Rostislav Zezula

Criticality conditions for a finite homogenized natural-uranium fueled reactor with prescribed thermal neutron flux

Aplikace matematiky, Vol. 15 (1970), No. 5, 328–338

Persistent URL: http://dml.cz/dmlcz/103304

CRITICALITY CONDITIONS FOR A FINITE HOMOGENIZED NATURAL-URANIUM FUELED REACTOR WITH PRESCRIBED THERMAL NEUTRON FLUX

ROSTISLAV ZEZULA

(Received July 15, 1969)

Let us consider a finite homogenized natural-uranium fueled reactor (with reflector) whose core Ω is described by means of two-group equations (in usual notation [1])

$$-\operatorname{div}(D \operatorname{grad} \Phi) + (\Sigma_U^a + \Sigma_M^a)\, \Phi = q \tag{1a}$$

$$-\operatorname{div}(\tau \operatorname{grad} q) + q = k\Sigma_U^a \Phi \tag{1b}$$

where $D = D(x) \in C^{(2)}(\Omega)$, $\tau = \tau(x) \in C^{(2)}(\Omega)$, $\Sigma_M^a = \Sigma_M^a(x) \in C^{(2)}(\Omega)$ are given functions, $\Omega \subset R_n$ is a given domain in the n-dimensional real Euclidean space R_n, $n > 1$ and $x = (x_1, x_2, \ldots, x_n) \in \Omega$ is the radiusvector. The influence of the reflector on the core is expressed by two given functions $\vartheta_1(\dot{\Omega})$, $\vartheta_2(\dot{\Omega})$ on the boundary $\dot{\Omega}$ of the core Ω.

By introducing the following notation for the relative fuel concentration

$$M = M(x) = \frac{\Sigma_U^a}{\Sigma_M^a} = \frac{\sigma_U^a}{\Sigma_M^a} N_U(x) \tag{2}$$

and by eliminating the slowing-down density $q = q(x) \in C^{(2)}(\Omega)$ we obtain from (1a), (1b), if $\tau \neq 0$, for the thermal neutron flux $\Phi = \Phi(x) \in C^{(4)}(\Omega)$ the equation

$$\begin{aligned} &\Delta(D\, \Delta\Phi) + \Delta(\operatorname{grad} D \,.\, \operatorname{grad} \Phi) - \Delta\Sigma_M^a[(M + 1)\, \Phi] - \frac{D}{\tau}\, \Delta\Phi - \\ &- \frac{\operatorname{grad} \tau}{\tau} \operatorname{grad}\left[-D\, \Delta\Phi - \operatorname{grad} D \,.\, \operatorname{grad} \Phi + \Sigma_M^a(M + 1)\, \Phi\right] - \\ &- \frac{1}{\tau}(\operatorname{grad} D \,.\, \operatorname{grad} \Phi) + \frac{1}{\tau}\, \Sigma_M^a[M(1 - k) + 1]\, \Phi = 0 \end{aligned} \tag{3}$$

where

$$k = k(M) \tag{4}$$

is a given function of $M = M(x)$, so that the equation (3) is nonlinear (quasilinear) in M.

By means of the well known formulae we can transform the equation (3) into the following one:

$$\begin{aligned}
&\Big\{D\,\Delta(\Delta\Phi) + \left(3 \operatorname{grad} D + \frac{D}{\tau}\operatorname{grad}\tau\right)\operatorname{grad}\Delta\Phi + \\
&+ \left[\Delta D + \frac{1}{\tau}\operatorname{grad}\tau \,.\, \operatorname{grad} D - \frac{D}{\tau} - \Sigma_M^a(M+1)\right]\Delta\Phi + \\
&+ \left(2 \operatorname{grad}\frac{\partial D}{\partial x_1} + \frac{1}{\tau}\frac{\partial D}{\partial x_1}\operatorname{grad}\tau\right)\operatorname{grad}\frac{\partial\Phi}{\partial x_1} + \ldots + \\
&+ \left(2 \operatorname{grad}\frac{\partial D}{\partial x_n} + \frac{1}{\tau}\frac{\partial D}{\partial x_n}\operatorname{grad}\tau\right) . \operatorname{grad}\frac{\partial\Phi}{\partial x_n} + \\
&+ \left[\operatorname{grad}\Delta D - \frac{1}{\tau}\Sigma_M^a(M+1)\operatorname{grad}\tau - 2(M+1)\operatorname{grad}\Sigma_M^a - 2\Sigma_M^a\operatorname{grad}(M+1)\right] . \\
&. \operatorname{grad}\Phi + \frac{\operatorname{grad}\tau}{\tau}\left(\frac{\partial\Phi}{\partial x_1}\operatorname{grad}\frac{\partial D}{\partial x_1} + \ldots + \frac{\partial\Phi}{\partial x_n}\operatorname{grad}\frac{\partial D}{\partial x_n}\right)\Big\} + \\
&+ \frac{\Phi}{\tau}\{\Sigma_M^a[M(1-k)+1] - (M+1)\operatorname{grad}\tau \,.\, \operatorname{grad}\Sigma_M^a - (M+1)\,\tau\Sigma_M^a - \\
&- \operatorname{grad}(M+1)\,(\Sigma_M^a\operatorname{grad}\tau + 2\tau\operatorname{grad}\Sigma_M^a) - \tau\Sigma_M^a\,\Delta(M+1)\} = 0
\end{aligned} \tag{5}$$

from which we obtain the implication

$$\begin{aligned}
&\Phi = \text{const} \neq 0 \Rightarrow \{\Sigma_M^a[M(1-k)+1] - (M+1)\operatorname{grad}\tau\operatorname{grad}\Sigma_M^a - \\
&- (M+1)\,\tau\,\Delta\Sigma_M^a - \operatorname{grad}(M+1)\,(\Sigma_M^a\operatorname{grad}\tau + 2\tau\operatorname{grad}\Sigma_M^a) - \\
&- \tau\Sigma_M^a\,\Delta(M+1)\} = 0\,.
\end{aligned} \tag{6}$$

This implication gives us the following necessary condition for the relative fuel concentration $M(x)$ producing a flat thermal neutron flux $\Phi = \text{const}$ in the reactor core Ω

$$\begin{aligned}
&\Delta M + \left(\frac{\operatorname{grad}\tau}{\tau} + 2\,\frac{\operatorname{grad}\Sigma_M^a}{\Sigma_M^a}\right)\operatorname{grad} M + \left[\frac{k(M)-1}{\tau} + \frac{\operatorname{grad}\tau}{\tau} . \frac{\operatorname{grad}\Sigma_M^a}{\Sigma_M^a} + \right. \\
&\left. + \frac{\Delta\Sigma_M^a}{\Sigma_M^a}\right] M + \left(\frac{1}{\tau} - \frac{\operatorname{grad}\tau}{\tau}\,\frac{\operatorname{grad}\Sigma_M^a}{\Sigma_M^a} - \frac{\Delta\Sigma_M^a}{\Sigma_M^a}\right) = 0
\end{aligned} \tag{7}$$

and in case of the symmetry of the core Ω we have for (7) the condition

$$\mathrm{d}M(0) = 0 \tag{7a}$$

which together with the boundary condition on the boundary $\dot{\Omega}$ of the core Ω

$$M(\dot{\Omega}) = \vartheta_1(\dot{\Omega}) \tag{7b}$$

determines (under physically plausible assumptions) for the given function $\vartheta_1(\dot{\Omega}) \in \in C(\dot{\Omega})$ uniquely the function $M \in C^2(\Omega)$. We see also that under the usual conditions

$$\Sigma^a_M = \text{const} > 0\,, \quad \tau = \text{const} > 0 \tag{7c}$$

the equation (7) reduces to the well known nonlinear elliptic equation (Goertzel's equation [1])

$$\Delta M + \frac{k(M) - 1}{\tau} M - \frac{1}{\tau} = 0 \tag{7d}$$

which together with the initial condition (7a) gives (in one-dimensional reactor geometries) for $M(0) = M_0$ a nonlinear (quasilinear) Cauchy's problem, investigated (for one-dimensional geometries) in [1], [2], [3], [4], [5], [6]. Sufficient conditions for the existence of a unique solution of linear and quasilinear elliptic boundary value problems are given e.g., in [7].

Let us now suppose that this necessary condition for the thermal flux flattening in a symmetric reactor core is fulfilled, i.e., that there is a relative fuel concentration $M(x)$ which is the unique solution of the Dirichlet's problem (7), (7b) satisfying (7a). Then it follows from the equation (5) and from the symmetry of the problem considered that the thermal neutron flux Φ in the core with this relative fuel concentration $M(x)$ necessarily must obey the (evidently linear) biharmonic equation

$$\begin{aligned} L(M)\,\Phi \equiv \Big\{ & D\,\Delta(\Delta\Phi) + \left(3 \operatorname{grad} D + \frac{D}{\tau} \operatorname{grad} \tau\right) \operatorname{grad} \Delta\Phi + \\ & + \left[\Delta D + \frac{1}{\tau} \operatorname{grad} \tau \,.\, \operatorname{grad} D - \frac{D}{\tau} - \Sigma^a_M(M+1)\right] \Delta\Phi + \\ & + \left[\left(2 \operatorname{grad} \frac{\partial D}{\partial x_1} + \frac{1}{\tau}\frac{\partial D}{\partial x_1} \operatorname{grad} \tau\right) \operatorname{grad} \frac{\partial \Phi}{\partial x_1} + \ldots + \right. \\ & \left. + \left(2 \operatorname{grad} \frac{\partial D}{\partial x_n} + \frac{1}{\tau}\frac{\partial D}{\partial x_n} \operatorname{grad} \tau\right) \operatorname{grad} \frac{\partial \Phi}{\partial x_n}\right] + \\ & + \left[\operatorname{grad} \Delta D - \frac{1}{\tau}\Sigma^a_M(M+1) \operatorname{grad} \tau - 2(M+1) \operatorname{grad} \Sigma^a_M - 2\Sigma^a_M \operatorname{grad}(M+1)\right] . \\ & . \operatorname{grad} \Phi + \frac{\operatorname{grad} \tau}{\tau}\left(\frac{\partial \Phi}{\partial x_1} \operatorname{grad} \frac{\partial D}{\partial x_1} + \ldots + \frac{\partial \Phi}{\partial x_n} \operatorname{grad} \frac{\partial D}{\partial x_n}\right)\Big\} = 0 \end{aligned} \tag{8}$$

with the symmetry relations

(8a) $$\mathrm{d}\Phi(0) = 0\,, \quad \mathrm{d}^{(3)}\Phi(0) = 0$$

and with the boundary conditions (for given functions $\psi(\dot{\Omega}) \in C(\dot{\Omega})$, $\omega(\dot{\Omega}) \in C(\dot{\Omega})$)

(8b) $$\Phi(\dot{\Omega}) = \omega(\dot{\Omega})\,, \quad \left.\frac{\partial \Phi}{\partial n}\right|_{\dot{\Omega}} = \psi(\dot{\Omega})$$

where $\partial n = \varepsilon \vec{n}$ denotes an infinitesimal translation in the direction of the outer normal $\vec{n}$, $\|\vec{n}\| = 1$ to the interface $\dot{\Omega}$ between the core and reflector. Conversely, from the relations (7), (7b) for M and (8), (8b) for Φ it follows that the equation (5) holds.

If we suppose that the Dirichlet's boundary value problem (8), (8b) has on the given Ω for every $\psi(\dot{\Omega}) \in C(\dot{\Omega})$, $\omega(\dot{\Omega}) \in C(\dot{\Omega})$ the unique solution $\Phi = \Phi_M(\psi, \omega)$ which satisfies (8a), then a s u f f i c i e n t c o n d i t i o n for the flattening of the thermal neutron flux $\Phi = \Phi_M = \Phi_M(0, \Phi_0)$ in the reactor core is given by the equations

(9) $$\psi(\dot{\Omega}) = 0\,, \quad \omega(\dot{\Omega}) = \Phi_0 = \text{const} > 0\,.$$

We shall show now that this sufficient flux-flattening condition together with the reactor criticality condition determines the critical core Ω^*.

Let us consider a two-parametrical system of symmetrical surfaces (with the parameters N_0, P_0)

(10) $$\dot{\Omega} = \dot{\Omega}(N_0, P_0)$$

whose elements $\dot{\Omega}(x_1, \ldots, x_n; N_0, P_0)$ are given by the parametric formulae

(11) $$x_i = x_i(s_1, \ldots, s_{n-1}; N_0, P_0) \quad (i = 1, 2, \ldots, n)\,, \quad n > 1\,.$$

Then we see that by the Dirichlet's problem (7), (7b) there corresponds (for given $\vartheta_1(s_1, \ldots, s_{n-1}; N_0, P_0)$) to all values of the real parameters N_0, P_0 a unique value $M_0 = M(0)$ (where 0 is the center of symmetry of $\dot{\Omega}$):

(12) $$M_0 = f(N_0, P_0)$$

Let us make the following assumptions:

I) The two-parametrical system of symmetrical surfaces $\dot{\Omega} \equiv \dot{\Omega}(N_0, P_0)$ can be chosen in such a way that solution $M(x_1, \ldots, x_n; N_0, P_0)$ of the Dirichlet's problem (7), (7b) fulfils for all N_0, P_0 the condition

(13) $$G_1(s_1, \ldots, s_{n-1}; N_0, P_0) \equiv$$
$$\equiv \frac{\partial M[x_1(s_1, \ldots, s_{n-1}; N_0, P_0), \ldots, x_n(s_1, \ldots, s_{n-1}; N_0, P_0)]}{\partial n} - \vartheta_2(s_1, \ldots, s_{n-1}; N_0, P_0) =$$
$$= g_1(s_1, \ldots, s_{n-1}; N_0, P_0) \,.\, H_1(N_0, P_0)$$

where the function $g_1(s_1, \ldots, s_{n-1}; N_0, P_0)$ is bounded and is not identically zero:

$$(13a) \qquad |g_1(s_1, \ldots, s_{n-1}; N_0, P_0)| \leqq \text{const}\,, \quad g_1(s_1, \ldots, s_{n-1}; N_0, P_0) \not\equiv 0$$

and $\vartheta_2(\dot{\Omega}) \in C(\dot{\Omega})$ is a given function on the boundary $\dot{\Omega}$ of the core Ω.

II) The implicit "criticality condition"

$$(14) \qquad H_1(N_0, P_0) = 0$$

can be explicitely solved in the variable N_0

$$(15) \qquad N_0 = h_1(P_0)\,.$$

Then it follows from (13), (14), (15) that for the values $N_0 = h_1(P_0)$ the criticality condition

$$(16) \qquad \frac{\partial M[x_1(s_1, \ldots, s_{n-1}; h_1(P_0), P_0), \ldots, x_n(s_1, \ldots, s_{n-1}; h_1(P_0), P_0)]}{\partial n} =$$

$$= \vartheta_2(s_1, \ldots, s_{n-1}; h_1(P_0), P_0)$$

is fulfilled, so that

$$(16a) \qquad \dot{\Omega} = \dot{\Omega}[h_1(P_0), P_0]$$

is a one-dimensional family of "possible critical shapes" of the reactor core and

$$(17) \qquad M_0(P_0) = f[h_1(P_0), P_0]$$

is the corresponding maximal relative fuel concentration in this critical core.

If we denote by $M_{P_0} = M(x_1, x_2, \ldots, x_n; h_1(P_0), P_0)$ the solution of the Dirichlet's problem (7), (7b) for a possible critical shape $\dot{\Omega}[h_1(P_0), P_0]$ of the reactor core, and by $\Phi_{M,P_0} = \Phi_M(x_1, \ldots, x_n; h_1(P_0), P_0)$ the corresponding solution of the Dirichlet's problem

$$(18) \qquad L(M_{P_0})\,\Phi = 0\,, \quad \Phi\{\dot{\Omega}[h_1(P_0), P_0]\} = \Phi_0 = \text{const}$$

$$(18a) \qquad \frac{\partial \Phi_M[x_1(s_1, \ldots, s_{n-1}; h_1(P_0), P_0), \ldots, x_n(s_1, \ldots, s_{n-1}; h_1(P_0), P_0)]}{\partial n} =$$

$$= \psi\{\dot{\Omega}[h_1(P_0), P_0]\}$$

and if we make further assumptions:

III) The one-parametrical system of possible critical surfaces $\dot{\Omega} = \dot{\Omega}[h_1(P_0), P_0]$ can be chosen in such a way that the function

$$G_2(s_1, \ldots, s_{n-1}; h_1(P_0), P_0) \equiv$$

$$\equiv \frac{\partial \Phi_M[x_1(s_1, \ldots, s_{n-1}; h_1(P_0), P_0), \ldots, x_n(s_1, \ldots, s_{n-1}; h_1(P_0), P_0)]}{\partial n}$$

can be splitted into a product of two functions

$$G_2(s_1, \ldots, s_{n-1}; h_1(P_0), P_0) = g_2(s_1, \ldots, s_{n-1}; h_1(P_0), P_0)\, H_2[h_1(P_0), P_0] \tag{19}$$

where the function $g_2(s_1, \ldots, s_{n-1}; h_1(P_0), P_0)$ is bounded and not identically zero:

$$|g_2(s_1, \ldots, s_{n-1}; h_1(P_0), P_0)| \leqq \text{const}\,, \quad g_2(s_1, \ldots, s_{n-1}; h_1(P_0), P_0) \not\equiv 0\,; \tag{19a}$$

IV) the equation

$$H_2[h_1(P_0), P_0] = 0 \tag{20}$$

has in the given interval $P_0^{(1)} \leqq P_0 \leqq P_0^{(2)}$ the unique root P_0^*

$$H_2[h_1(P_0^*), P_0^*] = 0\,, \quad P_0^* \in \langle P_0^{(1)}, P_0^{(2)} \rangle\,; \tag{20a}$$

then we see that for the values of the parameters

$$P_0^*\,, \quad N_0^* = h_1(P_0^*) \tag{21}$$

the thermal neutron flux $\Phi = \Phi^*$ in the reactor core with the critical shape

$$\dot{\Omega} = \dot{\Omega}(N_0^*, P_0^*) \tag{21a}$$

fulfils evidently the sufficient conditions (9) for the flux flattening so that we have

(22)

$$\Phi_M^* = \Phi_M(x_1, \ldots, x_n; N_0^*, P_0^*) = \Phi_0 = \text{const}\,; \quad L(M_{P_0^*})\, \Phi_M^* = 0\,; \quad \left.\frac{\partial \Phi_M^*}{\partial n}\right|_{\dot{\Omega}(N_0^*, P_0^*)} = 0\,.$$

By the foregoing considerations we have proved the following

Theorem 1. *Let us suppose that for the functions* $M(x_1, \ldots, x_n)$, $\Phi(x_1, \ldots, x_n)$ *the following conditions are fulfilled*:

1. *The domain* Ω *of the functions* M, Φ *is bounded by a two-parametrical symmetric boundary* $\dot{\Omega} = \dot{\Omega}(N_0, P_0)$ (*given by the parametric formulae* (11)), *on which four continuous real functions* $\vartheta_1(\dot{\Omega})$, $\vartheta_2(\dot{\Omega})$, $\psi(\dot{\Omega})$, $\omega(\dot{\Omega})$ *are given.*

2. *There exists a function* $M = M(x_1, \ldots, x_n; N_0, P_0)$ *which is the unique solution of the Dirichlet's boundary value problem* (7), (7b) *on* Ω *and satisfies the symmetry condition* (7a) *and the assumptions* I), II).

3. *The Dirichlet's boundary value problem* (8), (8b) *has for this* M *and for every* $\psi(\dot{\Omega}) \in C(\dot{\Omega})$, $\omega(\dot{\Omega}) \in C(\dot{\Omega})$ *a unique solution* $\Phi = \Phi_M(\psi, \omega) = \Phi(x; N_0, P_0, M, \psi, \omega)$ *in* Ω (*and particularly the solution* $\Phi_{M,P_0} = \Phi_M(x_1, \ldots, x_n; h_1(P_0), P_0)$ *for* $\psi(\dot{\Omega}) = \psi\{\dot{\Omega}[h_1(P_0), P_0]\}$, $\omega(\dot{\Omega}) = \Phi_0 = \text{const}$) *which satisfies the symmetry conditions* (8a) *and the assumptions* III), IV).

Then among the "possible critical shapes" $\dot{\Omega} = \dot{\Omega}[h_1(P_0), P_0]$ of the reactor core given by (16a) *and depending on the real parameter P_0 there is the critical shape $\dot{\Omega}(N_0^*, P_0^*)$ given by* (20a), (21) *for which the function $\Phi_M^* = \Phi(x_1, \ldots, x_n; N_0^*, P_0^*, M_{P_0*})$ fulfils the sufficient flattening conditions* (22) *for $M_{P_0*} = M(x_1, \ldots \ldots, x_n; N_0^*, P_0^*)$.*

It can be directly seen that under the conditions (7c) and under the usual further condition

$$D = \text{const} > 0 \tag{23}$$

relation (8) for the thermal neutron flux Φ reduces to the linear biharmonic equation

(23a)

$$\tilde{L}(M)\, \Phi \equiv D\, \Delta(\Delta \Phi) - \left[\frac{D}{\tau} + \Sigma_M^a(M + 1)\right] \Delta \Phi - 2\Sigma_M^a \operatorname{grad}(M + 1) \operatorname{grad} \Phi = 0$$

with the symmetry conditions (8a) and with the boundary value conditions (8b), (or (9) for the flattened flux $\Phi = \Phi_0$), which together with the simplified relations (7d), (7a), (7b) for the relative fuel concentration distribution M can be solved evidently in the same manner and under the same assumptions as in Theorem 1. However, numerical solution will be much easier.

We shall consider now a problem induced by the equation (5) which is a far reaching generalization of the problem of the thermal flux flattening: for the given thermal neutron flux $\Phi(x_1, \ldots, x_n) > 0$ in the critical reactor core Ω the distribution $M(x_1, \ldots, x_n)$ of the fuel concentration is to be determined which induces this given flux $\Phi(x_1, \ldots, x_n)$ and obeys the following boundary value conditions [2]

$$M(\dot{\Omega}) = \tilde{\vartheta}_1(\dot{\Omega}; \Phi(\dot{\Omega}))\,, \tag{24}$$

$$\left.\frac{\partial M}{\partial n}\right|_{\dot{\Omega}} = \tilde{\vartheta}_2(\dot{\Omega}; \Phi(\dot{\Omega})) \tag{24a}$$

where $\tilde{\vartheta}_1 = \tilde{\vartheta}_1(\dot{\Omega}; \Phi(\dot{\Omega}))$, $\tilde{\vartheta}_2 = \tilde{\vartheta}_2(\dot{\Omega}, \Phi(\dot{\Omega}))$ are given continuous functions expressing the influence of the reflector on the core. From the equation (5) it follows that the function $M(x_1, \ldots, x_n)$ in this case has to obey the following nonlinear elliptic equation (in general nonhomogeneous):

$$\begin{aligned}(25)\qquad & \Delta M + \frac{1}{\tau}\left(\operatorname{grad} \tau + \frac{2\tau}{\Sigma_M^a} \operatorname{grad} \Sigma_M^a + \frac{2\tau}{\Phi} \operatorname{grad} \Phi\right) \operatorname{grad} M - \\ & - \left\{\frac{1}{\tau}\left[(1 - k(M)) - \frac{1}{\Sigma_M^a} \operatorname{grad} \tau \,.\, \operatorname{grad} \Sigma_M^a - \frac{\tau}{\Sigma_M^a} \Delta \Sigma_M^a\right] - \right. \\ & \left. - \frac{2}{\Phi \Sigma_M^a} \operatorname{grad} \Sigma_M^a \,.\, \operatorname{grad} \Phi - \frac{1}{\tau \Phi} \operatorname{grad} \tau \,.\, \operatorname{grad} \Phi - \frac{1}{\Phi} \Delta \Phi\right\} M =\end{aligned}$$

$$= \frac{1}{\Phi \Sigma_M^a} \Bigg\{ D\,\Delta(\Delta\Phi) + \left(3 \operatorname{grad} D + \frac{D}{\tau} \operatorname{grad} \tau\right) . \operatorname{grad} \Delta\Phi +$$

$$+ \left(\Delta D + \frac{1}{\tau} \operatorname{grad} \tau \ \operatorname{grad} D - \frac{D}{\tau} - \Sigma_M^a\right) \Delta\Phi + \left[\left(2 \operatorname{grad} \frac{\partial D}{\partial x_1} + \frac{1}{\tau}\frac{\partial D}{\partial x_1} \operatorname{grad} \tau\right) .\right.$$

$$\left. . \operatorname{grad} \frac{\partial \Phi}{\partial x_1} + \ldots + \left(2 \operatorname{grad} \frac{\partial D}{\partial x_n} + \frac{1}{\tau}\frac{\partial D}{\partial x_n} \operatorname{grad} \tau\right) . \operatorname{grad} \frac{\partial \Phi}{\partial x_n}\right] +$$

$$+ \left(\operatorname{grad} \Delta D - \frac{1}{\tau} \Sigma_M^a \operatorname{grad} \tau - 2 \operatorname{grad} \Sigma_M^a\right) . \operatorname{grad} \Phi +$$

$$+ \left[\left(\frac{1}{\tau} \operatorname{grad} \tau \,.\, \operatorname{grad} \frac{\partial D}{\partial x_1}\right) \frac{\partial \Phi}{\partial x_1} + \ldots + \left(\frac{1}{\tau} \operatorname{grad} \tau \,.\, \operatorname{grad} \frac{\partial D}{\partial x_n}\right) \frac{\partial \Phi}{\partial x_n}\right] +$$

$$+ \frac{\Sigma_M^a}{\tau} \left(1 - \frac{1}{\Sigma_M^a} \operatorname{grad} \tau \,.\, \operatorname{grad} \Sigma_M^a - \frac{\tau}{\Sigma_M^a} \Delta \Sigma_M^a\right) \Phi \Bigg\}$$

which under the usual conditions (7c), (23) assumes the simplified form (from which we see that for Φ for which its right-hand side vanishes identically it will be homogeneous)

$$\text{(25a)} \qquad \Delta M + \frac{2}{\Phi} \operatorname{grad} \Phi \,.\, \operatorname{grad} M - \left\{\frac{1}{\tau}\left[1 - k(M)\right] - \frac{1}{\Phi} \Delta\Phi\right\} M =$$

$$= \frac{1}{\Phi \Sigma_M^a} \left[D\,\Delta(\Delta\Phi) - \left(\frac{D}{\tau} + \Sigma_M^a\right) \Delta\Phi + \frac{\Phi \Sigma_M^a}{\tau}\right].$$

The equation (25) or (25a) for $M(x_1, \ldots, x_n)$ together with the boundary value condition (24) represents again a quasilinear Dirichlet's problem which for $\Phi =$ $=$ const obviously reduces to the Dirichlet's problem (7), (7b) or (7d), (7b) respectively. If we consider again the two-parametrical system of surfaces (10), (11), choose a fixed extreme fuel concentration value M_0 in (12) and make the assumptions I) and II) and a further assumption:

V) the equation (implied by (12))

$$\text{(26)} \qquad f[h_1(P_0), P_0] = M_0$$

has in the given interval $P_0^{(1)} \leqq P_0 \leqq P_0^{(2)}$ the unique root P_0^{**}:

$$\text{(26a)} \qquad P_0^{**} = F(M_0) \in \langle P_0^{(1)}, P_0^{(2)} \rangle\,, \quad N_0^{**} = h_1(P_0^{**})\,,$$

then we obtain by an analogous consideration as above

Theorem 2.: *Let us suppose that for the given thermal neutron flux* $\tilde{\Phi}(x_1, \ldots, x_n) \in \in C^{(4)}(\Omega)$ *which satisfies the symmetry conditions* (8a) *there exists a fuel concentration* $M(x_1, \ldots, x_n; \tilde{\Phi}) \in C^{(2)}(\Omega)$ *which obeys the following conditions:*

1. *The domain* Ω *of the functions* M, $\tilde{\Phi}$ *is bounded by a two-parametrical symmetric boundary* $\dot{\Omega} = \dot{\Omega}(N_0, P_0)$ *(given by the parametric formulae* (11), *so that* $M = M(x_1, \ldots, x_n; \tilde{\Phi}, N_0, P_0)$), *on which two continuous functions* $\vartheta_1(\dot{\Omega}; \tilde{\Phi}(\dot{\Omega}))$, $\vartheta_2(\dot{\Omega}, \tilde{\Phi}(\dot{\Omega}))$ *are given.*

2. *The function* $M(x_1, \ldots, x_n; \tilde{\Phi}, N_0, P_0)$ *is the unique solution of the Dirichlet's boundary value problem* (25) *or* (25a), (24), *which assumes the fixed extreme fuel concentration* M_0 *at the center of symmetry* 0, *satisfies the symmetry relations* (7a) *and the assumptions I), II), V).*

Then among the "possible critical shapes" $\dot{\Omega} = \dot{\Omega}[h(P_0), P_0]$ *of the reactor core given by* (16a) *and depending on the real parameter* P_0 *there is a critical shape* $\dot{\Omega}(N_0^{**}, P_0^{**})$ *given by* (26a) *for which the fuel concentration* $M(x_1, \ldots, x_n; \tilde{\Phi}, N_0^{**}, P_0^{**})$ *induces in the reactor core* $\dot{\Omega}$ *the given thermal neutron flux* $\tilde{\Phi}(x_1, \ldots, x_n)$

We have still to show under which assumptions in the both above mentioned problems the usual two-group equations in the reflector Λ [1]

$$-\Delta\Phi_R + \frac{1}{L_R^2}\,\Phi_R = \frac{1}{D_R}\,q_R \tag{27}$$

$$-\Delta q_R + \frac{1}{\tau_R}\,q_R = 0 \tag{27a}$$

with the usual boundary conditions (expressing the continuity of the thermal and the fast neutron fluxes and currents) on the interface $\dot{\Omega}$ between the core Ω and the reflector Λ (where we have $\Phi(\dot{\Omega}) = \omega(\dot{\Omega})$, $(\partial\Phi/\partial n)|_{\dot{\Omega}} = \psi(\dot{\Omega})$, $M(\dot{\Omega}) = \vartheta_1$ and $(\partial M(\dot{\Omega})/\partial n) = \vartheta_2$)

$$\Phi_R(\dot{\Omega}) + \frac{D_R}{D}\frac{\partial\Phi_R}{\partial n}\bigg|_{\dot{\Omega}} = \Phi(\dot{\Omega}) + \frac{\partial\Phi}{\partial n}\bigg|_{\dot{\Omega}} \tag{28}$$

$$q_R(\dot{\Omega}) + \frac{\tau_R}{\tau}\frac{\partial q_R}{\partial n}\bigg|_{\dot{\Omega}} = \Phi(\dot{\Omega})\,\Sigma_M^a \left\{\frac{(\xi\Sigma_s)_R}{(\xi\Sigma_s)_A}\,[1 + M(\dot{\Omega})] + \frac{\partial M}{\partial n}\bigg|_{\dot{\Omega}}\right\} - \tag{28a}$$

$$- \frac{(\xi\Sigma_s)_R}{(\xi\Sigma_s)_A}\,\mathrm{div}\,(D\,\mathrm{grad}\,\Phi)\bigg|_{\dot{\Omega}} + \frac{\partial\Phi}{\partial n}\bigg|_{\dot{\Omega}} .\,\Sigma_M^a[1 + M(\dot{\Omega})] - \frac{\partial}{\partial n}\,[\mathrm{div}\,(D\,\mathrm{grad}\,\Phi)]\bigg|_{\dot{\Omega}}$$

and with the usual further conditions on the external face of the reflector

$$\Phi_R(\dot{\Lambda}) = 0 \tag{29}$$

$$q_R(\dot{\Lambda}) = 0 \tag{29a}$$

can be solved. Let us consider again a two-parametrical system of surfaces (with the parameters S_0, T_0)

$$\dot{\Lambda} = \dot{\Lambda}(S_0, T_0) \tag{30}$$

whose elements $\dot{\Lambda}(y_1, \ldots, y_n; S_0, T_0)$ are given by the parametric formulae

$$y_i = y_i(t_1, \ldots, t_{n-1}; S_0, T_0) \quad (i = 1, 2, \ldots, n)\,, \; n > 1\,. \tag{31}$$

If we make the following assumptions:

VI) Both the Newton's boundary value problems (27), (28) and (27a), (28a) have unique solutions Φ_R^* and q_R^* respectively on Λ.

VII) For these solutions the system of two boundary value conditions (29), (29a) assumes the form

$$\Phi_R^*[\dot{\Lambda}(S_0, T_0)] \equiv \varphi(t_1, \ldots, t_{n-1}; S_0, T_0)\, H_3(S_0, T_0) = 0 \tag{32}$$

$$q_R^*[\dot{\Lambda}(S_0, T_0)] \equiv Q(t_1, \ldots, t_{n-1}; S_0, T_0)\, H_4(S_0, T_0) = 0 \tag{32a}$$

where the coupled system of equations

$$H_3(S_0, T_0) = 0\,, \quad H_4(S_0, T_0) = 0 \tag{33}$$

has the unique solution $S_0 = S_0^*$, $T_0 = T_0^*$, i.e.,

$$H_3(S_0^*, T_0^*) = 0\,, \quad H_4(S_0^*, T_0^*) = 0\,, \tag{34}$$

then there obviously holds the following

Theorem 3.: *Under the assumptions VI), VII) there exists a unique outer surface $\dot{\Lambda} = \dot{\Lambda}(S_0^*, T_0^*)$ of the reflector Λ of the reactor whose core Ω has the prescribed thermal neutron flux $\Phi(x)$, $x \in \Omega$.*

Remark 1. We see immediately that we can fulfil the assumptions (13), (19) or (32), (32a) by putting e.g., $s_1 = s_2 = \ldots = s_{n-2} = N_0$, $s_{n-1} = P_0$ or $t_1 = t_2 = \ldots = t_{n-2} = S_0$, $t_{n-1} = T_0$, respectively.

Remark 2. For numerical solution of the problem considered, one can use e.g., finite difference methods, or finite element methods.

References

[1] *V. Bartošek, R. Zezula:* Flat Flux in a Slab reactor with Natural Uranium. Report ÚJV ČSAV 1310 (1965).

[2] *R. Zezula:* A sufficient condition for the flattening of thermal neutron flux and some related problems (in onedimensional geometries). Apl. Mat. 14 (1969), 134—145.

[3] *A. Miasnikov, R. Zezula:* Radial flux flattening in a cylindrical reactor with natural uranium (In Czech — Internal Report OTRF, ÚJV ČSAV).

[4] *V. Bartošek, R. Zezula:* Flat flux in a slab reactor with natural uranium. Journal of Nuclear Energy Parts A/B, 1966, vol. 20, pp. 129—139.

[5] *M. Hron, V. Lelek:* Flux flattening by means of a nonuniform fuel distribution in a slab reactor with finite reflector. Report ÚJV ČSAV 1660 (1966).

[6] *V. Bartošek, R. Zezula:* Stability of flat thermal flux in a slab reactor. Apl. Mat. 13 (1968), 367—375.

[7] *Ladyzhenskaya O. A., Ural'ceva N. N.:* Linear and quasilinear equations of elliptic type. Nauka, Moscow 1964 (in russian).

Souhrn

PODMÍNKY KRITIČNOSTI PRO KONEČNÝ HOMOGENIZOVANÝ REAKTOR NA PŘÍRODNÍ URAN S PŘEDEPSANÝM TOKEM TEPELNÝCH NEUTRONŮ

ROSTISLAV ZEZULA

V článku se matematicky formuluje (v dvougrupovém difuzním přiblížení a pro vícerozměrné geometrie) následující problém z teorie jaderných reaktorů: Pro zadaný průběh toku Φ tepelných neutronů v aktivní zoně Ω konečného homogenizovaného reaktoru určit rozložení koncentrace paliva $M(x)$ v Ω, které tento tok Φ vytváří. Jsou udány podmínky (zejména na tvar hranice $\dot{\Omega}$ jádra reaktoru Ω resp. hranice $\dot{\Lambda}$ jeho reflektoru Λ) postačující pro existenci jediného řešení tohoto problému, a zejména též pro existenci jediného řešení ve speciálním případě vyrovnaného toku tepelných neutronů $\Phi = \Phi_0 =$ konst v aktivní zoně Ω reaktoru, který má praktický význam, neboť dává minimum kritické hmoty.

Author's address: Dr. *Rostislav Zezula,* CSc., Matematický ústav Karlovy university, Sokolovská 83, Praha 8 - Karlín.