.
2. There exists such an integer k $(k > 3)$ that
   the herbaria $\boldsymbol{H}_2, \ldots, \boldsymbol{H}_{k-1}$ are regular,
   the herbarium $\boldsymbol{H}_k$ is semi-singular,
   the herbaria $\boldsymbol{H}_{k+1}, \boldsymbol{H}_{k+2}, \ldots$ are singular.

It suffices to remind oneself of what has been said about the atlases (1.5) in paragraph 1.

The herbarium $\boldsymbol{H}'_i$ $(i > 1)$, that is also the herbarium $\boldsymbol{H}_i$ is the set of that herbs the index of which is positive and smaller or equal to i. This can be easily seen.

Finally let us establish the herbaria

$$\boldsymbol{S}_3, \boldsymbol{S}_4, \ldots$$

$$\boldsymbol{T}_3, \boldsymbol{T}_4, \ldots$$

in the following manner: $\boldsymbol{S}_i$ or $\boldsymbol{T}_i$ is the set of that herbs of $^-\boldsymbol{H}_i$ the index of which is smaller than i or equal to i, respectively. Thus

$$^-\boldsymbol{H}_i = \boldsymbol{S}_i \cup \boldsymbol{T}_i \quad (i = 3, 4, \ldots) \tag{2.16}$$

with disjoint summands.

It may be shown that

$$\boldsymbol{S}_{i+1} = {}^-\boldsymbol{H}_{i+1} \cap {}^-\boldsymbol{H}_i \tag{2.17}$$

$$\boldsymbol{T}_{i+1} = {}^-\boldsymbol{H}_{i+1} - \boldsymbol{S}_{i+1} \quad (i = 3, 4, \ldots) \tag{2.18}$$

although at the first sight only

$$\boldsymbol{S}_{i+1} = {}^-\boldsymbol{H}_{i+1} \cap \boldsymbol{H}_i$$

may be stated. If namely $R \in {}^{-}\boldsymbol{H}_{i+1} \cap \boldsymbol{H}_i$, then $R' < R$ is false for $R' \in \boldsymbol{H}_{i+1}$ and consequently also for $R' \in \boldsymbol{H}_i$, since

$$\boldsymbol{H}_i \subset \boldsymbol{H}_{i+1} \tag{2.19}$$

according to (2.15), (2.10). Thus $R \in {}^{-}\boldsymbol{H}_i$ having in mind that $R \in \boldsymbol{H}_i$. This means

$${}^{-}\boldsymbol{H}_{i+1} \cap \boldsymbol{H}_i \subset {}^{-}\boldsymbol{H}_{i+1} \cap {}^{-}\boldsymbol{H}_i$$

which necessitates (2.17) .Then (2.18) is clear.

3. ALGORITHM

According to what has been hitherto said it is clear enough that there is no problem in the main to state an algorithm by means of which one could ascertain for any integer $n > 1$ whether it is possible or not to staine regularly every map consisting of n countries. It suffices namely to know for the given n whether the atlas $\boldsymbol{A}_n$ is regular or semi-singular or singular, that is whether the herbarium $\boldsymbol{H}_n$ is regular or semi-singular or singular. The sequence of these herbaria may be constructed in a recurrent way which has apparently an algorithmic character and admits the application of a computer.

However, the practical kernel of the problem lies in the fact that the storage and the time of the computer must be saved to the highest degree. These demands are not satisfied well enough when building up the sequence (2.7). The algorithm we introduce in this paragraph is much more economical in both lines. All herbaria recorded in the storage are simple and the pollination is applied no more than once to any herb whereas the construction of (2.7) possesses none of these two properties.

Algorithm. *Let n be an integer greater than* 4. *Build up the sequence*

$$\boldsymbol{V}_3, \boldsymbol{W}_3;\ \boldsymbol{V}_4, \boldsymbol{W}_4; \ldots, \boldsymbol{V}_p, \boldsymbol{W}_p \tag{3.1}$$

consisting of pairs of herbaria in the following way:

1. $\boldsymbol{V}_3 = \boldsymbol{2}$; $\boldsymbol{W}_3 = [\boldsymbol{2}] - \boldsymbol{2}$.

2. *Are the herbaria* $\boldsymbol{V}_i$, $\boldsymbol{W}_i$ $(i \geqq 3)$ *already known, establish* ${}^{-}[\boldsymbol{W}_i]$. *Is* $i = n - 2$ *or is* ${}^{-}[\boldsymbol{W}_i]$ *not regular, put* $p = i$. *Otherwise put*

$$\boldsymbol{V}_{i+1} = {}^{-}(\boldsymbol{V}_i \cup \boldsymbol{W}_i \cup {}^{-}[\boldsymbol{W}_i]) \cap (\boldsymbol{V}_i \cup \boldsymbol{W}_i) \tag{3.2}$$

$$\boldsymbol{W}_{i+1} = {}^{-}(\boldsymbol{V}_i \cup \boldsymbol{W}_i \cup {}^{-}[\boldsymbol{W}_i]) - \boldsymbol{V}_{i+1}\,. \tag{3.3}$$

Then every map consisting of n countries may be regularly stained by at most four colours if and only if the herbarium ${}^{-}[\boldsymbol{W}_p]$ *is regular.*

Before we start proving the assertion stated in this algorithm we introduce two lemmas.

Lemma 1. *Let the following be true for the herbs* R_1, R_2, R_2';

1. R_1 *is regular*;
2. $R_1 \leqq R_2$;
3. R_2' *has originated from* R_2 *by pollination.*

Then a herb R_1' *exists which has originated from* R_1 *by pollination and fulfils*

$$R_1' \leqq R_2' . \tag{3.4}$$

Proof. The case $R_1 = R_2$ is clear. Further let us assume $R_1 < R_2$. The inflorescences C_1, C_2, C_2' and the integer s exist in such a way that

$$R_1 = |C_1|, \quad R_2 = |C_2|, \quad \emptyset \neq C_1 \subset C_2, \quad C_1 \neq C_2,$$
$$R_2' = |C_2'|, \quad C_2' = sC_2, \quad 1 < s \leqq \text{ord } C_2 + 2.$$

Hence

$$\emptyset \neq sC_1 \subset sC_2,$$

since C_1 is regular and

$$1 < s \leqq \text{ord } C_1 + 2$$

holds. Denoting $C_1' = sC_1$, $R_1' = |C_1'|$ we have

$$R_1' = |C_1'|, \quad R_2' = |C_2'|, \quad \emptyset \neq C_1' \subset C_2'$$

which means (3.4).

Lemma 2. *If the herbarium* $^-\boldsymbol{H}_i$ $(i > 1)$ *is regular, then*

$$\boldsymbol{S}_{i+1} = {}^-(\boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^-[\boldsymbol{T}_i]) \cap (\boldsymbol{S}_i \cup \boldsymbol{T}_i),$$
$$\boldsymbol{T}_{i+1} = {}^-(\boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^-[\boldsymbol{T}_i]) - \boldsymbol{S}_{i+1}.$$

Proof. From (2.16), (2.17), (2.18) it follows that it is sufficient to prove

$${}^-(\boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^-[\boldsymbol{T}_i]) = {}^-\boldsymbol{H}_{i+1}. \tag{3.5}$$

1. Let us assume

$$R \in {}^-(\boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^-[\boldsymbol{T}_i]). \tag{3.6}$$

This means that

$$R \in \boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^-[\boldsymbol{T}_i] \tag{3.7}$$

and

$$\text{if } R' \in \boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^-[\boldsymbol{T}_i] \text{ then } R' < R \text{ is false}. \tag{3.8}$$

We seek to show

$$R \in {}^-\boldsymbol{H}_{i+1} \tag{3.9}$$

i.e.

$$R \in \boldsymbol{H}_{i+1} \tag{3.10}$$

and

$$\text{if } R' \in \boldsymbol{H}_{i+1} \text{ then } R' < R \text{ is false .} \tag{3.11}$$

From (2.4) it follows

$$\boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^{-}[\boldsymbol{T}_i] \subset \boldsymbol{H}_{i+1} \tag{3.12}$$

which necessitates (3.10). Thus only (3.11) is to be proved.

First let $R' \in \boldsymbol{H}_i$ (see (2.19)). Then such an $R'' \in {}^{-}\boldsymbol{H}_i$ exists that $R'' \leqq R'$. Assuming $R' < R$ we should have

$$R'' \in \boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^{-}[\boldsymbol{T}_i]\,, \quad R'' < R$$

which contradicts (3.8).

Secondly let $R' \in \boldsymbol{H}_{i+1} - \boldsymbol{H}_i$ so that ind $R' = i + 1$. Then such a herb R_2 exists that R' has originated from R_2 by pollination and ind $R_2 <$ ind R' holds (see (2.5)). This means $R_2 \in \boldsymbol{H}_i$. There exists a herb $R_1 \in {}^{-}\boldsymbol{H}_i$ fulfilling $R_1 \leqq R_2$. This herb is regular since $\boldsymbol{H}_i$ is regular. According to lemma 1 a herb R_1' exists which has originated from R_1 by pollination and fulfils

$$R_1' \leqq R'\,. \tag{3.13}$$

Now let us make the assumption

$$R' < R\,. \tag{3.14}$$

Since $R_1 \in {}^{-}\boldsymbol{H}_i$ we have

$$\text{either } R_1 \in \boldsymbol{S}_i \text{ or } R_1 \in \boldsymbol{T}_i\,. \tag{3.15}$$

In the first case ind $R_1 < i$ and ind $R_1' \leqq i$ by (2.4), i.e. $R_1' \in \boldsymbol{H}_i$. There exists a herb $R_0' \in {}^{-}\boldsymbol{H}_i$ fulfilling

$$R_0' \leqq R_1'\,. \tag{3.16}$$

From (3.16), (3.13), (3.14) it follows

$$R_0' \in \boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^{-}[\boldsymbol{T}_i]\,, \quad R_0' < R$$

which contradicts (3.8). Considering the second eventuality in (3.15) we have $R_1' \in [\boldsymbol{T}_i]$. There exists a herb $R_0'' \in {}^{-}[\boldsymbol{T}_i]$ fulfilling $R_0'' \leqq R_1'$. Hence

$$R_0'' \in \boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^{-}[\boldsymbol{T}_i]\,, \quad R_0'' < R$$

which is again a contradiction with (3.8). Thus the assumption (3.14) must be false for $R' \in \boldsymbol{H}_{i+1}$ and consequently (3.11) is valid. In this way

$${}^{-}(\boldsymbol{S}_i \cup \boldsymbol{T}_i \cup {}^{-}[\boldsymbol{T}_i]) \subset {}^{-}\boldsymbol{H}_{i+1}$$

is proved.

2. Let us assume that (3.9) holds. Then (3.10) and (3.11) are true. We seek to prove (3.6), i.e. (3.7) and (3.8). But (3.8) is a consequence of (3.11) and (3.12) so that only (3.7) remains.

First let $R \in \mathbf{H}_i$. Simultaneously

$$\text{if } R' \in \mathbf{H}_i \text{ then } R' < R \text{ is false}$$

by (3.11). Thus $R \in {}^{-}\mathbf{H}_i$ and (3.7) holds.

Secondly let $R \in \mathbf{H}_{i+1} - \mathbf{H}_i$. Then such a herb $R_2 \in \mathbf{H}_i$ exists that R has originated from R_2 by pollination. Further a herb $R_1 \in {}^{-}\mathbf{H}_i$ exists fulfilling $R_1 \leqq R_2$. Finally a herb R' exists according to lemma 1 which has originated from R_1 by pollination and fulfills

$$R' \leqq R\,. \tag{3.17}$$

It is $R' \in \mathbf{H}_{i+1}$ since $R_1 \in \mathbf{H}_i$. By (3.11) $R' < R$ is false so that $R' = R$ must hold (see (3.17)). Thus R has originated from R_1 by pollination. Since $R_1 \in {}^{-}\mathbf{H}_i$,

$$\text{either } R_1 \in \mathbf{S}_i \text{ or } R_1 \in \mathbf{T}_i \tag{3.18}$$

is true. The first eventuality cannot occur. It would be namely ind $R_1 < i$, ind $R \leqq i$ (see (2.4)), $R \in \mathbf{H}_i$ in contradiction with what has been assumed. In the second case of (3.18)

$$R \in [\mathbf{T}_i]\,. \tag{3.19}$$

We shall show that

$$\text{if } R'' \in [\mathbf{T}_i] \text{ then } R'' < R \text{ is false}\,. \tag{3.20}$$

Let $R'' \in [\mathbf{T}_i]$. Then a herb $R^* \in {}^{-}[\mathbf{T}_i]$ fulfilling $R^* \leqq R''$ exists. If $R'' < R$ were correct, we should have $R^* < R$ and $R^* \in \mathbf{S}_i \cup \mathbf{T}_i \cup {}^{-}[\mathbf{T}_i]$ – a contradiction with (3.8). From (3.19) and (3.20) it follows $R \in {}^{-}[\mathbf{T}_i]$ and (3.7) holds. Thus

$${}^{-}(\mathbf{S}_i \cup \mathbf{T}_i \cup {}^{-}[\mathbf{T}_i]) \supset {}^{-}\mathbf{H}_{i+1}$$

is proved and the proof of lemma 2 is completed.

Proof of the assertion stated in the algorithm. Let us assume that the integer n is greater than 4 and that the sequence (3.1) has been constructed in the way described in the algorithm. Obviously

$$3 \leqq p \leqq n - 2\,. \tag{3.21}$$

First we shall show that the following proposition (U_i) holds for every integer i fulfilling

$$3 \leqq i \leqq p\,.$$

Proposition (U_i). *The herbarium* ${}^{-}\mathbf{H}_i$ *is regular and*

$$\mathbf{S}_i = \mathbf{V}_i\,, \quad \mathbf{T}_i = \mathbf{W}_i \tag{3.22}$$

hold.

The proposition (U_3) may be verified by constructing $^{-}\boldsymbol{H}_3$, $\boldsymbol{S}_3$, $\boldsymbol{T}_3$. If (U_i) is correct for a particular i fulfilling

$$3 \leqq i < p ,$$

then the herbaria $\boldsymbol{V}_{i+1}$, $\boldsymbol{W}_{i+1}$ are defined and (3.2), (3.3) are true. From lemma 2 and (3.22) it follows

$$\boldsymbol{S}_{i+1} = \boldsymbol{V}_{i+1} , \quad \boldsymbol{T}_{i+1} = \boldsymbol{W}_{i+1} .$$

The herbaria $\boldsymbol{V}_i$, $\boldsymbol{W}_i$ are regular by (3.22), $^{-}[\boldsymbol{W}_i]$ is regular since $i \neq p$ (see point 2 in the algorithm). According to (3.2), (3.3) the herbaria $\boldsymbol{V}_{i+1}$, $\boldsymbol{W}_{i+1}$, i.e. the herbaria $\boldsymbol{S}_{i+1}$, $\boldsymbol{T}_{i+1}$ are regular and consequently $^{-}\boldsymbol{H}_{i+1}$ is also regular which was to be proved.

Since (U_p) is true, $^{-}\boldsymbol{H}_p$ is regular, $\boldsymbol{T}_p = \boldsymbol{W}_p$. By lemma 2 (see (3.5)) we get

$$^{-}\boldsymbol{H}_{p+1} = {}^{-}(^{-}\boldsymbol{H}_p \cup {}^{-}[\boldsymbol{W}_p]) . \tag{3.23}$$

Now let us assume that $^{-}[\boldsymbol{W}_p]$ is regular. Then $p = n - 2$ according to point 2 of the algorithm. By (3.23) $^{-}\boldsymbol{H}_{p+1}$ is regular so that $\boldsymbol{H}_{p+1}$ is also regular. Thus $\boldsymbol{H}_{p+2} = \boldsymbol{H}_n$ cannot be singular having in mind what has been said about the sequence (2.7) in paragraph 2. This means that every map consisting of n countries may be regularly stained by at most four colours.

Secondly let us assume that $^{-}[\boldsymbol{W}_p]$ is not regular. Then neither $^{-}\boldsymbol{H}_{p+1}$ nor $\boldsymbol{H}_{p+1}$ are regular and therefore they are semi-singular since $\boldsymbol{H}_p$ is regular. By (3.21) $p + 1 < < n$ which means that $\boldsymbol{H}_n$ is singular. Thus in this case it is not possible to staine every map consisting of n countries in the required way. The proof is completed.

As far as the economy of the algorithm is concerned the following is to be pointed out. It is clear that all the herbaria recorded in the storage when building up the sequence (3.1) are simple. Further if the pollination is applied to a herb R, then R belongs to some $\boldsymbol{W}_i$ $(3 \leqq i \leqq p)$. Denote j the smallest integer for which $R \in \boldsymbol{W}_j$. Then ind $R = j$ since $\boldsymbol{W}_j = \boldsymbol{T}_j$. Consequently $R \in \boldsymbol{T}_i$, i.e. $R \in \boldsymbol{W}_i$ cannot hold for any i greater than j. This means that none herb is pollinated more than once during the construction of (3.1).

Souhrn

O MOŽNOSTI POUŽITÍ SAMOČINNÉHO POČÍTAČE PŘI ŘEŠENÍ PROBLÉMU ČTYŘ BAREV

BORIS GRUBER

V práci je uveden algoritmus, jímž lze pro každé přirozené n zjistit, zda každou mapu skládající se z n zemí je možno pravidelně obarvit nejvýše čtyřmi barvami.

Author's address: RNDr. *Boris Gruber*, CSc., matematicko-fyzikální fakulta Karlovy university, Ke Karlovu 3, Praha 2.