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SOME STATISTICAL ASPECTS OF THE ESTIMATION OF PARAMETERS OF A LINEAR CONFORM TRANSFORMATION

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1. INTRODUCTION

In the two-dimensional Euclidean space E_x we know the coordinates

$$\mathbf{x}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad i = 1, \ldots, N; N > 2$$

of different points P_i. The transformation T of the space E_x onto the space E_ι – which is also a two-dimensional and Euclidean one – is given by the relation

$$[T(\mathbf{x}) \equiv]\, \vec{\eta} = (\mathbf{E}, \mathbf{M}) \begin{pmatrix} \mathbf{q} \\ \mathbf{a} \end{pmatrix} \tag{1}$$

where

$$\vec{\eta} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 1, 0 \\ 0, 1 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} x, & y \\ y, & -x \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

The components of the vectors $\mathbf{q}$ and $\mathbf{a}$ are unknown, but one can determine the position of the points $T(P_i) = \vec{\eta}_i$, $i = 1, \ldots, N$ by means of indirect measurement (further details in the following).

Our aim is to determine the estimates $\hat{\mathbf{q}}$, $\hat{\mathbf{a}}$ of the vectors $\mathbf{q}$, $\mathbf{a}$ and to investigate the statistical properties of these estimates and eventually to point out some consequences following from them for the transformation T.

When we consider the results of measurements of the vectors $\vec{\eta}_i$; $i = 1, \ldots, N$, which are normally distributed with a covariance matrix independent of the index i and stochastically independent, then the solution is known and is given e.g. in [1] pp. 248 and foll.

The aim of this paper is to generalize the solutions in case of stochastically dependent results of measurements of the vector $\vec{\eta}_i$ and to point out geometrical interpretations of the obtained results.

Remark. The above mentioned problem of multidimensional regression occurs often in mathematical cartography.

2. SYMBOLS AND ASSUMPTIONS

We arrange the coordinates of the points $T(P_i)$ in $2N$-dimensional column vector $\boldsymbol{H}(\boldsymbol{H}' = (\eta'_1, \ldots, \eta'_N))$. It is impossible to measure the components of the vector $\boldsymbol{H}$ directly. Measurements can be carried out only for the components of the vector $\bar{\boldsymbol{Z}}$ (of the order $m \times 1$; $m > 2N$) for which $\bar{\boldsymbol{Z}} = \mathbf{G}\boldsymbol{H}$ holds where the matrix $\mathbf{G}$ with known elements is of the order $m \times 2N$ and has the rank $h(\mathbf{G}) = 2N$. The result of the measurement $\boldsymbol{Z}$ of the vector $\bar{\boldsymbol{Z}}$ is the realization of a random vector with the distribution $N(\mathbf{G}\boldsymbol{H}, \sigma^2\mathbf{P}^{-1})$. The matrix $\mathbf{P}$ is a diagonal matrix of the weights $p_i > 0$ of the measurement of the i-th component of the vector $\bar{\boldsymbol{Z}}$ and therefore the dispersion of measurement of this components is σ^2/p_i. The parameter σ (the standard deviation of measurement with unit weight) is assumed in the following to be unknown. The estimate of the vector $\vec{\eta}_i$ is denoted by $\mathbf{y}_i$ and the $2N$-dimensional vector created by the subvectors $\mathbf{y}_i$ is denoted by $\boldsymbol{Y}$. According to [10] p. 145 it holds: $\boldsymbol{Y} = (\mathbf{G}'\mathbf{P}\mathbf{G})^{-1}\,\mathbf{G}'\mathbf{P}\boldsymbol{Z}$ is a normal regular vector with the distribution $N(\boldsymbol{H}; \sigma^2(\mathbf{G}'\mathbf{P}\mathbf{G})^{-1})$. (See also the definition of a normal regular vector in [10] p. 45 and in [1] p. 29.)

We denote the matrix $\sigma^2(\mathbf{G}'\mathbf{P}\mathbf{G})^{-1}$ by Σ and we suppose that at least one submatrix $\Sigma_{kl} = M[(\mathbf{y}_k - \vec{\eta}_k)(\mathbf{y}_k - \vec{\eta}_k)']$ for $k \neq l$ is different from the zero matrix, i.e. no stochastic independence of random vectors $\mathbf{y}_1, \ldots, \mathbf{y}_N$ occurs. (In the following we shall point out some simple consequences in case of stochastic independence.)

Denoting

$$\begin{pmatrix} \mathbf{E}, & \ldots, & \mathbf{E} \\ \mathbf{M}_1, & \ldots, & \mathbf{M}_N \end{pmatrix} = \mathbf{R}$$

where

$$\mathbf{M}_k = \begin{pmatrix} x_k & y_k \\ y_k & -x_k \end{pmatrix}$$

we may denote the distribution of vector $\boldsymbol{Y}$ symbolically by

$$\boldsymbol{Y}, \ldots, N\left(\mathbf{R}'\begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{a} \end{pmatrix}, \Sigma\right). \tag{2}$$

The density of probability of the random vector $\boldsymbol{Y}$ is ([1], p. 29):

$$n\left(\boldsymbol{Y} \mid \mathbf{R}'\begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{a} \end{pmatrix}, \Sigma\right) = f(\boldsymbol{Y}; \boldsymbol{q}, \boldsymbol{a}) = \tag{2a}$$

$$= \{1/[(2\pi)^N\, |\Sigma|^{1/2}]\} \exp\left\{-\frac{1}{2}\left[\left(\boldsymbol{Y} - \mathbf{R}'\begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{a} \end{pmatrix}\right)' \Sigma^{-1} \left(\boldsymbol{Y} - \mathbf{R}'\begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{a} \end{pmatrix}\right)\right]\right\}.$$

For the sake of brevity we denote the vector $\begin{pmatrix} \mathbf{q} \\ \mathbf{a} \end{pmatrix}$ by the symbol $\mathbf{t}$. If f is a function of the components of the vector $\mathbf{t}$, whose partial derivatives of the first degree exist, then the symbol $\partial f / \partial \mathbf{t}$ denotes the column vector whose i-th component is $\partial f / \partial t_i$ (see also [4]).

Lemma 1. *The matrix of the positive definite form is regular.*

Proof. The lemma is a consequence of the statement in [9] p. 181.

Lemma 2. *Let* $\mathbf{A}$ *be a matrix of the order* $n \times n$ *of a positive definite form and let the rank of the matrix* $\mathbf{B}$ *of the order* $m \times n$ *be* $h(\mathbf{B}) = m < n$, *then* $\mathbf{BAB}'$ *is a matrix of a positive definite form.*

Proof. As $\mathbf{Z}'\mathbf{BAB}'\mathbf{Z} \geqq 0$ holds obviously for each m-dimensional vector $\mathbf{Z}$ it will be sufficient to prove the implication $\mathbf{Z}'\mathbf{BAB}'\mathbf{Z} = 0 \Rightarrow \mathbf{Z} = 0$. Thus let $\mathbf{Z}'\mathbf{BAB}'\mathbf{Z} = 0$. With regard to the positive definiteness of the matrix $\mathbf{A}$ there must be $\mathbf{B}'\mathbf{Z} = \mathbf{0}$ and with regard to the rank of the matrix $\mathbf{B}$ there must be $\mathbf{Z} = \mathbf{0}$.

Lemma 3. *The rank of the matrix* $\mathbf{R}$ *is* 4.

Proof. According to the theorem of Laplace ([7] p. 296) let us develop the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{E} & \mathbf{E} \\ \mathbf{M}_1 & \mathbf{M}_2 \end{pmatrix}$$

with respect to the first two rows. We get:

$$|\mathbf{A}| = -(r_1^2 + r_2^2 - 2r_1 r_2 \cos \varphi) = -\varrho_2^2(P_1, P_2) \neq 0$$

where ϱ_2 is the metric in the two-dimensional space E_x, $r_i = \varrho_2(P_0, P_i)$, $i = 1, 2$, P_0 is a point in E_x with zero coordinates and φ is the angle between the radiusvectors of the points P_1 and P_2.

Theorem 1. *There exists just one maximum likelihood estimate of the vector* $\mathbf{t}$. *It is given by the solution of the system*

$$\mathbf{R}\Sigma^{-1}\mathbf{R}'\hat{\mathbf{t}} = \mathbf{R}\Sigma^{-1}\mathbf{Y}, \tag{3}$$

which can be written in the form

$$\begin{pmatrix} \sum_i \sum_j \Sigma^{ij}, & \sum_i \sum_j \Sigma^{ji}\mathbf{M}_i \\ \sum_i \sum_j \mathbf{M}_j \Sigma^{ji}, & \sum_i \sum_j \mathbf{M}_j \Sigma^{ji} \mathbf{M}_i \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{a} \end{pmatrix} = \begin{pmatrix} \sum_i \sum_j \Sigma^{ji}\mathbf{y}_i \\ \sum_i \sum_j \mathbf{M}_j \Sigma^{ji} \mathbf{y}_i \end{pmatrix} \tag{3a}$$

where Σ^{ij} *is the submatrix* Σ^{-1} *of the elements of the* $(2i - 1)$*-st and* $2i$*-th rows and of the* $(2j - 1)$*-st and* $2j$*-th columns.*

Proof. The likelihood equations ([3] pp. 477–478) with respect to (2a) are given by the relation

$$(4) \qquad \partial(\ln L(\mathbf{t}))/\partial\mathbf{t} =$$
$$= (\partial/\partial\mathbf{t})\{\ln\langle\{1/[(2\pi)^N |\Sigma|^{1}/^{2}]\} . \exp[-\tfrac{1}{2}(\mathbf{Y} - \mathbf{R}'\mathbf{t})' \Sigma^{-1}(\mathbf{Y} - \mathbf{R}'\mathbf{t})]\rangle\} = 0^{(4\times 1)} .$$

The left hand side of the above equation may be rearranged:

$$(\partial/\partial\mathbf{t})\langle\ln\{1/[(2\pi)^N |\Sigma|^{1}/^{2}]\} - \tfrac{1}{2}[\mathbf{Y}'\Sigma^{-1}\mathbf{Y} - \mathbf{t}'\mathbf{R}\Sigma^{-1}\mathbf{Y} -$$
$$- \mathbf{Y}'\Sigma^{-1}\mathbf{R}'\mathbf{t} + \mathbf{t}'\mathbf{R}\Sigma^{-1}\mathbf{R}'\mathbf{t}]\rangle = \mathbf{R}\Sigma^{-1}\mathbf{Y} - \mathbf{R}\Sigma^{-1}\mathbf{R}'\mathbf{t} .$$

If we show that the matrix $\mathbf{C} = \mathbf{R}\Sigma^{-1}\mathbf{R}'$ is the matrix of a positive definite form, we show with respect to Lemma 1 the existence of only one solution of the equation of likelihood (4). According to the assumption we have in E_x N various points P_i, thus with respect to Lemma 3 the rank of the matrix is 4 and with respect to Lemma 2 and Lemma 1 the existence of only one solution is proved. To complete the proof it is sufficient to show that for $\mathbf{t}$ of (3) the function L attains its maximum. With respect to the form of the function $L = f$ from (2a) and of the theorem on the local minimum [5] p. 505 it is sufficient to show that the matrix $\|-\partial^2 \ln L/\partial t_i\, \partial t_j\|$ is the matrix of a positive definite form. The last matrix, however, is equal to $\mathbf{C}$ which proves the theorem.

Remark 1. As can be seen from (3), if we do not know the parameter σ^2 in the matrix Σ, it is still possible to determine $\hat{\mathbf{t}}$.

Remark 2. When comparing the expression (3a) with the solution of the problem of linear regression between the variables x_i and y_i (with the normal distribution $N(c_1 + c_2 x_i; \sigma^2/p_i)$) we see that the role of the coordinate x_i, of the result of the measurement y_i, of the weight of measurement p_i of the value $c_1 + c_2 x_i$ and of the constant c_1 and c_2 in our problem were taken over by the submatrix $\mathbf{M}_i$, the vector $\mathbf{y}_i$, the expression $\sigma^2 \sum_j \Sigma^{ij}$ and by $\mathbf{q}$ and $\mathbf{a}$, respectively.

Lemma 4. *If* $\mathbf{x}$ *is a normal vector* $N(\vec{\mu}, \Sigma)$, *then* $\mathbf{t} = \mathbf{D}\mathbf{x}$ *is a normal vector* $N(\mathbf{D}\vec{\mu}, \mathbf{D}\Sigma\mathbf{D}')$.

Proof is in [1] p. 41.

Theorem 2. *The estimate* $\hat{\mathbf{t}}$ *is a normally distributed* $N(\mathbf{t}, \mathbf{C}^{-1})$.

Proof. With respect to (3) we have $\hat{\mathbf{t}} = \mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{Y}$ and thus with respect to [1] p. 39 and Lemma 4 we have that $\hat{\mathbf{t}}$ is normally distributed with the vector of the mean values $M(\hat{\mathbf{t}}) = \mathbf{C}^{-1}\mathbf{R}\Sigma^{-1} M(\mathbf{Y}) = \mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{R}'\mathbf{t} = \mathbf{t}$ and with the covariance matrix $M[(\hat{\mathbf{t}} - \mathbf{t})(\hat{\mathbf{t}} - \mathbf{t})'] = \mathbf{C}^{-1}\mathbf{R}\Sigma^{-1} M[(\mathbf{Y} - \mathbf{R}'\mathbf{t})(\mathbf{Y} - \mathbf{R}'\mathbf{t})'] \Sigma^{-1}\mathbf{R}'\mathbf{C}^{-1} = \mathbf{C}^{-1}$, respectively.

Corollary. *The estimate* $\hat{\mathbf{t}}$ *from* (3) *is statistically unbiased.*

Theorem 3. *The estimate* $\hat{\mathbf{t}}$ *from* (3) *is jointly efficient.*

Proof. With respect to [6] p. 28 it is sufficient to prove that

$$\|M(-\partial^2 \ln L/\partial t_i \, \partial t_j)\| = \mathbf{C}\,.$$

With respect to the last two sentences in the proof of Theorem 1, the last equality is obvious.

4. STATISTICAL PROPERTIES OF VECTORS OF CORRECTIONS AND OF VECTORS OF TRANSFORMED COORDINATES

When relation (1) is applied, further random vectors occur and it may be useful to know their statistical properties.

Definition 1. *The vector* $\mathbf{V}$ *given by the relation* $\mathbf{V} = \mathbf{R}'\hat{\mathbf{t}} - \mathbf{Y}$ *is called the vector of corrections of the components of the vector* $\mathbf{Y}$.

Definition 2. *A random vector* $\mathbf{x}$ *with p components, with a vector of the mean values* $M(\mathbf{x}) = \vec{\mu}$ *and with the covariance matrix* $M[(\mathbf{x} - \vec{\mu})(\mathbf{x} - \vec{\mu})'] = \Sigma$ *is called normal if*

1. *there is a matrix* $\mathbf{A}$ *of the order* $p \times r$, *so that the rank* $h(\Sigma) = r$;
2. *there is an r-dimensional normal regular vector* $\mathbf{y}$;
3. *the following holds:* $\mathbf{x} = \mathbf{A}\mathbf{y} + \lambda$ *where* λ *is an r-dimensional vector. If* $r < p$, $\mathbf{x}$ *is a singular normal vector of the rank r.*

Lemma 5. *Let us have two linear vector functions of the n-dimensional normal vector* $\mathbf{x}$: $\mathbf{y}_1 = \mathbf{A}_1\mathbf{x}$, $\mathbf{y}_2 = \mathbf{A}_2 x$ *where* $\mathbf{A}_1$ *is a matrix of the order* $m_1 \times n$ *and* $\mathbf{A}_2$ *is a matrix of the order* $m_2 \times n$. *Then the equality* $\mathbf{A}_1\Sigma\mathbf{A}_2' = \mathbf{0}$ *is a necessary and sufficient condition for the stochastic independence of the vectors* $\mathbf{y}_1$ *and* $\mathbf{y}_2$, Σ *being the covariance matrix of the vector* $\mathbf{x}$.

Proof is in [10] p. 57.

Lemma 6. *If the symmetrical matrix* $\mathbf{B}$ *is positively definite, then there is such a regular matrix* $\mathbf{E}$ *that* $\mathbf{EBE}' = \mathbf{I}$ (*unit matrix*).

Proof is in [1] p. 457.

Lemma 7. *Let the p-dimensional vector* $\mathbf{x}$ *be normally distributed* $\dots N(\vec{\mu}, \Sigma)$. *Then* $\mathbf{y} = \mathbf{Cx}$ *is, for a regular matrix* $\mathbf{C}$, *normally distributed* $\dots N(\mathbf{C}\vec{\mu}, \mathbf{C}\Sigma\mathbf{C}')$.

Proof is in [1] p. 32.

Lemma 8. *For each symmetrical matrix* $\mathbf{A}$ *of a quadratic form, there is such an orthogonal matrix* $\mathbf{Q}$ *that* $\mathbf{QAQ}'$ *is a diagonal matrix.*

Proof is in [9] p. 227.

Lemma 9. *If* $\mathbf{A}_{m,m}$ *is a regular matrix of the order* $m \times m$, *then the rank of the matrix* $\mathbf{AB}$ *where* $\mathbf{B}$ *is a matrix of the order* $m \times n$, *is equal to the rank of the matrix* $\mathbf{B}$.

Proof is in [9] p. 101.

Lemma 10. *For the rank of the matrix* $\mathbf{A} = \mathbf{BC}$ *the relation* $h(\mathbf{A}) \leqq \min\{h(\mathbf{B}), h(\mathbf{C})\}$ *holds.*

Proof is in [9] p. 101.

Theorem 4. *Vector* $\mathbf{V}$ *is a normal singular vector of the rank* $2N - 4$.

Proof. With respect to Definition 1 and Theorem 1 we have $\mathbf{V} = (\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1} - \mathbf{I})\,\mathbf{Y}$ where $\mathbf{I}$ is the unit matrix. With respect to Lemma 4, $\mathbf{V}$ is a normal vector with the vector of the mean values $M(\mathbf{V}) = (\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1} - \mathbf{I})\,\mathbf{R}'\mathbf{t} = (\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{R}' - \mathbf{R}')\,\mathbf{t} = (\mathbf{R}' - \mathbf{R}')\,\mathbf{t} = 0$ and with the covariance matrix $M(\mathbf{V}\,.\,\mathbf{V}') = (\mathbf{I} - \mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1})\,.\,\Sigma(\mathbf{I} - \Sigma^{-1}\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}) = \Sigma - 2\mathbf{R}'\mathbf{C}^{-1}\mathbf{R} + \mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{R}'\mathbf{C}^{-1}\mathbf{R} = \Sigma - \mathbf{R}'\mathbf{C}^{-1}\mathbf{R}$. With respect to Lemma 6 there is such a regular matrix $\mathbf{F}$ that $\Sigma = \mathbf{F}^{-1}\mathbf{F}'^{-1}$. Further, with respect to Lemma 9 the following holds: $h(\Sigma - \mathbf{R}'\mathbf{C}^{-1}\mathbf{R}) = h(\mathbf{F}^{-1}\mathbf{F}'^{-1} - \mathbf{R}'\mathbf{C}^{-1}\mathbf{R}) = h(\mathbf{F}^{-1}(\mathbf{I} - \mathbf{FR}'\mathbf{C}^{-1}\mathbf{RF}')\,\mathbf{F}'^{-1}) = h(\mathbf{I} - \mathbf{FR}'\mathbf{C}^{-1}\mathbf{RF}')$. Let us denote $\mathbf{U} = \mathbf{FR}'\mathbf{C}^{-1}\mathbf{RF}'$. Obviously $\mathbf{U}' = \mathbf{U}$ and $\mathbf{U}^2 = \mathbf{U}$. With respect to Lemma 8 there is an orthogonal matrix $\mathbf{H}$ such that $\mathbf{HUH}' = \mathbf{D}$ is a diagonal matrix. $\mathbf{D}^2 = \mathbf{HUH}'\mathbf{HUH}' = \mathbf{HU}^2\mathbf{H}' = \mathbf{HUH}' = \mathbf{D}$ holds, which is possible only if there are only 0 or 1 on the diagonal of the matrix. The number of ones is obviously equal to the rank of the matrix $\mathbf{D}$ and $h(\mathbf{D}) = h(\mathbf{U})$ with respect to Lemma 9. With respect to Lemma 10 we have further $h(\mathbf{U}) = h(\mathbf{FR}'\mathbf{C}^{-1}\mathbf{RF}') \leqq 4$. As $\mathbf{RF}'\mathbf{UFR}' = \mathbf{RF}'\,.\,(\mathbf{FR}'\mathbf{C}^{-1}\mathbf{RF}')\,\mathbf{FR}' = \mathbf{R}\Sigma^{-1}\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{R}' = \mathbf{C}$, it holds $h(\mathbf{C}) \leqq h(\mathbf{U})$ with respect to Lemma 10. With respect to Theorem 1 $h(\mathbf{C}) = 4$, hence $h(\mathbf{U}) = 4$. Further, with respect to Lemma 9 $h(\mathbf{I} - \mathbf{U}) = h[\mathbf{H}(\mathbf{I} - \mathbf{U})\,\mathbf{H}'] = h(\mathbf{I} - \mathbf{D}) = 2N - 4$. In the proof of Lemma 4 it is shown that in such a case $\mathbf{V}$ is a singular normal vector with the rank $2N - 4$.

Theorem 5. *The vectors* $\hat{\mathbf{t}}$ *and* $\mathbf{V}$ *are statistically independent.*

Proof. With respect to Theorem 1 and Definition 1 the following holds: $\mathbf{V} = (\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1} - \mathbf{I})\,\mathbf{Y}$; $\hat{\mathbf{t}} = \mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{Y}$. As $(\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1} - \mathbf{I})\,\Sigma\Sigma^{-1}\mathbf{R}'\mathbf{C}^{-1} = \mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{R}'\mathbf{C}^{-1} - \mathbf{R}'\mathbf{C}^{-1} = \mathbf{R}'\mathbf{C}^{-1} - \mathbf{R}'\mathbf{C}^{-1} = \mathbf{0}$, Theorem is proved by means of Lemma 5.

Theorem 6. *The random variable* $\mathbf{V}'\Sigma^{-1}\mathbf{V}$ *has* χ^2*-distribution with* $2N - 4$ *degrees of freedom.*

Proof. With respect to Theorem 4 $\mathbf{V}'\Sigma^{-1}\mathbf{V} = \mathbf{Y}'(\mathbf{I} - \Sigma^{-1}\mathbf{R}'\mathbf{C}^{-1}\mathbf{R})\,\Sigma^{-1}(\mathbf{I} - \mathbf{R}'\mathbf{C}^{-1}\,.\,\mathbf{R}\Sigma^{-1})\,\mathbf{Y} = \mathbf{Y}'(\Sigma^{-1} - \Sigma^{-1}\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1})\,\mathbf{Y} = (\mathbf{Y} - \mathbf{R}'\mathbf{t})'\,(\Sigma^{-1} - \Sigma^{-1}\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1})\,.\,(\mathbf{Y} - \mathbf{R}'\mathbf{t})$. With respect to Lemma 6 there is such a regular matrix $\mathbf{F}$ that $\mathbf{F}\Sigma\mathbf{F}' = \mathbf{I}$. If the matrix from the proof of Theorem 4 is considered as $\mathbf{U}$ we have $\mathbf{V}'\Sigma^{-1}\mathbf{V} = \mathbf{w}'(\mathbf{I} - \mathbf{U})\,\mathbf{w}$ where $\mathbf{w} = \mathbf{F}(\mathbf{Y} - \mathbf{R}'\mathbf{t})$ is, according to Lemma 7, a normally distributed vector $N(\mathbf{O}, \mathbf{I})$. We have further $\mathbf{V}'\Sigma^{-1}\mathbf{V} = \mathbf{w}'\mathbf{H}'(\mathbf{I} - \mathbf{HUH}')\,\mathbf{Hw} = \mathbf{Z}'(\mathbf{I} - \mathbf{D})\,\mathbf{Z}$ where $\mathbf{Z}$ is, according to Lemma 7 and Lemma 8, a normally distributed $N(\mathbf{O}, \mathbf{I})$ and $\mathbf{H}$ is an orthogonal matrix from the proof of Theorem 4. Thus $\mathbf{V}'\Sigma^{-1}\mathbf{V} = Z_{i_1}^2 + \ldots + Z_{i_{2N-4}}^2$ where $i_1, \ldots, i_{2N-4}$ are the indices of those components of the vector $\mathbf{Z}$ in the rows of which there are zeros along the diagonal of the matrix $\mathbf{D}$. With respect to the definition of the χ^2-distribution, Theorem is proved.

Corollary. *With respect to the assumption it holds* $\Sigma = \sigma^2(\mathbf{G}'\mathbf{PG})^{-1}$ *where* σ *is an unknown parameter. According to Theorem* 6 $\mathbf{V}'\mathbf{G}'\mathbf{PGV} = \sigma^2\chi^2_{2N-4}$ *and thus*

1. $\hat{\sigma}^2 = [1/(2N - 4)]\ \mathbf{V}'\mathbf{G}'\mathbf{PGV}$ *is, regarding the properties of the* χ^2*-distribution, an unbiased estimate of the a priori unknown dispersion* σ^2.
2. *The confidence interval for the parameter* σ *is given by the relation*

$$P\{\sqrt{(\mathbf{V}'\mathbf{G}'\mathbf{PGV}/\alpha_2)} \leqq \sigma \leqq \sqrt{(\mathbf{V}'\mathbf{G}'\mathbf{PGV}/\alpha_1)}\} = \\ = \int_{\alpha_1}^{\alpha_2} [(u^{[(2N-4)/2]-1} e^{-u/2})/(2^{(2N-4)/2}\ \Gamma[(2N - 4)/2])]\ \mathrm{d}u\,.$$

Remark. To determine the vector $\mathbf{Y}$ one may use also the vector $\mathbf{V}_1 = \mathbf{GY} - \mathbf{Z}$. In [10] p. 151 and in the following it is shown that the random variable $\mathbf{V}_1'\mathbf{PV}_1 = \sigma^2\chi^2_{m-2N}$. As it is statistically independent of $\mathbf{V}'\mathbf{G}'\mathbf{PGV}$, we have $\mathbf{V}_1'\mathbf{PV}_1 + \mathbf{V}'\mathbf{G}'\mathbf{P}\,.\,\mathbf{GV} = \sigma(\chi^2_{2N-4} + \chi^2_{m-2N}) = \sigma^2\chi^2_{m-4}$. With respect to the inequality $m > 2N$ this may be used for a better estimate of the parameter σ.

Theorem 7. *The random vector* $\mathbf{R}'\hat{\mathbf{t}}$ *(vector of the transformed coordinates of the points* $P_1, \ldots, P_N$*) is a singular normal vector* $N(\mathbf{R}'\mathbf{t};\ \mathbf{R}'\mathbf{C}^{-1}\mathbf{R})$ *of the rank* 4.

Proof. With respect to Theorem 2 and Lemma 4 the following holds: $\mathbf{R}'\hat{\mathbf{t}} = \mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{Y}$ is a normal vector with the vector of the mean values $M(\mathbf{R}'\hat{\mathbf{t}}) =$

$= \mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{R}'\mathbf{t} = \mathbf{R}'\mathbf{t}$ and with the covariance matrix $M[\mathbf{R}'(\hat{\mathbf{t}} - \mathbf{t})(\hat{\mathbf{t}} - \mathbf{t})' \mathbf{R}] = \mathbf{R}'\mathbf{C}^{-1}\mathbf{R}$. With respect to Definition 2 and to the proof of Theorem 4 it is sufficient to show that the rank of the matrix $\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}$ is $h(\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}) = 4$. With respect to Lemma 10 evidently $h(\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}) \leqq 4$ and with respect to Lemma 9 and Lemma 10 $h(\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}) = h(\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}) \geqq h(\mathbf{R}'\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\mathbf{R}') = h(\mathbf{R}') = 4$ (with respect to Lemma 3), which completes the proof.

Definition 3. *Let us call the vector $\Delta = \mathbf{Rt} - \mathbf{Y}$ the vector of errors.*

According to the assumption concerning the distribution of the random vector $\mathbf{Y}$, Δ is obviously a normaly distributed $N(\mathbf{O}, \Sigma)$. By the expression $(\mathbf{E}, \mathbf{M}_i)\,\mathbf{t} - \mathbf{y}_i$, the vector of error Δ_i at the point $T(P_i)$ is denoted, thus in E_y it is the vector from the measured position of the point $T(P_i)$ to the true position $T(P_i)$, i.e. $\Delta' = (\Delta'_1, \ldots, \Delta'_N)$. Similarly we decompose $\mathbf{V}$, $\mathbf{V}' = (\mathbf{v}'_1, \ldots, \mathbf{v}'_N)$.

Lemma 11. *The symmetrical matrix is only then the matrix of a positive definite form, if all its principal subdeterminants are positive.* For the proof see [9] p. 181.

Lemma 12. *If the square matrix $\mathbf{A}$ can be divided into submatrices*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

where $\mathbf{A}_{11}$ is square and regular, then

$$|\mathbf{A}| = |\mathbf{A}_{11}|\,|\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}|\,.$$

For the proof see [1] p. 463.

Lemma 13. *Let the components of the vector $\mathbf{x}$ be divided into two parts $\mathbf{x}_1, \mathbf{x}_2$; $\mathbf{x}' = (\mathbf{x}'_1, \mathbf{x}'_2)$. Analogously the components of the vectors of the mean values $\vec{\mu}$ are divided into two vectors $\vec{\mu}_1, \vec{\mu}_2$, and we assume further that the covariance matrix Σ of the vector $\mathbf{x}$ is regular and is divided into corresponding submatrices $\Sigma_{11}, \Sigma_{12}, \Sigma_{22}$ with respect to the division of vector $\mathbf{x}$, which are covariance matrices of vectors $\mathbf{x}_1$; $\mathbf{x}_1$ and $\mathbf{x}_2$; $\mathbf{x}_2$. Then, if $\mathbf{x}$ is a normal vector, the conditional distribution $\mathbf{x}_1$ at the given $\mathbf{x}_2$ is also normal with the vector of the mean values $\vec{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \vec{\mu}_2)$ and with the covariance matrix $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$.*

For the proof see [1] pp. 43–45.

Theorem 8. *If the matrix $\Sigma_{ii} - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\begin{pmatrix}\mathbf{E}\\ \mathbf{M}_i\end{pmatrix}$ is regular, then the vector of error Δ_i at the point $T(P_i)$ conditioned by the knowledge of the corresponding vector of the correction $\mathbf{v}_i$ is regularly normally distributed with the mean value $M(\Delta_i \mid \mathbf{v}_i) = \mathbf{v}_i$ and with the covariance matrix $M(\Delta_i\Delta'_i \mid \mathbf{v}_i) = (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\begin{pmatrix}\mathbf{E}\\ \mathbf{M}_i\end{pmatrix}$.*

Proof. Since

$$M(\mathbf{v}_i\Delta_i') = M\{[\Delta_i - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\Delta]\,\Delta_i'\} =$$

$$= M(\Delta_i\Delta_i') - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\begin{bmatrix} M(\Delta_1\Delta_i') \\ \vdots \\ M(\Delta_N\Delta_i') \end{bmatrix} =$$

$$= \Sigma_{ii} - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\mathbf{R}\Sigma^{-1}\begin{bmatrix} \Sigma_{1i} \\ \vdots \\ \Sigma_{Ni} \end{bmatrix} = \Sigma_{ii} - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\mathbf{R}\begin{bmatrix} \mathbf{O} \\ \vdots \\ \mathbf{E} \\ \vdots \\ \mathbf{O} \end{bmatrix} =$$

$$= \Sigma_{ii} - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\begin{bmatrix} \mathbf{E} \\ \mathbf{M}_i \end{bmatrix},$$

Lemma 4 and Theprem 4 implies that $(\Delta_i', \mathbf{v}_i')$ is a normal vector with the zero vector of the mean values and with the covariance matrix $\mathbf{B}$

$$\mathbf{B} = \begin{bmatrix} \Sigma_{ii}; & \Sigma_{ii} - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\begin{pmatrix} \mathbf{E} \\ \mathbf{M}_i \end{pmatrix} \\ \Sigma_{ii} - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\begin{pmatrix} \mathbf{E} \\ \mathbf{M}_i \end{pmatrix}; & \Sigma_{ii} - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\begin{pmatrix} \mathbf{E} \\ \mathbf{M}_i \end{pmatrix} \end{bmatrix}.$$

Hence, in order that the covariance matrix may have the rank 4 (i.e. for the sake of regularity of distribution) it is sufficient to show that its determinant is positive. With respect to Lemma 12, Lemma 11 and the assumptions of Theorem there is

$$|\mathbf{B}| = |\Sigma_{ii}|\left|\Sigma_{ii} - (\mathbf{E}, \mathbf{M}_i)\,\mathbf{C}^{-1}\begin{pmatrix} \mathbf{E} \\ \mathbf{M}_i \end{pmatrix}\right| > 0\,.$$

With respect to Lemma 13, theorem is proved.

5. GEOMETRICAL INTERPRETATION

The transformation (1) can be interpreted as the shifting of the origin of coordinates S to the point $T(S)$ with the coordinates $\boldsymbol{q}$, as a change of the original metric by multiplying it by the number $m = \sqrt{(\boldsymbol{a}'\boldsymbol{a})}$ (scale) and as a rotation of the point field by the value $\alpha = \arctan(a_2/a_1)$. After such a transformation an arbitrary point $P \in E_x$ will have the position $T(P)$ given by the vector $\mathbf{y} = \boldsymbol{q} + \mathbf{M}\boldsymbol{a}$.

Since the parameters $\mathbf{t}$ of the transformation (1) are estimated by means of the random vector $\boldsymbol{Y}$, the estimates $\hat{\boldsymbol{q}}$, $\hat{m}$, $\hat{\alpha}$, $\hat{\mathbf{y}}$ will be also random variables. The accuracy of the estimate will be characterized by the corresponding dispersion, eventually by the covariance matrix or the confidence interval and the confidence domain.

Lemma 14. *If* $\mathbf{y}$ *is the m-dimensional normal regular vector* $N(\mathbf{O}, \Sigma)$, *then the random variable* $\mathbf{y}'\Sigma^{-1}\mathbf{y}$ *has a* χ^2*-distribution with m degrees of freedom.*

For the proof see [1] p. 77.

Lemma 15. *If* $\mathbf{x}$ *is the random vector with the normal distribution* $N(\vec{\mu}, \Sigma)$, *then the joint distribution of an arbitrary group of components of the vector* $\mathbf{x}$ *is a multidimensional normal distribution with mean values, dispersion and covariances equal to the corresponding elements in* $\vec{\mu}$ *and* Σ.

For the proof see [1] p. 38.

In the following we assume the matrix $\mathbf{C}$ to be divided into submatrices of the order 2×2

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix}$$

or the matrix $\mathbf{C}^{-1}$ similarly divided into submatrices, using the superscripts $\mathbf{C}^{ij}$.

Theorem 9. *The true position of the point* $T(S)$ *is with probability* P *in the ellipse*

$$(5) \qquad (\mathbf{q} - \hat{\mathbf{q}})' (\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}) (\mathbf{q} - \hat{\mathbf{q}}) = -2 \ln (1 - P) .$$

Proof. With respect to Lemma 15 and Theorem 2 the two-dimensional vector $\mathbf{q}$ is a normal vector $N(\mathbf{q}, \mathbf{C}^{11})$. We have further

$$(a) \qquad \mathbf{C}_{11}\mathbf{C}^{11} + \mathbf{C}_{12}\mathbf{C}^{21} = \mathbf{I} \Rightarrow \mathbf{C}^{11} = \mathbf{C}_{11}^{-1} - \mathbf{C}_{11}^{-1}\mathbf{C}_{12}\mathbf{C}^{21}$$

$$(b) \qquad \mathbf{C}_{21}\mathbf{C}^{11} + \mathbf{C}_{22}\mathbf{C}^{21} = \mathbf{O} \Rightarrow \mathbf{C}^{21} = -\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\mathbf{C}^{11}$$

From (a) and (b) we have $\mathbf{C}^{11} = \mathbf{C}_{11}^{-1} + \mathbf{C}_{11}^{-1}\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\mathbf{C}^{11}$. By rearrangement we have $(\mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21})\,\mathbf{C}^{11} = \mathbf{I}$. Considering further that the probability integral for the χ^2-distribution with two degrees of freedom satisfies the following

$$\int_0^{k^2} \tfrac{1}{2} e^{-x/2}\, \mathrm{d}x = 1 - e^{-k^2/2}$$

then with respect to Lemma 14 the Theorem is proved.

Remark. If we estimate the parameter σ by means of the vector $\mathbf{V}$ (see Theorem 6 and the following Corollary) we obtain as the confidence ellipse instead of (5) the following ellipse (see also [10] p. 317):

$$\text{(5a)} \qquad (\mathbf{q} - \hat{\mathbf{q}})' \, \sigma^2 (\mathbf{C}_{11} - \mathbf{C}_{12} \mathbf{C}_{22}^{-1} \mathbf{C}_{21}) (\mathbf{q} - \hat{\mathbf{q}}) = \\ = \mathbf{V}' \sigma^2 \Sigma^{-1} \mathbf{V} \left[\left(\sqrt[N-2]{\frac{1}{1 - P}} \right) - 1 \right]$$

which results from the definition of Fisher's random variable, Theorem 5 and the properties of the probability integral of Fisher's random variable with 2 and $2N - 4$ degrees of freedom, respectively.

$$P = \int_0^{\gamma_0} \{ \Gamma(N-1) / [\Gamma(N-2) \, \Gamma(1)] \} \, [1 + F/(N-2)]^{-(N-1)} \, \mathrm{d}F = \\ = 1 - \left(1 + \frac{\gamma_0}{N-2} \right)^{-N+2} .$$

Further one must realize that if in the matrix $\Sigma = \sigma^2 (\mathbf{G}'\mathbf{P}\mathbf{G})^{-1}$ only the parameter σ^2 is not known, then $\sigma^{-2}\Sigma$ is evidently a matrix with known elements.

Conditions (A). *Let $\Sigma_{kl} = \mathbf{O}$ be independent of k for $k \neq l$ and let $\Sigma_{kk} = \Sigma_0$ not depend on k. Let further the point S (the origin of the system of coordinates in E_x) be situated at the centre of gravity of the points $P_1, \ldots, P_N$.*

Conditions (B). *Let $\Sigma_{kl} = \mathbf{O}$ for $k \neq l$ and $\Sigma_{kk} = \sigma^2 \mathbf{I}$. Let the point S again be situated at the centre of gravity of the points $P_1, \ldots, P_N$.*

Corollary 1. *Let Conditions* (A) *be satisfied. Then the submatrix $\mathbf{C}_{11} = N\Sigma_0^{-1}$ and $\mathbf{C}_{12} = \sum_i \sum_j \Sigma^{ij} \mathbf{M}_j = \Sigma_0^{-1} \sum_j \mathbf{M}_j = \Sigma_0^{-1} \mathbf{O} = \mathbf{O}$. The confidence domain* (5) *is similar to the confidence domain of the determination of the position of the point $T(P_i)$, but the half-axes of this confidence ellipse are multiplied by the number $1/\sqrt{N}$.* (See also in [8].)

Corollary 2. *Let Conditions (B) be satisfied. The confidence ellipse* (5) *then changes into a circle with the centre at $\hat{\mathbf{q}}$ and with the radius $\sigma \sqrt{\{[-2 \ln (1 - P)]/N\}}$. In the case of* (5a) *the radius is equal to $(\mathbf{V}'\mathbf{V}/N)^{1/2} \, ([1/(1 - P)]^{1/(N-2)} - 1)^{1/2}$.*

Lemma 16. 1. *Let in a certain neighbourhood of the point $\hat{\mathbf{a}} = \mathbf{a}$ the function $H(\hat{\mathbf{a}})$ be continuous and let it have continuous partial derivatives of the first and second order with respect to the argument a_1, a_2.*

2. *Let for all possible values of $\hat{\mathbf{a}}$ $|H| < CN^p$ hold, where C and p are positive constants.*

3. *There is a number $a > 0$ independent of N such that the set of the points*

$P_1, \ldots, P_N$ *is outside the circle with the radius a and with the centre at the centre of gravity T of these points.*

4. *There are numbers b, r, independent from N, for which* $0 < b < O_1(N) \leqq \leqq O_2(N) \leqq r$ *holds where* $O_1(N)$ *or* $O_2(N)$, *are the values of the minimum or maximum half-axis respectively of the ellipsoid* $\mathbf{Y}'\Sigma^{-1}\mathbf{Y} = 1$ *in the 2N-dimensional Euclidean space.*

If we denote by H_0, H_1, H_2 the values which the function H and its first partial derivatives assume at the point $\mathbf{a}$, then we have for the mean values $M(H)$ and the dispersion $\sigma^2(H)$ of the random variable $H(\hat{\mathbf{a}})$

$$M(H) = H_0 + O(N^{-1}), \tag{6}$$

$$\sigma^2(H) = (H_1, H_2)\, \mathbf{C}^{22} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} + O(N^{-3/2}). \tag{7}$$

The definition of the function O see in [3] p. 121.

Proof. The Lemma results from a modification of the Lemma in [3] p. 339. Therefore in the following we shall make use of the proof presented there. Let $\Sigma = r^2\mathbf{I}$ where r^2 is the constant from Condition 4. Thus

$$\mathbf{C}^{22} = r^2 \begin{bmatrix} 1/(\sum_i \varrho^2(P_i, T), & 0 \\ 0, & 1/(\sum_i \varrho^2(P_i, T)) \end{bmatrix}$$

and consequently $\sigma^2(\hat{a}_1) \leqq r^2/(Na^2) = O(N^{-1})$. Similarly $\sigma^2(\hat{a}_2) \leqq O(N^{-1})$ and $M[(\hat{a}_1 - a_1)(\hat{a}_2 - a_2)] = O(N^{-1})$.

If Condition 4 is satisfied and if the random vector $\mathbf{Y}$ from (2) is replaced by the vector $\mathbf{z}$ with the normal distribution $N(\mathbf{R}'\mathbf{t}, r^2\mathbf{I})$, then the estimate of the position of the point $T(P_i)$; $i = 1, \ldots, N$ is replaced by an estimate with an undiminished generalized dispersion (on generalized dispersion see [1] pp. 231 to 240). When estimating the values a_1 and a_2 by using the vector $\mathbf{z}$ and respecting Conditions 3 and 4, we have $a_1 = O(N^{-1})$, $a_2 = O(N^{-1})$ and $M[(\hat{a}_1 - a_1)(\hat{a}_2 - a_2)] = O(N^{-1})$; all the more must these equalities hold if the vector $\mathbf{Y}$ is used.

According to the inequality of TCHEBYSHEFF [3] p. 179 and with respect to the normality of the vector $\mathbf{a}$ (Theorem 2) for each $\varepsilon > 0$ the following holds

$$P\{\hat{a}_1 : |\hat{a}_1 - a_1| \geqq \varepsilon\} = P\{\hat{a}_1 : |\hat{a}_1 - a_1|^{2k} \geqq \varepsilon^{2k}\} \leqq$$
$$\leqq M(\hat{a}_1 - a_1)^{2k}/\varepsilon^{2k} = 1 \,.\, 3 \,.\, 5, \ldots, (2k-1)\, \sigma^{2k}(\hat{a}_1)/\varepsilon^{2k} \leqq$$
$$\leqq \frac{1 \,.\, 3 \,.\, 5 \ldots (2k-1)\, r^{2k}}{N^k a^k \varepsilon^{2k}}.$$

An analogous inequality holds for $\hat{a}_2$. By the symbol Z we denote the set of all points $\mathbf{Y}$ in the $2N$-dimensional Euclidean space for which $|\hat{a}_1 - a_1| < \varepsilon$ and $|\hat{a}_2 - - a_2| < \varepsilon$ simultaneously holds. By the symbol Z' we denote the complement of this set.

If we now substitute in [3] on p. 340 for A the $1 \cdot 3 \cdot 5 \ldots (2k - 1)\, r^{2k}/a^k$, we can almost verbatim complete the proof of Lemma 16 by repeating the procedure from [3] pp. 340–342 starting from the relation (27.7.4).

Lemma 17. *Let in a certain neighbourhood of the point* $\mathbf{a}$ *the function* $H(\hat{\mathbf{a}})$ *be continuous, let it have continuous partial derivatives of the first and the second order with respect to the argument* $\hat{\mathbf{a}}_1$ *and* $\hat{\mathbf{a}}_2$ *and let Conditions* 3 *and* 4 *of Lemma* 16 *be satisfied. Then the random variable* $H(\hat{\mathbf{a}})$ *is asymptotically normally distributed with the mean value and with the dispersion equalling the main expression on the right-hand side of* (6) *and* (7).

Proof. Lemma results by some non-essential rearrangements of the Lemma in [3] p. 351 and therefore the proof is completed by the same argument.

Theorem 10. *If Conditions* 3 *and* 4 *of Lemma* 16 *are satisfied, then the random variable* $\hat{\alpha} = \arctan(\hat{a}_2/\hat{a}_1)$ *has the mean value and the dispersion*

$$M(\hat{\alpha}) = \operatorname{arctgn}(a_2/a_1) + O(N^{-1})$$

$$\sigma^2(\hat{\alpha}) = \left(-\frac{\sin\alpha}{m}, \frac{\cos\alpha}{m}\right) \mathbf{C}^{22} \begin{pmatrix} -\dfrac{\sin\alpha}{m} \\ \dfrac{\cos\alpha}{m} \end{pmatrix} + O(N^{-3/2}),$$

while being asymptotically normal according to Lemma 17.

Proof. It suffices to consider the relations $m^2 = a_1^2 + a_2^2$, $\tan\alpha = a_2/a_1$, Lemma 16 and Lemma 17.

Corollary. *If Conditions* (*B*) *are satisfied, then*

$$\sigma^2(\hat{\alpha}) = \sigma^2 / (m^2 \sum_{i=1}^{N} \varrho^2(P_i, T)) + O(N^{-3/2}).$$

Theorem 11. *The random variable* $\hat{m} = \sqrt{(a_1^2 + a_2^2)}$ *is, when Conditions* 3 *and* 4 *of Lemma* 16 *are satisfied, asymptotically normal and*

$$M(\hat{m}) = m + O(N^{-1})$$

$$\sigma^2(\hat{m}) = (\cos\alpha, \sin\alpha)\, \mathbf{C}^{22} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + O(N^{-3/2}) = m^2\, \sigma^2(\hat{\alpha}) + O(N^{-3/2}).$$

Proof. It is sufficient to consider Lemma 16 and Lemma 17 and Theorem 10 respectively.

Corollary. *If Conditions (B) are satisfied, then*

$$\sigma_2(\hat{m}) = \sigma^2 / \sum_{i=1}^{N} \varrho^2(P_i, T).$$

Definition 4. *The distribution function*

$$F(u, v) = \frac{1}{2\pi} \iint_{x^2+y^2<u^2} e^{-1/2[(x-v)^2+y^2]} \, dx \, dy =$$
$$= \frac{1}{2\pi} \iint_{(x-v)^2+y^2<u^2} e^{-1/2(x^2+y^2)} \, dx \, dy,$$

where v is a parameter, is called the Rayleigh-Rice function ([2] p. VI).

Theorem 12. *For the distribution function $G(u)$ of the random variable $\hat{m}$ we have $F(u_1; v) \leqq G(u) = P(\hat{m} < u) \leqq F(u_2; v)$ where $F(u; v)$ is the Rayleigh-Rice distribution function, $u_1(u_2)$ is the minor (major) half-axis of the ellipse $\mathbf{r}'\mathbf{F}\mathbf{F}'\mathbf{r} = u^2$ and the parameter $v = \sqrt{[\mathbf{a}'(\mathbf{C}^{22})^{-1}\,\mathbf{a}]}$. For the matrix $\mathbf{F}$ the following must hold: $\mathbf{F}(\mathbf{C}^{22})^{-1}\,\mathbf{F}' = \mathbf{I}$.*

Proof. If we consider the substitution $\hat{\mathbf{a}} = \mathbf{a} + \mathbf{F}'\mathbf{r}$ where $\mathbf{F}(\mathbf{C}^{22})^{-1}\,\mathbf{F}' = \mathbf{I}$ (according to Lemma 5) in the integral

$$P(\hat{m} < u) = \iint_{\hat{\mathbf{a}}'\hat{\mathbf{a}}<u^2} n(\hat{\mathbf{a}} \mid \mathbf{a}, \mathbf{C}^{22}) \, d\hat{\mathbf{a}},$$

we have

$$P(\hat{m} < u) = \iint_{\Omega} n(\mathbf{r} \mid \mathbf{O}; \mathbf{I}) \, dr$$

where

$$\Omega = \{\mathbf{r} : (\mathbf{r} + \mathbf{F}'^{-1}\mathbf{a})'\, \mathbf{F}\mathbf{F}'(\mathbf{r} + \mathbf{F}'^{-1}\mathbf{a}) \leqq u^2\},$$

i.e. the interior of an ellipse, the centre of which is at the distance $v = \sqrt{(\mathbf{a}'\mathbf{F}^{-1}\mathbf{F}'^{-1}\mathbf{a})} = \sqrt{(\mathbf{a}'(\mathbf{C}^{22})^{-1}\,\mathbf{a})}$ from the origin, which completes the proof according to Definition 4.

Corollary. *If Conditions (B) are satisfied we have*

$$G(u) = F(u\sigma / \sum_{i=1}^{N} \varrho^2(P_i; T);\; m\sigma / \sum_{i=1}^{N} \varrho^2(P_i, T)).$$

If we denote $\hat{\mathbf{y}} = (\mathbf{E}, \mathbf{M})\,\hat{\mathbf{t}}$ there holds

Theorem 13. *The point $T(B)$ with the coordinates $\mathbf{y} = (\mathbf{E}, \mathbf{M})\,\mathbf{t}$ is with the probability P in the ellipse*

$$(8)\quad (\mathbf{y} - \hat{\mathbf{y}})' (\mathbf{C}^{11} + \mathbf{M}\mathbf{C}^{21} + \mathbf{C}^{12}\mathbf{M} + \mathbf{M}\mathbf{C}^{22}\mathbf{M})^{-1} (\mathbf{y} - \hat{\mathbf{y}}) = -2 \ln (1 - P)\,.$$

Proof is analogous to that of Theorem 9.

Corollary. *If Conditions (B) hold, then the ellipse (8) changes into a circle with the centre at $\hat{\mathbf{y}}$ and with the radius*

$$\sigma \sqrt{[-2 \ln (1 - P)]}\, \sqrt{[1/N + \varrho^2(B, T)/\sum_{i=1}^{N} \varrho^2(P_i, T)]}\,.$$

If we estimate the parameter σ by means of the vector $\mathbf{V}$, we replace the expression $\sigma \sqrt{[-2 \ln (1 - P)]}$ by

$$\sqrt{\left\{\frac{\mathbf{V}'\mathbf{V}}{N}\left[\sqrt[N-2]{\left(\frac{1}{1 - P}\right)} - 1\right]\right\}}\,.$$

Theorem 14. *If the vector $\mathbf{V}_i = (\mathbf{E}, \mathbf{M}_i)\,\hat{\mathbf{t}} - \mathbf{y}_i$ is known, the end point of the true error Δ_i at the point $T(P_i)$ is with the probability P in the ellipse*

$$(\Delta_i - \mathbf{V}_i)' \left[(\mathbf{E}, \mathbf{M}_i)\, \mathbf{C}^{-1} \begin{pmatrix} \mathbf{E} \\ \mathbf{M}_i \end{pmatrix}\right]^{-1} (\Delta_i - \mathbf{V}_i) = -2 \ln (1 - P)\,.$$

Proof. Considering Theorem 8, the proof is analogous to that of Theorem 9.

Corollary. *If Conditions (B) are satisfied, then the confidence domain for the end point of the vector Δ_i is a circle with the centre at the end point of the vector $\mathbf{V}_i$ with the radius*

$$\sigma \sqrt{[-2 \ln (1 - P)]}\, \sqrt{[1/N + \varrho^2(P_i, T)/\sum_{i=1}^{N} \varrho^2(P_i, T)]}\,.$$

6. CONCLUSION

The linear conform transformation T is characterized from the geometrical point of view by a shifting of the coordinates $\mathbf{q}$, by a rotation α and by a change of scale m. After these steps an arbitrary point P of the first system is mapped into the point $T(P)$. The parameters of the transformation $q_1, q_2, a_1, a_2, m = \sqrt{(a_1^2 + a_2^2)}; \alpha = \operatorname{arctgan}(a_2/a_1)$ can be determined if we know the coordinates of at least two so-called identical points P_1, P_2 in the first system, or $T(P_1), T(P_2)$ in the second system.

In the present paper relations are derived that make it possible to calculate an efficient estimate of parameters and to consider the estimates $\mathbf{q}$, α and m and the position $T(P)$ defined by means of parameters estimated under the following conditions:

1. The coordinates of the identical points P_i $i = 1, \ldots, N$; $N > 2$ are given.

2. The coordinates of the points $T(P_i)$; $i = 1, \ldots, N$ are determined as the realization of the random vector $\mathbf{Y}$ with $2N$ componente having a non-diagonal covariance matrix (the so-called joint determination of coordinates by indirect measurement).

Until now this problem, frequently occurring in mathematical cartography, has been approached by the method of the least squares, which, under the above conditions, does not yield any efficient estimates.

References

[1] *Андерсон Т.*: Введение в многомерный статистический анализ, Госиздат физико-математической литературы, Москва 1963 (Transl. f. Engl. Anderson T.: An introduction to multivariate statistical analysis, New York, John Wiley and sons).

[2] *Барк Л. С., Большев Л. Н., Кузнецов П. И., Черенков А. П.*: Таблицы распределения Релея-Райса, Вычислительный центр АН СССР, Москва 1964.

[3] *Cramer H.*: Metody matematyczne w statystce, Państwowe wydawnictwo naukowe, Warszawa 1958.

[4] *Dwyer P. S., Macphail M. S.*: Symbolic matrix derivatives, The annals of mathematical statistics, Volume XIX, pp. 517—534 (1948).

[5] *Jarník V.*: Diferenciální počet II, NČSAV, Praha 1953.

[6] *Kendall M. G.* and *Stuart A.*: The advanced theory of statistics, Volume 2, Charles Griffin, London 1961.

[7] *Kořínek V.*: Základy algebry, NČSAV, Praha 1956.

[8] *Kubáček L.*: Confidence region in Helmert transformation, Studia geof. et geod. 10 (1966), pp. 124—136.

[9] *Курош А. Г.*: Курс высшей алгебры, Госиздат физико-математической литературы, Москва 1959.

[10] *Линник Ю. В.*: Метод наименших квадратов и основы математикостатистической теории обработки наблюдений, Госиздат физико-математической литературы, Москва 1962.

Súhrn

NIEKTORÉ ŠTATISTICKÉ ASPEKTY PRI ODHADE PARAMETROV LINEÁRNEJ KONFORMNEJ TRANSFORMÁCIE

LUBOMÍR KUBÁČEK

V práci sú odvodené vzťahy pre výpočet efektívnych odhadov parametrov lineárnej konformnej transformácie T a pre ocenenie niektorých funkcií týchto parametrov.

Pritom sú uvažované nasledujúce predpoklady:

1. Sú dané súradnice tzv. identických bodov P_i; $i = 1, \ldots, N$; $N > 2$. (Za identický bod považujeme každý bod P_i dvojrozmerného euklidovského priestoru E_x, ktorého súradnice v E_x poznáme, pričom súčasne poznáme štatistický odhad súradníc bodu $T(P_i)$.)

2. Súradnice bodov $T(P_i)$; $i = 1, \ldots, N$ sú odhadované realizáciou náhodného vektora $\boldsymbol{Y}$ s $2N$ komponentami. Vektor $\boldsymbol{Y}$ má nediagonálnu kovariačnú maticu (tzn. ide o združené určenie súradníc pomocou nepriameho merania).

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