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A METHOD OF INVERTING MATRICES

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In the present paper a method of computing the inverse of a matrix is proposed by means of computing inverses of matrices having simpler form than the original matrix. In particular the case is discussed when the auxiliary matrix inverted in each step of the process is triangular and of a lower order than the matrix of the previous step.

1.

Let $\mathbf{A}$ be a given square matrix of order n. Let us denote $\mathbf{A} = \mathbf{A}_0$ and decompose the matrix $\mathbf{A}_0$ into the sum of two matrices

$$\mathbf{A}_0 = \mathbf{M}_0 + \mathbf{N}_0 , \tag{1}$$

so that the inverse $\mathbf{M}_0^{-1}$ of the matrix $\mathbf{M}_0$ exists. Instead of (1) it is possible to write

$$\mathbf{A}_0 = \mathbf{M}_0(\mathbf{E} + \mathbf{M}_0^{-1}\mathbf{N}_0) . \tag{2}$$

It is clear that if $\mathbf{A}_0$ is regular then also $(\mathbf{E} + \mathbf{M}_0^{-1}\mathbf{N}_0)$ is regular, and

$$\mathbf{A}_0^{-1} = (\mathbf{E} + \mathbf{M}_0^{-1}\mathbf{N}_0)^{-1}\, \mathbf{M}_0^{-1} \tag{3}$$

holds.

Let us denote

$$\mathbf{E} + \mathbf{M}_0^{-1}\mathbf{N}_0 = \mathbf{A}_1 \tag{4}$$

so that

$$\mathbf{A}_0^{-1} = \mathbf{A}_1^{-1}\mathbf{M}_0^{-1} . \tag{5}$$

The matrix $\mathbf{A}_1$ can be similarly decomposed into the sum of two matrices, $\mathbf{A}_1 = \mathbf{M}_1 + \mathbf{N}_1$, so that $\mathbf{M}_1^{-1}$ exists. We obtain a relation similar to (3) for $\mathbf{A}_1^{-1}$:

$$\mathbf{A}_1^{-1} = (\mathbf{E} + \mathbf{M}_1^{-1}\mathbf{N}_1)^{-1}\, \mathbf{M}_1^{-1} . \tag{6}$$

Using (6) for $\boldsymbol{A}_1^{-1}$ in (5) we obtain

$$\boldsymbol{A}_0^{-1} = (\boldsymbol{E} + \boldsymbol{M}_1^{-1}\boldsymbol{N}_1)^{-1}\, \boldsymbol{M}_1^{-1}\boldsymbol{M}_0^{-1}\,. \tag{7}$$

If we continue in this manner, putting always

$$\boldsymbol{A}_k = (\boldsymbol{E} + \boldsymbol{M}_{k-1}^{-1}\boldsymbol{N}_{k-1}) \tag{8}$$

and decomposing $\boldsymbol{A}_k$ into the sum of two matrices,

$$\boldsymbol{A}_k = \boldsymbol{M}_k + \boldsymbol{N}_k \tag{9}$$

($\boldsymbol{M}_k$ regular), we obtain

$$\boldsymbol{A}_k^{-1} = (\boldsymbol{E} + \boldsymbol{M}_k^{-1}\boldsymbol{N}_k)^{-1}\, \boldsymbol{M}_k^{-1}\,,$$

and

$$\boldsymbol{A}_0^{-1} = \boldsymbol{A}_{k+1}^{-1}\boldsymbol{M}_k^{-1}\boldsymbol{M}_{k-1}^{-1} \ldots \boldsymbol{M}_1^{-1}\, \boldsymbol{M}_0^{-1}\,. \tag{10}$$

If we manage to choose the matrices $\boldsymbol{M}_k$ in the decompositions (9) so that we could invert them easily, and if we reach after a certain finite number of steps a matrix $\boldsymbol{A}_{k+1}$ in (10), the inverse of which is also easily computed, then we can get, with the aid of relation (10), the inverse $\boldsymbol{A}^{-1} = \boldsymbol{A}_0^{-1}$ of the given matrix $\boldsymbol{A}$.

In the next paragraph a choice of matrices $\boldsymbol{M}_k$ ($k = 0, 1, \ldots$) is described enabling us to find the inverse $\boldsymbol{A}^{-1}$ according to (10) for $k = n - 1$.

2.

Let us choose the decomposition of the matrix $\boldsymbol{A} = \boldsymbol{A}_0 = (a_{ij}^{(0)})$ into the sum of two matrices $\boldsymbol{M}_0$ and $\boldsymbol{N}_0$ in the following way:

$$\boldsymbol{M}_0 = \begin{pmatrix} a_{11}^{(0)} & & & & \\ a_{21}^{(0)} & 1 & & & \\ a_{31}^{(0)} & a_{32}^{(0)} & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \\ a_{n1}^{(0)} & a_{n2}^{(0)} & a_{n3}^{(0)} & \ldots & 1 \end{pmatrix}, \quad \boldsymbol{N}_0 = \begin{pmatrix} 0 & a_{12}^{(0)} & a_{13}^{(0)} & \ldots & a_{1n}^{(0)} \\ & a_{22}^{(0)} - 1 & a_{23}^{(0)} & \ldots & a_{2n}^{(0)} \\ & & a_{33}^{(0)} - 1 & \ldots & a_{3n}^{(0)} \\ & & & \cdots & \cdots \\ & & & & a_{nn}^{(0)} - 1 \end{pmatrix}.$$

Let us suppose $a_{11}^{(0)} \neq 0$. (The case where this condition is not fulfilled will be investigated in the next paragraph.) Then the matrix $\boldsymbol{M}_0$ is regular, and it is relatively easy to find its inverse. In addition, the 1's on the diagonal make the computations even simpler. The matrix $\boldsymbol{M}_0^{-1}$ is again a lower triangular one with its diagonal elements $1/a_{11}^{(0)}, 1, \ldots, 1$. In the matrix $\boldsymbol{N}_0 = (n_{ij}^{(0)})$ there is $n_{11}^{(0)} = 0$. Thus the matrix $\boldsymbol{M}_0^{-1}\boldsymbol{N}_0$ is a square matrix having as its first column a column of zeros so that the first column of the matrix $\mathbf{A}_1 = \mathbf{E} + \mathbf{M}_0^{-1}\mathbf{N}_0$ equals to the first column of the identity matrix.

In the decomposition of the matrix $\mathbf{A}_1 = (a_{ij}^{(1)})$ into the sum $\mathbf{M}_1 + \mathbf{N}_1$ we choose $\mathbf{M}_1 = (m_{ij}^{(1)})$ and $\mathbf{N}_1 = (n_{ij}^{(1)})$ in such a way that there is $m_{ii}^{(1)} = 1$, $i = 1, 3, \ldots, n$, $m_{22}^{(1)} = a_{22}^{(1)}$; let us again suppose $a_{22}^{(1)} \neq 0$. Then $\mathbf{M}_1$ is regular with its first column equal (to the first column of $\mathbf{A}_1$ and therefore) to the first column of the identity matrix. The matrix $\mathbf{M}_1^{-1}$ is again a lower triangular matrix with its first column equal to the first column of the identity matrix and with the diagonal elements $1, 1/a_{22}^{(1)}, 1, \ldots \ldots, 1$.

As $m_{11}^{(1)} = 1 = a_{11}^{(1)}$ and $m_{22}^{(1)} = a_{22}^{(1)}$, there is $n_{11}^{(1)} = 0$ and $n_{22}^{(1)} = 0$. Therefore the square matrix $\mathbf{M}_1^{-1}\mathbf{N}_1$ has zeros in its first two columns in the diagonal and below, the matrix $\mathbf{A}_2 = \mathbf{E} + \mathbf{M}_1^{-1}\mathbf{N}_1$ having its first two columns equal to the first two columns of the identity matrix with the exception of the overdiagonal element in the second column.

In this way we proceed further. In general, in the k-th step we choose the matrix $\mathbf{M}_k$ in the decomposition of the matrix $\mathbf{A}_k = (a_{ij}^{(k)})$ into the sum of two matrices $\mathbf{M}_k = (m_{ij}^{(k)})$ and $\mathbf{N}_k = (n_{ij}^{(k)})$ as follows:

$$\begin{aligned}
(11)\qquad & m_{ii}^{(k)} = 1\,, \quad i = 1, \ldots, n\,, \quad i \neq k + 1\,, \\
& m_{k+1,k+1}^{(k)} = a_{k+1,k+1}^{(k)} \quad (\text{supposing } a_{k+1,k+1}^{(k)} \neq 0)\,, \\
& m_{ij}^{(k)} = a_{ij}^{(k)}\,, \quad i = 2, \ldots, n\,; \quad j = 1, \ldots, i - 1\,, \\
& m_{ij}^{(k)} = 0\,, \quad i = 1, \ldots, n - 1\,; \quad j = i + 1, \ldots, n\,.
\end{aligned}$$

Then the elements of the matrix $\mathbf{N}_k$ fulfil:

$$\begin{aligned}
(12)\qquad & n_{ii}^{(k)} = 0\,, \quad i = 1, 2, \ldots, k + 1\,, \\
& n_{ii}^{(k)} = a_{ii}^{(k)} - 1\,, \quad i = k + 2, \ldots, n\,, \\
& n_{ij}^{(k)} = a_{ij}^{(k)}\,, \quad i = 1, \ldots, n - 1\,; \quad j = i + 1, \ldots, n\,, \\
& n_{ij}^{(k)} = 0\,, \quad i = 2, \ldots, n\,; \quad j = 1, \ldots, i - 1\,.
\end{aligned}$$

According to this choice of matrices $\mathbf{M}_k$ $(k = 0, 1, 2, \ldots)$, the partition of the matrices $\mathbf{A}_k$, $\mathbf{M}_k$, $\mathbf{N}_k$ into blocks is possible as follows:

$$(13)\qquad \mathbf{A}_k = \begin{pmatrix} \mathbf{P}_k & \mathbf{Q}_k \\ \mathbf{O}_{n-k} & \mathbf{R}_{n-k} \end{pmatrix},$$

$$(14)\qquad \mathbf{M}_k = \begin{pmatrix} \mathbf{E}_k & \mathbf{Q}_k \\ \mathbf{O}_{n-k} & \mathbf{L}_{n-k} \end{pmatrix},$$

$$(15)\qquad \mathbf{N}_k = \begin{pmatrix} \widetilde{\mathbf{P}}_k & \mathbf{Q}_k \\ \mathbf{O}_{n-k} & \mathbf{U}_{n-k} \end{pmatrix},$$

where

$$\text{(16)} \qquad \mathbf{P}_k = \begin{pmatrix} 1 & a_{12}^{(k)} & \dots & a_{1k}^{(k)} \\ & 1 & \dots & a_{2k}^{(k)} \\ & & \dots\dots\dots & \\ & & 1 & a_{k-1,k}^{(k)} \\ & & & 1 \end{pmatrix} \quad \text{(triangular, order } k\text{)},$$

$$\text{(17)} \qquad \mathbf{Q}_k = \begin{pmatrix} a_{1,k+1}^{(k)} & \dots & a_{1.n}^{(k)} \\ \dots & \dots & \dots \\ a_{k,k+1}^{(k)} & \dots & a_{k,n}^{(k)} \end{pmatrix} \quad \text{(rectangular, type } k,\ n-k\text{)},$$

$$\text{(18)} \qquad \mathbf{R}_{n-k} = \begin{pmatrix} a_{k+1,k+1}^{(k)} & \dots & a_{k+1,n}^{(k)} \\ \dots & \dots & \dots \\ a_{n,k+1}^{(k)} & \dots & a_{n,n}^{(k)} \end{pmatrix} \quad \text{(square, order } n-k\text{)},$$

(supposing $a_{k+1,k+1}^{(k)} \neq 0$), $\mathbf{O}_{n-k}$ and $\mathbf{O}_k$ is the rectangular zero matrix of the type $n-k$, k and k, $n-k$, respectively. $\mathbf{E}_k$ is the identity matrix of order k,

$$\text{(19)} \qquad \mathbf{L}_{n-k} = \begin{pmatrix} a_{k+1,k+1}^{(k)} & & & \\ a_{k+2,k+1}^{(k)} & 1 & & \\ \dots & \dots & & \\ a_{n-1,k+1}^{(k)} & \dots & 1 & \\ a_{n,k+1}^{(k)} & \dots & a_{n,n-1}^{(k)} & 1 \end{pmatrix} \quad \text{(triangular, order } n-k\text{)},$$

$$\text{(20)} \qquad \widetilde{\mathbf{P}}_k = \mathbf{P}_k - \mathbf{E}_k = \begin{pmatrix} 0 & a_{12}^{(k)} & \dots & a_{1k}^{(k)} \\ & 0 & \dots & a_{2k}^{(k)} \\ & & \dots\dots\dots & \\ & & 0 & a_{k-1,k}^{(k)} \\ & & & 0 \end{pmatrix},$$

$$\text{(21)} \qquad \mathbf{U}_{n-k} = \mathbf{R}_{n-k} - \mathbf{L}_{n-k} = \begin{pmatrix} 0 & a_{k+1,k+2}^{(k)} & \dots & a_{k+1,n}^{(k)} \\ & a_{k+2,k+2}^{(k)} - 1 & \dots & a_{k+2,n}^{(k)} \\ & & \dots\dots\dots & \\ & & & a_{n,n}^{(k)} - 1 \end{pmatrix}.$$

For the inverse $\mathbf{M}_k^{-1}$ of the matrix $\mathbf{M}_k$ of the form (14) we have

$$\text{(22)} \qquad \mathbf{M}_k^{-1} = \begin{pmatrix} \mathbf{E}_k & \mathbf{O}_k \\ \mathbf{O}_{n-k} & \mathbf{L}_{n-k}^{-1} \end{pmatrix}.$$

Thus it is clear that for finding the inverse $\mathbf{M}_k^{-1}$ of the triangular matrix $\mathbf{M}_k$ of order n we need only to compute the inverse $\mathbf{L}_{n-k}^{-1}$ of the triangular matrix $\mathbf{L}_{n-k}$ of order $n-k$. Let us denote by b_{ij} the elements of the matrix $\mathbf{L}_{n-k}^{-1}$. With respect to the form

(19) of the matrix $\mathbf{L}_{n-k}$, the matrix $\mathbf{L}_{n-k}^{-1}$ will have the form

$$(23) \qquad \mathbf{L}_{n-k}^{-1} = \begin{pmatrix} \dfrac{1}{a_{k+1,k+1}^{(k)}} & & \\ b_{k+2,k+1} & 1 & \\ \cdots & \cdots & \cdots \\ b_{n,k+1} & \cdots & 1 \end{pmatrix}.$$

For the product $\mathbf{M}_k^{-1}\mathbf{N}_k$ we obtain, using (22) and (15),

$$(24) \qquad \mathbf{M}_k^{-1}\mathbf{N}_k = \begin{pmatrix} \mathbf{E}_k & \mathbf{O}_k \\ \mathbf{O}_{n-k} & \mathbf{L}_{n-k}^{-1} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{P}}_k & \mathbf{Q}_k \\ \mathbf{O}_{n-k} & \mathbf{U}_{n-k} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{P}}_k & \mathbf{Q}_k \\ \mathbf{O}_{n-k} & \mathbf{L}_{n-k}^{-1}\ \mathbf{U}_{n-k} \end{pmatrix}.$$

Thus it is clear that for computing the product of the matrices $\mathbf{M}_k^{-1}$ and $\mathbf{N}_k$ of order n it is sufficient only to compute the product of the (triangular) matrices $\mathbf{L}_{n-k}^{-1}$ and $\mathbf{U}_{n-k}$ of order $n - k$. According to (21), the first column of the product $\mathbf{L}_{n-k}^{-1}\mathbf{U}_{n-k}$ is a column of zeros.

The matrix $\mathbf{A}_{k+1}$, found by formula (8), will have – according to (24) and with the corresponding partition of the identity matrix into blocks, $\mathbf{E} = \begin{pmatrix} \mathbf{E}_k & \mathbf{O}_k \\ \mathbf{O}_{n-k} & \mathbf{E}_{n-k} \end{pmatrix}$ – the form

$$\mathbf{A}_{k+1} = \mathbf{E} + \mathbf{M}_k^{-1}\mathbf{N}_k = \begin{pmatrix} \mathbf{E}_k + \tilde{\mathbf{P}}_k & \mathbf{Q}_k \\ \mathbf{O}_{n-k} & \mathbf{E}_{n-k} + \mathbf{L}_{n-k}^{-1}\mathbf{U}_{n-k} \end{pmatrix},$$

i.e., with respect to (20),

$$(25) \qquad \mathbf{A}_{k+1} = \begin{pmatrix} \mathbf{P}_k & \mathbf{Q}_k \\ \mathbf{O}_{n-k} & \mathbf{S}_{n-k} \end{pmatrix}$$

where $\mathbf{S}_{n-k} = \mathbf{E}_{n-k} + \mathbf{L}_{n-k}^{-1}\mathbf{U}_{n-k}$. At the same time, the matrix $\mathbf{S}_{n-k}$ has the form

$$(26) \qquad \mathbf{S}_{n-k} = \begin{pmatrix} 1 & a_{k+1,k+2}^{(k+1)} & \cdots & a_{k+1,n}^{(k+1)} \\ 0 & a_{k+2,k+2}^{(k+1)} & \cdots & a_{k+2,n}^{(k+1)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n,k+2}^{(k+1)} & \cdots & a_{n,n}^{(k+1)} \end{pmatrix}.$$

If we now change the matrix $\mathbf{A}_{k+1}$ in such a way that we shift in (25) the horizontal line of partition one row down and the vertical line one column to the right, we get – regarding the form (26) of the matrix $\mathbf{S}_{n-k}$ –

$$(27) \qquad \mathbf{A}_{k+1} = \begin{pmatrix} \mathbf{P}_{k+1} & \mathbf{Q}_{k+1} \\ \mathbf{O}_{n-k-1} & \mathbf{R}_{n-k-1} \end{pmatrix}$$

where the matrices $\mathbf{P}_{k+1}$, $\mathbf{Q}_{k+1}$, $\mathbf{R}_{n-k-1}$ have the form corresponding to (16), (17) and (18), so that (27) corresponds to (13) when passing from k to $k + 1$.

As can be seen from (25), the matrices $\mathbf{P}_k$ and $\mathbf{Q}_k$ remain unchanged when passing from $\mathbf{A}_k$ to $\mathbf{A}_{k+1}$ (i.e. they become parts of the matrices $\mathbf{P}_{k+1}$ and $\mathbf{Q}_{k+1}$ of (27)). Thus we can rewrite (16) and (17) as follows:

$$(16') \qquad \mathbf{P}_k = \begin{pmatrix} 1 & a_{12}^{(1)} & \dots & a_{1k}^{(1)} \\ & 1 & \dots & a_{2k}^{(2)} \\ & & \dots\dots\dots & \\ & & 1 & a_{k-1,k}^{(k-1)} \\ & & & 1 \end{pmatrix}.$$

$$(17') \qquad \mathbf{Q}_k = \begin{pmatrix} a_{1,k+1}^{(1)} & \dots & a_{1,n}^{(1)} \\ a_{2,k+1}^{(2)} & \dots & a_{2,n}^{(2)} \\ \dots & \dots & \dots \\ a_{k,k+1}^{(k)} & \dots & a_{k,n}^{(k)} \end{pmatrix}.$$

From the form of $\mathbf{A}_{k+1}$ in (27) it can be also seen that for $k = n - 1$ there is

$$\mathbf{A}_n = \mathbf{P}_n ,$$

i.e. the matrix $\mathbf{A}_n$ is an upper triangular matrix (with all its diagonal elements equal to 1), and it is easy to invert it.

Thus, through the mentioned choice of matrices $\mathbf{M}_k$ and using formula (10) for $k = n - 1$, we reach the following expression of the inverse $\mathbf{A}^{-1}$ of the given matrix $\mathbf{A}$:

$$(28) \qquad \mathbf{A}^{-1} = \mathbf{A}_n^{-1}\mathbf{M}_{n-1}^{-1}\mathbf{M}_{n-2}^{-1} \dots \mathbf{M}_1^{-1}\mathbf{M}_0^{-1}$$

where $\mathbf{M}_k$ are lower triangular matrices of the form (14) and $\mathbf{A}_n$ is an upper triangular matrix with all its diagonal elements equal to 1. As it follows form the previous description, the computation of $\mathbf{A}^{-1}$ according to (20) consists in the following three steps:

(a) the successive computation of the inverses of lower triangular matrices of the form (19) and of order $n - k$ $(k = 0, 1, \dots, n - 1)$,

(b) the inversion of the upper triangular matrix $\mathbf{A}_n$ of order n and of the form (16′),

(c) the successive multiplication of these inverse matrices.

It is also possible to stop the algorithm even at $\mathbf{A}_{n-1}$, because this matrix, as can be seen from (27) for $k = n - 2$, is an upper triangular one, and can be inverted easily as well. Then

$$\mathbf{A}^{-1} = \mathbf{A}_{n-1}^{-1}\mathbf{M}_{n-2}^{-1} \dots \mathbf{M}_0^{-1} .$$

Formula (28) shows the connection between the computation of the inverse matrix using the process just described and the computation with the aid of the so called Banachiewicz method. The latter method uses the decomposition of the given matrix $\boldsymbol{A}$ into the product of two triangular matrices so that one of them has all its diagonal elements equal to 1. (Essentially it is one of the variants of the elimination method.)

If we denote $\boldsymbol{P}^{-1} = \boldsymbol{A}_n^{-1}$ and $\boldsymbol{L}^{-1} = \boldsymbol{M}_{n-1}^{-1} \ldots \boldsymbol{M}_1^{-1} \boldsymbol{M}_0^{-1}$, we obtain from (28)

$$\boldsymbol{A}^{-1} = \boldsymbol{P}^{-1} \boldsymbol{L}^{-1} \tag{29}$$

where $\boldsymbol{P}^{-1}$ is an upper triangular matrix with all its diagonal elements equal to 1, and $\boldsymbol{L}^{-1}$ a lower triangular matrix.

Thus for the matrix $\boldsymbol{A}$ there follows from (29)

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{P}\,, \tag{30}$$

which corresponds to the Banachiewicz decomposition.

The difference between both these methods consists in the fact that according to the Banachiewicz method we find at first the decomposition (30) of the matrix $\boldsymbol{A}$ by the successive computation of the matrices $\boldsymbol{L}$ and $\boldsymbol{P}$, then we invert the matrices $\boldsymbol{L}$ and $\boldsymbol{P}$, and the product of these inverse matrices equals to the desired $\boldsymbol{A}^{-1}$. According to the method described in this paper we get by successive computations directly the decomposition (29) of the desired inverse matrix $\boldsymbol{A}^{-1}$.

3.

Let us now investigate the case when for some k the condition $a_{k+1,k+1}^{(k)} \neq 0$ is not fulfilled. Let us suppose that we have reached, according to the described algorithm, the p-th step,

$$\boldsymbol{A}_0^{-1} = \boldsymbol{A}_p^{-1} \boldsymbol{M}_{p-1}^{-1} \ldots \boldsymbol{M}_1^{-1} \boldsymbol{M}_0^{-1}\,. \tag{31}$$

Let in this step be $a_{p+1,p+1}^{(p)} = 0$. Then for the matrix $\boldsymbol{M}_p$ from the decomposition $\boldsymbol{A}_p = \boldsymbol{M}_p + \boldsymbol{N}_p$, found according to the mentioned algorithm, the inverse does not exist. In this case we change the algorithm as follows: From the matrix $\boldsymbol{A}_p$ we proceed to the matrix

$$\overset{(1)}{\boldsymbol{A}}_p = \boldsymbol{A}_p \boldsymbol{T}_1 \tag{32}$$

where $\boldsymbol{T}_1$ is a matrix interchanging (by multiplication from the right) the $(p + 1)$-st column with some $(p + q)$-th column $(p + q \leqq n)$ so that the corresponding element on the diagonal of the matrix $\overset{(1)}{\boldsymbol{A}}_p$ is different from zero. If the given matrix $\boldsymbol{A}$ is regular then it is always possible to find such $\boldsymbol{T}_1$.

In the further procedure we find, using the original algorithm, the matrix $\overset{(1)}{\boldsymbol{A}}{}_p^{-1}$

instead of A_p^{-1}. Thus we put

$$\overset{(1)}{A}_p = \overset{(1)}{M}_p + \overset{(1)}{N}_p ,$$

$$\overset{(1)}{A}{}_p^{-1} = (E + \overset{(1)}{M}{}_p^{-1}\overset{(1)}{N}_p)^{-1}\,\overset{(1)}{M}{}_p^{-1} = \overset{(1)}{A}_{p+1}M_p^{-1} , \text{ etc.}$$

till we obtain

$$\overset{(1)}{A}{}_p^{-1} = \overset{(1)}{A}{}_n^{-1}\overset{(1)}{M}{}_{n-1}^{-1} \dots \overset{(1)}{M}{}_p^{-1} .$$

Hence we could find the matrix

$$A_p^{-1} = T_1\overset{(1)}{A}{}_p^{-1} \tag{33}$$

(by interchanging the $(p + 1)$-st row by the $(p + q)$-th row in the matrix $\overset{(1)}{A}{}_p^{-1}$) and then find $A^{-1} = A_0^{-1}$ from (31).

But this would not be advantageous, as the uniformity of the process would be broken (and, moreover, the number of necessary adresses used in a computer would increase, because it would be necessary to store separately the product $M_{p-1}^{-1}M_{p-2}^{-1} \dots$ $\dots M_1^{-1}M_0^{-1}$ and the product $\overset{(1)}{M}{}_{n-1}^{-1} \dots \overset{(1)}{M}{}_p^{-1}$).

Thus we choose another possibility: Having passed from the matrix A_p to the matrix $\overset{(1)}{A}_p$, we continue according to the original algorithm so that we do not compute the matrix A_0^{-1} according to (31), but the matrix $\overset{(1)}{A}{}_0^{-1}$ in the form

$$\begin{aligned}
\overset{(1)}{A}{}_0^{-1} &= \overset{(1)}{A}{}_{p+1}^{-1}\overset{(1)}{M}{}_p^{-1}M_{p-1}^{-1} \dots M_1^{-1}M_0^{-1} = \\
&= \overset{(1)}{A}{}_{p+2}^{-1}\overset{(1)}{M}{}_{p+1}^{-1}\overset{(1)}{M}{}_p^{-1}M_{p-1}^{-1} \dots M_1^{-1}M_0^{-1} = \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&= \overset{(1)}{A}{}_n^{-1}\overset{(1)}{M}{}_{n-1}^{-1} \dots \overset{(1)}{M}{}_p^{-1}M_{p-1}^{-1} \dots M_1^{-1}M_0^{-1} = \\
&= \overset{(1)}{A}{}_p^{-1}M_{p-1}^{-1} \dots M_1^{-1}M_0^{-1} .
\end{aligned} \tag{34}$$

If we then interchange the $(p + 1)$-st row by the $(p + q)$-th row of the matrix $\overset{(1)}{A}{}_0^{-1}$, i.e. if we multiply $\overset{(1)}{A}{}_0^{-1}$ by the matrix T_1 from the left, we obtain from (34), using (33) and (31),

$$T_1\overset{(1)}{A}{}_0^{-1} = T_1\overset{(1)}{A}{}_p^{-1}M_{p-1}^{-1} \dots M_0^{-1} = A_p^{-1}M_{p-1}^{-1} \dots M_0^{-1} = A_0^{-1} = A^{-1} ,$$

i.e. the desired inverse matrix A^{-1} of the given matrix A.

If during the computations according to (34) it happens again that for some $r\,(p < r < n)$ the $(r + 1)$-st diagonal element of the matrix $\overset{(1)}{A}_r$ is zero, we pass again by a suitable interchange of the $(r + 1)$-st column with some $(r + s)$-th column $(r + s \leqq n)$ of the matrix $\overset{(1)}{\mathbf{A}}_r$ to the matrix $\overset{(2)}{\mathbf{A}}_r = \overset{(1)}{\mathbf{A}}_r \mathbf{T}_2$ (the meaning of $\mathbf{T}_2$ is similar to that of $\mathbf{T}_1$), and we continue according to our algorithm so that instead of (34) we obtain at the end

$$
(35) \qquad \begin{aligned} \overset{(2)}{\mathbf{A}}{}_0^{-1} &= \overset{(2)}{\mathbf{A}}{}_n^{-1} \overset{(2)}{\mathbf{M}}{}_{n-1}^{-1} \dots \overset{(2)}{\mathbf{M}}{}_r^{-1} \overset{(1)}{\mathbf{M}}{}_{r-1}^{-1} \dots \overset{(1)}{\mathbf{M}}{}_p^{-1} \mathbf{M}_{p-1}^{-1} \dots \mathbf{M}_0^{-1} = \\ &= \overset{(2)}{\mathbf{A}}{}_r^{-1} \overset{(1)}{\mathbf{M}}{}_{r-1}^{-1} \dots \overset{(1)}{\mathbf{M}}{}_p^{-1} \mathbf{M}_{p-1}^{-1} \dots \mathbf{M}_0^{-1} . \end{aligned}
$$

If we interchange then the $(r + 1)$-st row by the $(r + s)$-th row and then the $(p + 1)$-st row by the $(p + q)$-th row of the matrix $\overset{(2)}{\mathbf{A}}{}_0^{-1}$, i.e. if we multiply $\overset{(2)}{\mathbf{A}}{}_0^{-1}$ by the product $\mathbf{T}_1 \mathbf{T}_2$ from the left, we obtain from (35)

$$
\begin{aligned} \mathbf{T}_1 \mathbf{T}_2 \overset{(2)}{\mathbf{A}}{}_0^{-1} &= \mathbf{T}_1 \mathbf{T}_2 \overset{(2)}{\mathbf{A}}{}_r^{-1} \overset{(1)}{\mathbf{M}}{}_{r-1}^{-1} \dots \overset{(1)}{\mathbf{M}}{}_p^{-1} \mathbf{M}_{p-1}^{-1} \dots \mathbf{M}_0^{-1} = \\ &= \mathbf{T}_1 \overset{(1)}{\mathbf{A}}{}_r^{-1} \overset{(1)}{\mathbf{M}}{}_{r-1}^{-1} \dots \overset{(1)}{\mathbf{M}}{}_p^{-1} \mathbf{M}_{p-1}^{-1} \dots \mathbf{M}_0^{-1} = \\ &= \mathbf{T}_1 \overset{(1)}{\mathbf{A}}{}_0^{-1} = \mathbf{A}_0^{-1} = \mathbf{A}^{-1} , \end{aligned}
$$

i.e. the desired inverse matrix $\mathbf{A}^{-1}$.

It is clear that the process just described of removing the "forbidden" zero element can be repeated in any steps where necessary. After the whole computation has been finished the desired inverse matrix is obtained by the corresponding permutation of rows.

Thus it follows that if the given matrix $\mathbf{A}$ is regular then it is always possible to obtain its inverse by the process proposed in this paper.

Rounding-off errors and some questions concerning the possibility of the practical use of this method will be discussed in a special paper.

Výtah

O JEDNOM ZPŮSOBU VÝPOČTU INVERSNÍ MATICE

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V tomto článku je navržen způsob výpočtu inversní matice pomocí výpočtu inversních matic jednoduššího tvaru než má původní matice. Jako zvláštní případ je zde podrobně probrán případ, kdy pomocná matice, která se v každém kroku postupně invertuje, je trojúhelníková, řádu nižšího než matice v předchozím kroku.

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