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ON THE EXISTENCE AND UNIQUENESS OF SOLUTION AND SOME VARIATIONAL PRINCIPLES IN LINEAR THEORIES OF ELASTICITY WITH COUPLE-STRESSES

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II. MINDLIN'S ELASTICITY WITH MICROSTRUCTURE AND THE FIRST STRAIN-GRADIENT THEORY

INTRODUCTION

The purpose of the present paper is to define the weak (generalized) solution of the boundary-value problems in some non-classical theories of elastostatics, to prove the existence and uniqueness of the weak solution and to formulate the principles of minimum potential energy and minimum complementary energy. We follow the method used in [1] for the case of Cosserat continuum.

In this paper we deal (A) with the MINDLIN's linear theory of elasticity with microstructure (see [2]) and (B) with the first strain-gradient theory. We restrict ourselves to the statical case and to the bounded bodies in three-dimensional space only.

As for [1], Section 1 of [1] represents the mathematical background for our present investigation, too. We shall not repeat it, but refer directly to its theorems and formulas, using the notation (I.1.1). (I.1.2) etc.

A. MINDLIN'S THEORY OF ELASTICITY WITH MICROSTRUCTURE

1. BASIC EQUATIONS

The linear case of Mindlin's elasticity theory with microstructure (see [2]) with homogeneous microdeformation coincides (as it was shown by R. A. TOUPIN in [3]) with the theory of linear oriented bodies with three directors.

The following basic equations are established in [2]. The equations of statical equilibrium are

$$\tau_{ij,i} + \sigma_{ij,i} + X_j = 0\,, \tag{1.1}$$

$$\mu_{ijk,i} + \sigma_{jk} + \Phi_{jk} = 0\,, \quad i, j, k = 1, 2, 3 \tag{1.2}$$

where $\tau_{ij} = \tau_{ji}$ denotes the classical stress tensor, σ_{ij} means the relative stress tensor and μ_{ijk} the couple-stress tensor. X_i and Φ_{jk} denote the body force vector and the body double-force tensor per unit volume, respectively.

The geometrical equations take the form

$$\varepsilon_{ij} = u_{(i,j)} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i})\,, \quad \gamma_{ij} = u_{j,i} - \varphi_{ij}\,, \tag{1.3}$$
$$\varkappa_{ijk} = \varphi_{jk,i}$$

where u_i denotes the displacement vector and φ_{ij} the tensor of the microdeformation. The comma stands for the partial differentiation as usual, the round brackets denote the symmetrization.

We suppose that the elastic energy per unit volume $\mathscr{A}$ depends on ε_{ij}, γ_{ij}, $\varkappa_{ijk}$ as follows

$$\mathscr{A}(\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}) = \tfrac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \tfrac{1}{2}b_{ijkl}\gamma_{ij}\gamma_{kl} + \tfrac{1}{2}a_{ijklmn}\varkappa_{ijk}\varkappa_{lmn} + \tag{1.4}$$
$$+ d_{ijklm}\gamma_{ij}\varkappa_{klm} + f_{ijklm}\varkappa_{ijk}\varepsilon_{lm} + g_{ijkl}\gamma_{ij}\varepsilon_{kl}$$

where

$$c_{ijkl} = c_{klij} = c_{jikl}\,, \quad b_{ijkl} = b_{klij}\,, \quad a_{ijklmn} = a_{lmnijk}\,, \tag{1.5}$$
$$f_{ijklm} = f_{ijkml}\,, \quad g_{ijkl} = g_{ijlk}$$

and c_{ijkl}, b_{ijkl}, a_{ijklmn}, d_{ijklm}, f_{ijklm}, g_{ijkl} are bounded and measurable functions in $\bar{\Omega} = \Omega \cup \Gamma$. Then the constitutive equations become

$$\begin{aligned} \tau_{pq} &\equiv \partial\mathscr{A}/\partial\varepsilon_{pq} = c_{pqij}\varepsilon_{ij} + g_{ijpq}\gamma_{ij} + f_{ijkpq}\varkappa_{ijk}\,, \\ \sigma_{pq} &\equiv \partial\mathscr{A}/\partial\gamma_{pq} = g_{pqij}\varepsilon_{ij} + b_{ijpq}\gamma_{ij} + d_{pqijk}\varkappa_{ijk}\,, \\ \mu_{pqr} &\equiv \partial\mathscr{A}/\partial\varkappa_{pqr} = f_{pqrij}\varepsilon_{ij} + d_{ijpqr}\gamma_{ij} + a_{pqrijk}\varkappa_{ijk}\,. \end{aligned} \tag{1.6}$$

Observe that for $\varphi_{(ij)} = 0$, $\Phi_{(ij)} = 0$ we obtain Cosserat continuum.

Let Ω be a bounded region with Lipschitz boundary Γ[1]) and $\Gamma = \Gamma_u \cup \Gamma_T \cup N$ a disjoint decomposition of Γ. Γ_u and Γ_T are either open in Γ or empty and N has the surface measure zero. Let $\bar{u}_i$, $\bar{\varphi}_{ij} \in W_2^{(1)}(\Omega)$, $\bar{T}_i$, $\bar{M}_{ij} \in L_2(\Gamma_T)$. We shall consider the following boundary conditions

$$\bar{u}_i = u_i\,, \quad \bar{\varphi}_{ij} = \varphi_{ij} \quad \text{on} \quad \Gamma_u\,, \tag{1.7}$$

[1]) See Section 1 in [1] for the definitions of a Lipschitz boundary, $W_2^{(1)}(\Omega)$, $L_2(\Gamma)$ etc.

$$(1.8 \qquad \bar{T}_i = T_i \equiv n_j(\tau_{ji} + \sigma_{ji})\,, \quad \bar{M}_{ij} = M_{ij} \equiv n_k \mu_{kij} \quad \text{on} \quad \Gamma_T\,.$$

Here n_i denotes the unit outward normal to Γ.

Now we shall derive the principle of virtual work. Let X_i, $\Phi_{jk} \in L_2(\Omega)$. We say that the array of tensors $\{\tau_{ij}, \sigma_{ij}, \mu_{ijk}\}$ represents a *statically admissible stress field*, if τ_{ij}, σ_{ij}, $\mu_{ijk} \in W_2^{(1)}(\Omega)$, (1.1), (1.2) are satisfied in Ω in the sense of $L_2(\Omega)$ and the boundary conditions (1.8) are met in the sense of $L_2(\Gamma_T)$. We say that the array $\{u_i, \varphi_{jk}\}$, u_i, $\varphi_{jk} \in W_2^{(1)}(\Omega)$ forms a *geometrically admissible deformation field*, if (1.7) is met in the sense of traces.

Let $\{\tau_{ij}, \sigma_{ij}, \mu_{ijk}\}$ and $\{u_i, \varphi_{jk}\}$ be a statically admissible stress field and a geometrically admissible deformation field, respectively. Then

$$0 = \int_\Omega [(\tau_{ij,i} + \sigma_{ij,i} + X_j)\, u_j + (\mu_{ijk,i} + \sigma_{jk} + \Phi_{jk})\, \varphi_{jk}]\, \mathrm{d}X\,.$$

Integrating by parts we obtain

$$(1.9) \qquad \int_\Omega [\tau_{ij}\varepsilon_{ij} + \sigma_{ij}\gamma_{ij} + \mu_{ijk}\varkappa_{ijk}]\, \mathrm{d}X = \int_\Omega (X_i u_i + \Phi_{jk}\varphi_{jk})\, \mathrm{d}X +$$
$$+ \int_{\Gamma_T} (\bar{T}_i u_i + \bar{M}_{jk}\varphi_{jk})\, \mathrm{d}\Gamma + \int_{\Gamma u} [(\tau_{ij} + \sigma_{ij})\, \bar{u}_j + \mu_{ijk}\bar{\varphi}_{jk}]\, n_i\, \mathrm{d}\Gamma\,.$$

Equation (1.9) which holds for any two admissible fields will be called the principle of virtual work.

Let us suppose that the form (1.4) is positive definite, i.e. there exists a number $c > 0$ such that for all $X \in \Omega$ there holds

$$(1.10) \qquad \mathscr{A}(\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}) \geqq c \sum_{i,j,k=1}^{3} (\varepsilon_{ij}^2 + \gamma_{ij}^2 + \varkappa_{ijk}^2)\,.$$

The matrix of the system (1.6) is symmetrical and it represents at the same time the matrix of the quadratic form (1.4). By virtue of (1.10), the absolute value of the determinant of this matrix is greater than a certain number $c' > 0$ for each $X \in \Omega$. Hence the inverse matrix exists and is symmetrical as well. We can solve (1.6) with respect to ε_{ij}, γ_{ij}, $\varkappa_{ijk}$:

$$(1.11) \qquad \begin{aligned} \varepsilon_{ij} &= q_{ijpq}\tau_{pq} + t_{pqij}\sigma_{pq} + s_{pqrij}\mu_{pqr}\,, \\ \gamma_{ij} &= t_{ijpq}\tau_{pq} + p_{pqij}\sigma_{pq} + r_{ijpqr}\mu_{pqr}\,, \\ \varkappa_{ijk} &= s_{ijkpq}\tau_{pq} + r_{pqijk}\sigma_{pq} + o_{ijkpqr}\mu_{pqr} \end{aligned}$$

where

$$q_{ijpq} = q_{pqij} = q_{jipq}\,, \quad p_{pqij} = p_{ijpq}\,, \quad o_{ijkpqr} = o_{pqrijk}\,,$$
$$s_{pqrij} = s_{pqrji}\,, \quad t_{ijpq} = t_{ijqp}\,.$$

q_{ijpq}, t_{pqij}, s_{pqrij}, p_{pqij}, r_{ijpqr}, o_{ijkpqr} are again bounded and measurable in $\bar{\Omega}$. Using (1.6) and (1.11) we can express the elastic energy in terms of stresses

$$(1.12) \qquad 2\mathscr{A}(\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}) = 2\tilde{\mathscr{A}}(\tau_{ij}, \sigma_{ij}, \mu_{ijk}) = q_{ijkl}\tau_{ij}\tau_{kl} + p_{ijkl}\sigma_{ij}\sigma_{kl} + \\ + 2r_{ijklm}\sigma_{ij}\mu_{klm} + 2s_{ijklm}\mu_{ijk}\tau_{lm} + 2t_{ijkl}\sigma_{ij}\tau_{kl} .$$

2. THE EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION OF THE BOUNDARY-VALUE PROBLEMS

Let us choose the quantities introduced previously in [1] as follows: $m = 12$, $\varkappa_s = 1$, $s = 1, 2, \ldots, 12$. Let us denote

$$\boldsymbol{u} = \{u_1, u_2, u_3, \varphi_{11}, \varphi_{22}, \varphi_{33}, \varphi_{12}, \varphi_{13}, \varphi_{23}, \varphi_{21}, \varphi_{31}, \varphi_{32}\} = \{u_i, \varphi_{jk}\}$$

and analogically

$$\boldsymbol{v} = \{v_i, \psi_{jk}\} .$$

Let W be defined as the space of all $\boldsymbol{u} = \{u_i, \varphi_{jk}\}$, $u_i, \varphi_{jk} \in W_2^{(1)}(\Omega)$ with the norm

$$|\boldsymbol{u}|_W^2 = \sum_{i,j=1}^{3} \left(|u_i|^2_{W_2^{(1)}(\Omega)} + |\varphi_{ij}|^2_{W_2^{(1)}(\Omega)}\right) .$$

V is the subspace of all elements $\boldsymbol{u} = \{u_i, \varphi_{jk}\} \in W$ which satisfy the homogeneous boundary conditions (1.7) (i.e. for $\bar{u}_i = 0$, $\bar{\varphi}_{ij} = 0$) in the sense of traces. The bilinear form $A(\boldsymbol{v}, \boldsymbol{u})$ on $W \times W$ is defined by

$$(2.1) \quad A(\boldsymbol{v}, \boldsymbol{u}) = \int_\Omega [c_{ijkl}\, \varepsilon_{ij}(\boldsymbol{v})\, \varepsilon_{kl}(\boldsymbol{u}) + b_{ijkl}\, \gamma_{ij}(\boldsymbol{v})\, \gamma_{kl}(\boldsymbol{u}) + a_{ijklmn}\, \varkappa_{ijk}(\boldsymbol{v})\, \varkappa_{lmn}(\boldsymbol{u}) + \\ + d_{ijklm}\{\gamma_{ij}(\boldsymbol{v})\, \varkappa_{klm}(\boldsymbol{u}) + \gamma_{ij}(\boldsymbol{u})\, \varkappa_{klm}(\boldsymbol{v})\} + f_{ijklm}\{\varkappa_{ijk}(\boldsymbol{v})\, \varepsilon_{lm}(\boldsymbol{u}) + \\ + \varkappa_{ijk}(\boldsymbol{u})\, \varepsilon_{lm}(\boldsymbol{v})\} + g_{ijkl}\{\gamma_{ij}(\boldsymbol{v})\, \varepsilon_{kl}(\boldsymbol{u}) + \gamma_{ij}(\boldsymbol{u})\, \varepsilon_{kl}(\boldsymbol{v})\}]\, \mathrm{d}X$$

where (1.5) holds and

$$\varepsilon_{ij}(\boldsymbol{v}) = v_{(i,j)}\,, \quad \gamma_{ij}(\boldsymbol{v}) = v_{j,i} - \psi_{ij}\,, \quad \varkappa_{ijk}(\boldsymbol{v}) = \psi_{jk,i}\,,$$
$$\varepsilon_{ij}(\boldsymbol{u}) = u_{(i,j)}\,, \quad \gamma_{ij}(\boldsymbol{u}) = u_{j,i} - \varphi_{ij}\,, \quad \varkappa_{ijk}(\boldsymbol{u}) = \varphi_{jk,i}\,.$$

According to (1.5), we have $A(\boldsymbol{v}, \boldsymbol{u}) = A(\boldsymbol{u}, \boldsymbol{v})$,

$$(2.2) \qquad \int_\Omega 2\mathscr{A}(\varepsilon_{ij}(\boldsymbol{u}), \gamma_{ij}(\boldsymbol{u}), \varkappa_{ijk}(\boldsymbol{u}))\, \mathrm{d}X = A(\boldsymbol{u}, \boldsymbol{u}) .$$

Further, let us define for $\boldsymbol{v} = \{v_i, \psi_{jk}\} \in \boldsymbol{W}$ the functionals

$$(2.3)\qquad f(\boldsymbol{v}) = \int_{\Omega} (X_i v_i + \Phi_{jk}\psi_{jk})\,\mathrm{d}X\,,$$

$$g(\boldsymbol{v}) = \int_{\Gamma_T} (\overline{T}_i v_i + \overline{M}_{jk}\psi_{jk})\,\mathrm{d}\Gamma\,.$$

We define the weak solution of the boundary-value problem as follows:

Let $\bar{\boldsymbol{u}} = \{\bar{u}_i, \bar{\varphi}_{jk}\} \in \boldsymbol{W}$ be such that the given data $\bar{u}_i$, $\bar{\varphi}_{jk}$ on Γ_u may be obtained by means of the embedding of $W_2^{(1)}(\Omega)$ into $L_2(\Gamma_u)$. We say that $\boldsymbol{u} \in \boldsymbol{W}$ *is the weak solution of the boundary-value problem, if* $\boldsymbol{u} - \bar{\boldsymbol{u}} \in \boldsymbol{V}$ *and for each* $\boldsymbol{v} = \{v_i, \psi_{jk}\} \in \boldsymbol{V}$ *there holds*

$$A(\boldsymbol{v}, \boldsymbol{u}) = f(\boldsymbol{v}) + g(\boldsymbol{v})$$

where $A(\boldsymbol{v}, \boldsymbol{u})$, $f(\boldsymbol{v})$, $g(\boldsymbol{v})$ *are defined by* (2.1), (2.3).

Similarly to the case of Cosserat bodies (see [1]), we could easily find the connection between (2.4) and the principle of "virtual" deformations. In fact, if $\boldsymbol{u} = \{u_i, \varphi_{jk}\}$ is a geometrically admissible deformation field and $\{\tau_{ij}(\boldsymbol{u}), \sigma_{ij}(\boldsymbol{u}), \mu_{ijk}(\boldsymbol{u})\}$ a statically admissible stress field, then writing the principle of virtual work first for these quantities, second for a "varied" field $\boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{v} \in \boldsymbol{V}$ and $\{\tau_{ij}(\boldsymbol{u}), \sigma_{ij}(\boldsymbol{u}), \mu_{ijk}(\boldsymbol{u})\}$, after subtracting we obtain (2.4). Next, interpreting $\boldsymbol{v} = \delta\boldsymbol{u} = \{\delta u_i, \delta\varphi_{jk}\}$, (2.4) may be called the principle of virtual deformation. On the other hand if $\boldsymbol{u} \in \boldsymbol{W}$ is a weak solution then $\tau_{ij}(\boldsymbol{u})$, $\sigma_{ij}(\boldsymbol{u})$, $\mu_{ijk}(\boldsymbol{u})$ need not belong to $W_2^{(1)}(\Omega)$.

Let the operators $N_l\boldsymbol{v}$, $l = 1, 2, \ldots, 45$ be

$$(2.5)\qquad \begin{array}{lll}
N_1\boldsymbol{v} = v_{1,1}\,, & N_2\boldsymbol{v} = v_{(1,2)}\,, & N_3\boldsymbol{v} = v_{(1,3)} \\
N_4\boldsymbol{v} = v_{2,2}\,, & N_5\boldsymbol{v} = v_{(2,1)}\,, & N_6\boldsymbol{v} = v_{(3,1)} \\
N_7\boldsymbol{v} = v_{3,3}\,, & N_8\boldsymbol{v} = v_{(2,3)}\,, & N_9\boldsymbol{v} = v_{(3,2)} \\
N_{10}\boldsymbol{v} = v_{1,1} - \psi_{11}\,, & N_{11}\boldsymbol{v} = v_{2,1} - \psi_{12}\,, & N_{12}\boldsymbol{v} = v_{3,1} - \psi_{13} \\
N_{13}\boldsymbol{v} = \psi_{11,1}\,, & N_{14}\boldsymbol{v} = \psi_{22,1} & N_{15}\boldsymbol{v} = \psi_{33,1} \\
N_{16}\boldsymbol{v} = \psi_{12,1}\,, & N_{17}\boldsymbol{v} = \psi_{13,1}\,, & N_{18}\boldsymbol{v} = \psi_{23,1} \\
N_{19}\boldsymbol{v} = \psi_{21,1}\,, & N_{20}\boldsymbol{v} = \psi_{31,1}\,, & N_{21}\boldsymbol{v} = \psi_{32,1} \\
N_{22}\boldsymbol{v} = v_{1,2} - \psi_{21}\,, & N_{23}\boldsymbol{v} = v_{2,2} - \psi_{22}\,, & N_{24}\boldsymbol{v} = v_{3,2} - \psi_{23} \\
N_{25}\boldsymbol{v} = \psi_{11,2}\,, & N_{26}\boldsymbol{v} = \psi_{22,2} & N_{27}\boldsymbol{v} = \psi_{33,2} \\
N_{28}\boldsymbol{v} = \psi_{12,2}\,, & N_{29}\boldsymbol{v} = \psi_{13,2} & N_{30}\boldsymbol{v} = \psi_{23,2} \\
N_{31}\boldsymbol{v} = \psi_{21,2}\,, & N_{32}\boldsymbol{v} = \psi_{31,2}\,, & N_{33}\boldsymbol{v} = \psi_{32,2} \\
N_{34}\boldsymbol{v} = v_{1,3} - \psi_{31}\,, & N_{35}\boldsymbol{v} = v_{2,3} - \psi_{32}\,, & N_{36}\boldsymbol{v} = v_{3,3} - \psi_{33} \\
N_{37}\boldsymbol{v} = \psi_{11,3} & N_{38}\boldsymbol{v} = \psi_{22,3}\,, & N_{39}\boldsymbol{v} = \psi_{33,3}
\end{array}$$

$$N_{40}\boldsymbol{v} = \psi_{12,3}\,, \qquad N_{41}\boldsymbol{v} = \psi_{13,3}\,, \qquad N_{42}\boldsymbol{v} = \psi_{23,3}$$

$$N_{43}\boldsymbol{v} = \psi_{21,3}\,, \qquad N_{44}\boldsymbol{v} = \psi_{31,3}\,, \qquad N_{45}\boldsymbol{v} = \psi_{32,3}\,.$$

We have

$$\sum_{i,j,k=1}^{3} \left(\varepsilon_{ij}^2(\boldsymbol{v}) + \gamma_{ij}^2(\boldsymbol{v}) + \varkappa_{ijk}^2(\boldsymbol{v})\right) = \sum_{l=1}^{45} (N_l \boldsymbol{v})^2$$

and (2.2), (1.10) imply the inequality (I.1.4). According to Theorem I.1.1 the *operators* $N_l\boldsymbol{v}$ *form a coercive system on* W, because the matrix (I.1.3) contains three diagonal matrices $\xi_i \boldsymbol{E}$ ($\boldsymbol{E}$ being the unit matrix of the 12-th order) so that its rank is 12 for each $\xi \in \boldsymbol{C}_3$, $\xi \neq 0$.

For $\boldsymbol{v} \in \mathscr{P}$ there holds

$$\varepsilon_{ij}(\boldsymbol{v}) = \gamma_{ij}(\boldsymbol{v}) = \varkappa_{ijk}(\boldsymbol{v}) = 0 \tag{2.6}$$

almost everywhere in Ω.

Hence (2.1) defines a bilinear form $[\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{u}}]$ on $W/\mathscr{P} \times W/\mathscr{P}$ and all the suppositions of Theorem I.1.2 are satisfied. Consequently, the *necessary and sufficient condition for the existence of a weak solution is*

$$\boldsymbol{p} \in \mathscr{P} \quad \Rightarrow \quad f(\boldsymbol{p}) + g(\boldsymbol{p}) = 0\,, \tag{2.7}$$

where according to (2.6) *and* (1.3)

$$\mathscr{P} = \{\boldsymbol{v} \equiv \{v_i, \psi_{jk}\} \in V,\, v_k = a_k + b_{kj}x_j,\, \psi_{jk} = b_{kj}\}\,,$$
$$a_k = \text{const.}\,, \quad b_{kj} = -b_{jk} = \text{const.}$$

The solution is determined except for an element $\boldsymbol{p} \in \mathscr{P}$. *The inequality* (I.1.7) *yields the continuous dependence of the solution upon the given data*: $\bar{u}_i$, $\bar{\varphi}_{jk}$, $\overline{T}_i$, $\overline{M}_{jk}$.

We are not going a detailed analysis of boundary-value problems, but we restrict ourselves to two important cases only:

1. Let Γ_u be empty, i.e. $\Gamma = \Gamma_T \cup N$. Then the condition (2.7) is equivalent to the system of equilibrium conditions

$$\int_\Omega X_i \,\mathrm{d}X + \int_\Gamma \overline{T}_i \,\mathrm{d}\Gamma = 0\,,$$

$$\int_\Omega (\varepsilon_{ijk}x_j X_k + \Phi_i)\,\mathrm{d}X + \int_\Gamma (\varepsilon_{ijk}x_j \overline{T}_k + M_i)\,\mathrm{d}\Gamma = 0$$

where

$$\Phi_i = \varepsilon_{ijk}\Phi_{jk}\,, \quad M_i = \varepsilon_{ijk}M_{jk}\,.$$

2. Let Γ_u contain a non-empty set open in Γ. Then $\mathscr{P} = \{0\}$, (2.7) is satisfied and there exists one and only one weak solution $\boldsymbol{u} \in W$.

Using Theorem I.1.3 it is possible to formulate the boundary-value problems so that their solutions are unique even in case $\Gamma = \Gamma_T$. We could easily prove that for linear independent functionals $p_i(\boldsymbol{v})$ introduced in Theorem I.1.3, the following systems may be taken:

$$\text{(2.8)} \qquad \text{(a)} \int_M v_i \, \mathrm{d}M \,, \quad \int_M \varepsilon_{ijk} x_k v_j \, \mathrm{d}M \quad \text{or}$$

$$\text{(b)} \int_M v_i \, \mathrm{d}M \,, \quad \int_M \psi_{jk} \, \mathrm{d}M \quad \text{or}$$

$$\text{(c)} \int_{\Omega^*} v_i \, \mathrm{d}X \,, \quad \int_{\Omega^*} \varepsilon_{ijk} v_{k,j} \, \mathrm{d}X \,.$$

Here $M = \Omega^*$ or Γ^*, where $\Omega^* \subset \Omega$ denotes an arbitrary set of a positive volume measure and $\Gamma^* \subset \bar{\Omega}$ a non-empty sum of a finite number of Lipschitz surfaces. (See [1] Section 3 for the definition of Lipschitz surface). Particularly, it is possible to choose $\Omega^* = \Omega$, $\Gamma^* \subset \Gamma$.

3. THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY AND THE PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY

Let us define $A(\boldsymbol{v}, \boldsymbol{u})$, $f(\boldsymbol{v})$ and $g(\boldsymbol{v})$ by (2.1) and (2.3) and suppose (2.7). Then Theorem I.1.2 holds and using the results of Section I.1 we can formulate the principle of minimum potential energy as follows:

The quadratic functional $\mathscr{L}(\boldsymbol{u})$ defined for $\boldsymbol{u} = \{u_i, \varphi_{jk}\} \in W$ by

$$\mathscr{L}(\boldsymbol{u}) = \int_\Omega [\mathscr{A}(\varepsilon_{ij}(\boldsymbol{u}), \gamma_{ij}(\boldsymbol{u}), \varkappa_{ijk}(\boldsymbol{u})) - (X_i u_i + \Phi_{jk}\varphi_{jk})] \, \mathrm{d}X - $$

$$- \int_{\Gamma_T} (\bar{T}_i u_i + \bar{M}_{jk}\varphi_{jk}) \, \mathrm{d}\Gamma$$

attains the minimum on the set

$$\bar{\boldsymbol{u}} \oplus V,$$

if and only if

$$\boldsymbol{u} = \mathring{\boldsymbol{u}} + \boldsymbol{p}$$

where $\mathring{\boldsymbol{u}}$ is the weak solution, $\boldsymbol{p} \in \mathscr{P}$.

When using subspaces V_p with $p_i(\boldsymbol{v})$ chosen for example as in (2.8), then the *functional $\mathscr{L}(\boldsymbol{u})$ attains the minimum on the set*

$$\bar{\boldsymbol{u}} \oplus V_p,$$

if and only if

$$u = \mathring{u} \,,$$

where $\mathring{u}$ is the unique weak solution in $\bar{u} \oplus V_p$.

Applying the same procedure as in [1], Section 4, we derive the principle of minimum complementary energy.

Let $\mathscr{T}$ be defined as the Banach space of the stress fields T

$$T = \{\tau_{ij}, \sigma_{ij}, \mu_{ijk}\}\,, \quad \tau_{ij}, \sigma_{ij}, \mu_{ijk} \in L_2(\Omega)$$

with the norm

$$|T|^2_{\mathscr{T}} = \sum_{i,j=1}^{3} \left(|\tau_{ij}|^2_{L_2(\Omega)} + |\sigma_{ij}|^2_{L_2(\Omega)}\right) + \sum_{i,j,k=1}^{3} |\mu_{ijk}|^2_{L_2(\Omega)} \,.$$

We could easily verify that the bilinear form

$$(3.2) \qquad (T', T'') = \int_\Omega \left[q_{ijkl}\tau'_{ij}\tau''_{kl} + p_{ijkl}\sigma'_{ij}\sigma''_{kl} + a_{ijklmn}\mu'_{ijk}\mu''_{lmn} + \right.$$
$$+ r_{ijklm}(\sigma'_{ij}\mu''_{klm} + \sigma''_{ij}\mu'_{klm}) + s_{ijklm}(\mu'_{ijk}\tau''_{lm} + \mu''_{ijk}\tau'_{lm}) +$$
$$\left. + t_{ijkl}(\sigma'_{ij}\tau''_{kl} + \sigma''_{ij}\tau'_{kl})\right] \mathrm{d}X \,, \quad T', T'' \in \mathscr{T}$$

defines a scalar product in $\mathscr{T}$ and the norm

$$|T|_{\mathscr{H}} = (T, T)^{1/2}$$

is equivalent to $|T|_{\mathscr{T}}$. We denote by $\mathscr{H}$ the Hilbert space of the stress fields $T \in \mathscr{T}$ with the scalar product (3.2). Let $\mathscr{H}_1 \subset \mathscr{H}$ denote the subset of such stress fields $T \in \mathscr{H}$ to which $u = \{u_i, \varphi_{jk}\} \in V$ exists such that using (1.3) equations (1.6) hold (i.e. $T = T(u)$). Furthermore, let $\mathscr{H}_2 \subset \mathscr{H}$ denote the subset of such stress fields T that for each $v = \{v_i, \psi_{jk}\} \in V$,

$$\int_\Omega [\tau_{ij}\, \varepsilon_{ij}(v) + \sigma_{ij}\, \gamma_{ij}(v) + \mu_{ijk}\, \varkappa_{ijk}(v)] \,\mathrm{d}X = 0$$

holds. It is easy to prove that $\mathscr{H}_1$ and $\mathscr{H}_2$ are orthogonal. Next let us consider an arbitrary stress field $T \in \mathscr{H}$ satisfying the equations of equilibrium (1.1), (1.2) and the statical boundary conditions on Γ_T in the weak sense, i.e. let for each $v = \{v_i, \psi_{jk}\} \in V$

$$(3.3) \qquad \int_\Omega [\tau_{ij}\, \varepsilon_{ij}(v) + \sigma_{ij}\, \gamma_{ij}(v) + \mu_{ijk}\, \varkappa_{ijk}(v)] \,\mathrm{d}X =$$
$$= \int_\Omega (X_i v_i + \Phi_{jk}\psi_{jk}) \,\mathrm{d}X + \int_{\Gamma_T} (\bar{T}_i v_i + \bar{M}_{jk}\psi_{jk}) \,\mathrm{d}\Gamma$$

hold. Denote $\mathring{T} = \{\mathring{\tau}_{ij}, \mathring{\sigma}_{ij}, \mathring{\mu}_{ijk}\} \equiv T(\mathring{u})$, i.e. $\mathring{T}$ is related to the weak solution $\mathring{u}$

through (1.3), (1.6). If we write

$$\mathring{\boldsymbol{u}} = \bar{\boldsymbol{u}} + \boldsymbol{w}$$

then

$$\mathring{T} = \boldsymbol{T}(\bar{\boldsymbol{u}}) + \boldsymbol{T}(\boldsymbol{w}), \quad \boldsymbol{T}(\boldsymbol{w}) \in \mathscr{H}_1 .$$

As $\mathring{T}$ meets (3.3), $\boldsymbol{T} - \mathring{\boldsymbol{T}} \in \mathscr{H}_2$ and we have

$$|\boldsymbol{T} - \boldsymbol{T}(\bar{\boldsymbol{u}})|^2_{\mathscr{H}} = |\boldsymbol{T} - \mathring{T}|^2_{\mathscr{H}} + |\boldsymbol{T}(\boldsymbol{w})|^2_{\mathscr{H}} .$$

Consequently, $|\boldsymbol{T} - \boldsymbol{T}(\bar{\boldsymbol{u}})|^2_{\mathscr{H}}$ attains the minimum on the set of all $\boldsymbol{T} \in \mathscr{H}$ which meet (3.3), if and only if $\boldsymbol{T} = \mathring{\boldsymbol{T}}$. The same assertion is true for the functional

$$\tilde{\mathscr{S}}(\boldsymbol{T}) = \tfrac{1}{2}\{|\boldsymbol{T} - \boldsymbol{T}(\bar{\boldsymbol{u}})|^2_{\mathscr{H}} - |\boldsymbol{T}(\bar{\boldsymbol{u}})|^2_{\mathscr{H}}\} = \tfrac{1}{2}(\boldsymbol{T}, \boldsymbol{T}) - (\boldsymbol{T}, \boldsymbol{T}(\bar{\boldsymbol{u}})) .$$

Hence the principle of minimum complementary energy follows:

The quadratic functional

$$\tilde{\mathscr{S}}(\boldsymbol{T}) = \int_\Omega [\tilde{\mathscr{A}}(\tau_{ij}, \sigma_{ij}, \mu_{ijk}) - \{\tau_{ij}\, \varepsilon_{ij}(\bar{u}) + \sigma_{ij}\, \gamma_{ij}(\bar{u}) + \mu_{ijk}\, \varkappa_{ijk}(\bar{u})\}]\, \mathrm{d}X$$

where $\tilde{\mathscr{A}}$ *is defined by* (1.12) *and*

$$\varepsilon_{ij}(\bar{\boldsymbol{u}}) = \bar{u}_{(i,j)}, \quad \gamma_{ij}(\bar{\boldsymbol{u}}) = \bar{u}_{j,i} - \bar{\varphi}_{ij}, \quad \varkappa_{ijk}(\bar{\boldsymbol{u}}) = \bar{\varphi}_{j,i} ,$$

attains the minimum on the set of $\boldsymbol{T} \in \mathscr{T}$ *which satisfy* (3.3), *if and only if*

$$|\boldsymbol{T} - \mathring{T}|_{\mathscr{T}} = 0 .$$

If moreover the weak solution $\mathring{\boldsymbol{u}}$ is such that $\boldsymbol{T}(\mathring{\boldsymbol{u}})$ is a statically admissible stress field, then using the principle of virtual work as in Section I.4, we are able to establish an alternative form of the principle:

The quadratic functional

$$\mathscr{S}(\boldsymbol{T}) = \int_\Omega \tilde{\mathscr{A}}(\tau_{ij}, \sigma_{ij}, \mu_{ijk})\, \mathrm{d}X - \int_{\Gamma u} (T_i \bar{u}_i + M_{jk}\bar{\varphi}_{jk})\, \mathrm{d}\Gamma$$

where $\tilde{\mathscr{A}}$ *is defined by* (1.12) *and*

$$T_i = n_i(\tau_{ij} + \sigma_{ij}), \quad M_{jk} = n_i \mu_{ijk} ,$$

attains the minimum on the set of statically admissible stress fields, if and only if

$$|\boldsymbol{T} - \mathring{T}|_{\mathscr{T}} = 0 .$$

B. NON-SIMPLE BODIES: THE FIRST STRAIN-GRADIENT THEORY

The elastic energy $\mathscr{A}$ of non-simple bodies depends on the derivatives of the displacement vector up to the n-th order, $n > 1$. In case that $n = 2$, we speak of the first strain-gradient theory. For the detailed analysis of this case see [5] where three equivalent alternatives are discussed. For our purpose we choose one of them only, which is investigated more thoroughly in [4]. However, analogous results could be obtained also for the other alternatives.

4. THE BASIC EQUATIONS

The following basic equations are mostly presented in [4]. We assume that the elastic energy per unite volume has the form

$$\mathscr{A}(\varepsilon_{ij}; \varepsilon_{ij,k}) \quad \text{where} \quad \varepsilon_{ij} = u_{(i,j)} . \tag{4.1}$$

The statical equations of equilibrium are

$$\tau_{ij,i} - \mu_{ijk,ik} + X_j = 0\,, \quad i, j, k = 1, 2, 3\,. \tag{4.2}$$

Here $\tau_{ij} = \tau_{ji}$, μ_{ijk} denotes the stress tensor and the couple-stress tensor, respectively. X_j is the body force vector per unit volume.

We suppose that for anisotropic bodies (4.1) has the form

$$\mathscr{A}(\varepsilon_{ij}, \varkappa_{ijk}) = \tfrac{1}{2}k_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \tfrac{1}{2}m_{ijklmn}\varkappa_{ijk}\varkappa_{lmn} + n_{ijklm}\varepsilon_{ij}\varkappa_{klm} \tag{4.3}$$

where

$$k_{ijkl} = k_{klij} = k_{jikl}\,, \quad m_{ijklmn} = m_{lmnijk}\,, \quad n_{ijklm} = n_{jiklm}\,, \quad \varkappa_{ijk} \equiv \varepsilon_{jk,i}$$

and k_{ijkl}, m_{ijklmn}, n_{ijklm} are bounded and measurable functions in $\bar{\Omega} = \Omega \cup \Gamma$.

Then the constitutive equations become

$$\begin{aligned} \tau_{ij} &\equiv \partial\mathscr{A}/\partial\varepsilon_{ij} = k_{ijpq}\varepsilon_{pq} + n_{ijpqr}\varkappa_{pqr}\,, \\ \mu_{ijk} &\equiv \partial\mathscr{A}/\partial\varkappa_{ijk} = n_{pqijk}\varepsilon_{pq} + m_{ijkpqr}\varkappa_{pqr}\,. \end{aligned} \tag{4.4}$$

Moreover, we suppose that the form (4.3) is positive definite, i.e. there exists such a number $c > 0$ that for all $X \in \Omega$ there holds

$$\mathscr{A}(\varepsilon_{ij}, \varkappa_{ijk}) \geqslant c \sum_{i,j,k=1}^{3} (\varepsilon_{ij}^2 + \varkappa_{ijk}^2)\,. \tag{4.5}$$

By virtue of (4.5) we can solve (4.4) with respect to ε_{ij}, $\varkappa_{ijk}$. Similarly to Section 1, substituting these inverted equations into (4.3) we obtain the elastic energy $\tilde{\mathscr{A}}(\tau_{ij}, \mu_{ijk})$ expressed by stresses.

Further, we assume that Ω is a bounded region with Lipschitz boundary Γ which consists of a finite numbers of smooth[2]) surfaces S_α. Let b_β be intersections of two adjoined smooth surfaces and $B = \bigcup_\beta b_\beta$. Let $\Gamma = \Gamma_u \cup \Gamma_T \cup B \cup N$ be a disjoint decomposition of Γ, Γ_u and Γ_T being either open in Γ or empty, B and N are sets of zero surface measure. Suppose that B has a finite one-dimensional measure.

Let $\bar{P}_j, \bar{R}_j \in L_2(\Gamma_T)$, $\bar{u}_j \in W_2^{(2)}(\Omega)$, $\bar{Q}_j \in L_2(B)$. We consider the following boundary conditions:

$$
\begin{aligned}
&(4.6) \qquad && \bar{P}_j = P_j \equiv n_k[\tau_{kj} - (\mu_{ijk} + \mu_{lij}n_l n_k - \mu_{ljk}n_l n_i)_{,i}] \quad \text{on} \quad \Gamma_T\,,\\
& && \bar{R}_j = R \equiv \mu_{ijk}n_i n_k \quad \text{on} \quad \Gamma_T\,,\\
&(4.7) && \bar{u}_j = u_j\,, \quad \bar{\omega}_j \equiv \bar{u}_{j,l}n_l = u_{j,l}n_l \equiv \omega_j \quad \text{on} \quad \Gamma_u\,,\\
&(4.8) && \bar{Q}_j = Q_j \equiv \langle \mu_{ijk}n_i s_k \rangle \quad \text{on} \quad B
\end{aligned}
$$

where

$$
s_k = \varepsilon_{klm} t_l n_m\,,
$$

$\langle\ \rangle$ denotes the difference of limits from both sides of b_β and t_l represents the unit tangent vector to b_β . n_i denotes the outward normal to Γ, which is uniquely defined on $\Gamma \dot{-} B \dot{-} N$. ω_j is the normal component of the displacement gradient. Moreover, we suppose that $X_i \in L_2(\Omega)$.

We say that the couple $\{\tau_{ij}, \mu_{ijk}\}$ where $\tau_{ij} \in W_2^{(1)}(\Omega)$, $\mu_{ijk} \in W_2^{(2)}(\Omega)$ is a *statically admissible stress field*, if (4.2) are satisfied in the sense of $L_2(\Omega)$ and boundary conditions (4.6), (4.8) hold in the sense of $L_2(\Gamma_T)$ and $L_2(B)$. We say that $\boldsymbol{u}$ is a *geometrically admissible displacement field*, if $u_i \in W_2^{(2)}(\Omega)$ and the boundary conditions (4.7) are satisfied in the sense of traces. Let $\{\tau_{ij}, \mu_{ijk}\}$ and $\boldsymbol{u}$ be a statically admissible stress field and a geometrically admissible displacement field, respectively. Then using (4.2), (4.6)–(4.8), we derive

$$
\int_\Omega (\tau_{ij,i} - \mu_{ijk,ik} + X_j)\, u_j \,\mathrm{d}X + \int_{\Gamma_T} [(\bar{P}_j - P_j)\, u_j + (\bar{R}_j - R_j)\,\omega_j] \,\mathrm{d}\Gamma +
$$
$$
+ \int_{\Gamma u} [(\bar{u}_j - u_j)\, P_j + (\bar{\omega}_j - \omega_j)\, R_j] \,\mathrm{d}\Gamma + \int_B (\bar{Q}_j - Q_j)\, u_j \,\mathrm{d}B \doteq 0\,.
$$

Then following the procedure in [4], but going in the opposite direction, using

---

[2]) We say that S_α is a smooth surface, if it may be described by means of a continuous function $f_\alpha(X)$, $X \in \bar{O}_\alpha$, continuously differentiable in the interior of O_α, where $\bar{O}_\alpha$ is a closed two-dimensional region.

Stokes theorem and integrating by parts we obtain the relation

$$(4.9) \qquad \int_\Omega (\tau_{ij}\varepsilon_{ij} + \mu_{ijk}\varkappa_{ijk})\,\mathrm{d}X = \int_\Omega X_i u_i\,\mathrm{d}X +$$
$$+ \int_{\Gamma u} (P_i\bar{u}_i + R_i\bar{\omega}_i)\,\mathrm{d}\Gamma + \int_{\Gamma T} (\bar{P}_i u_i + \bar{R}_i\omega_i)\,\mathrm{d}\Gamma + \int_B \bar{Q}_i u_i\,\mathrm{d}B$$

which holds for arbitrary admissible fields and may be called the principle of virtual work.

5. THE EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION OF THE BOUNDARY-VALUE PROBLEMS

Let us choose the quantities introduced in Section I.1 as follows: $m = 3$, $\varkappa_s = 2$, $s = 1, 2, 3$. Denote $\boldsymbol{u} = \{u_1, u_2, u_3\}$, $\boldsymbol{v} = \{v_1, v_2, v_3\}$. Let $\boldsymbol{W}$ be defined as the space of $\boldsymbol{u} = \{u_1, u_2, u_3\}$, $u_i \in W_2^{(2)}(\Omega)$, with the norm

$$|\boldsymbol{u}|_W^2 = \sum_{i=1}^{3} |u_i|^2_{W_2^{(2)}(\Omega)}$$

where

$$|u_i|^2_{W_2^{(2)}(\Omega)} = \int_\Omega u_i^2\,\mathrm{d}X + \int_\Omega \sum_{j,k=1}^{3} u_{i,jk}^2\,\mathrm{d}X\,.$$

$\boldsymbol{V}$ is the subspace of all $\boldsymbol{v} \in \boldsymbol{W}$ which satisfy the homogeneous boundary conditions (4.7) (i.e. for $\bar{u}_j = 0$, $\bar{\omega}_j = 0$) in the sense of traces[3])

Let the bilinear form $A(\boldsymbol{v}, \boldsymbol{u})$ on $\boldsymbol{W} \times \boldsymbol{W}$ be defined by

$$(5.1) \qquad A(\boldsymbol{v}, \boldsymbol{u}) = \int_\Omega [k_{ijkl}\,\varepsilon_{ij}(\boldsymbol{v})\,\varepsilon_{kl}(\boldsymbol{u}) + m_{ijklmn}\,\varkappa_{ijk}(\boldsymbol{v})\,\varkappa_{lmn}(\boldsymbol{u}) +$$
$$+ n_{ijklm}\{\varepsilon_{ij}(\boldsymbol{v})\,\varkappa_{klm}(\boldsymbol{u}) + \varepsilon_{ij}(\boldsymbol{u})\,\varkappa_{klm}(\boldsymbol{v})\}]\,\mathrm{d}X$$

where

$$\varepsilon_{ij}(\boldsymbol{v}) = v_{(i,j)}\,, \quad \varkappa_{ijk}(\boldsymbol{v}) = \varepsilon_{jk,i}(\boldsymbol{v})\,,$$
$$\varepsilon_{ij}(\boldsymbol{u}) = u_{(i,j)}\,, \quad \varkappa_{ijk}(\boldsymbol{u}) = \varepsilon_{jk,i}(\boldsymbol{u})\,.$$

[3]) Note that $u_{j,l} \in W_2^{(1)}(\Omega)$ so that they may be embedded into $L_2(\Gamma_u)$ and the space $W_2^{(2)}(\Omega)$ may be embedded into the space $C(\bar{\Omega})$ of functions continuous on $\bar{\Omega} = \Omega \cup \Gamma$. For $\boldsymbol{v} \in \boldsymbol{V}$,

$$\frac{\partial \boldsymbol{v}}{\partial n} = 0$$

holds on Γ_u in the sense of $L_2(\Gamma_u)$. Moreover, $\boldsymbol{v} = \boldsymbol{0}$ on Γ_u in the sense of continuous functions.

Obviously

$$A(\boldsymbol{v}, \boldsymbol{u}) = A(\boldsymbol{u}, \boldsymbol{v}) , \quad A(\boldsymbol{u}, \boldsymbol{u}) = 2 \int_\Omega \mathscr{A}(\varepsilon_{ij}(\boldsymbol{u}), \varkappa_{ijk}(\boldsymbol{u}))\, \mathrm{d}X .$$

Further, let us define for $\boldsymbol{v} \in W$ the functionals

$$f(\boldsymbol{v}) = \int_\Omega X_i u_i \, \mathrm{d}X , \tag{5.2}$$

$$g(\boldsymbol{v}) = \int_\Omega (\bar{P}_i v_i + \bar{R}_i \omega_i(\boldsymbol{v}))\, \mathrm{d}\Gamma + \int_B \bar{Q}_i v_i \, \mathrm{d}B \tag{5.3}$$

where

$$\omega_i(\boldsymbol{v}) = v_{i,l} n_l .$$

According to Section I.1, we define the weak solution of the boundary-value problem as follows:

Let $\bar{\boldsymbol{u}} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\} \in W$ represent the given data on Γ_u through embedding of $W_2^{(2)}(\Omega)$ into $C(\bar{\Omega})$ and of $W_2^{(1)}(\Omega)$ into $L_2(\Gamma_u)^3$). We say that $\boldsymbol{u} = \{u_1, u_2, u_3\}$ is the *weak solution of the boundary-value problem, if* $\boldsymbol{u} - \bar{\boldsymbol{u}} \in V$ *and for each* $\boldsymbol{v} = \{v_1, v_2, v_3\} \in V$ *there holds*

$$A(\boldsymbol{v}, \boldsymbol{u}) = f(\boldsymbol{v}) + g(\boldsymbol{v})$$

where $A(\boldsymbol{v}, \boldsymbol{u}), f(\boldsymbol{v}), g(\boldsymbol{v})$ *are defined by* (5.1)–(5.3).

Similarly to Section 2 we could show that if $\boldsymbol{u}$ is the geometrically admissible displacement field such that the corresponding $\tau_{ij}(\boldsymbol{u})$, $\mu_{jkl}(\boldsymbol{u})$ form a statically admissible field, then (5.4) is valid.

Choosing $N_l \boldsymbol{v}$, $l = 1, 2, \ldots, 36$ in the form

$$N_{1-9}\boldsymbol{v} = v_{(i,j)} , \quad N_{10-36}\boldsymbol{v} = v_{(i,j)k} \tag{5.5}$$

we obtain

$$\sum_{l=1}^{36} (N_l \boldsymbol{v})^2 = \sum_{i,j,k=1}^{3} (\varepsilon_{ij}^2(\boldsymbol{v}) + \varkappa_{ijk}^2(\boldsymbol{v})) .$$

According to the supposition of positive definiteness of the quadratic form (4.3), equation (I.1.4) is satisfied and Theorem I.1.1 yields *coerciveness of the system* (5.5). Indeed, let us form the matrix $N_{ls}\xi$: now $|\alpha| = 2$ and $N_{ls}\xi$ takes the quadratic form

$$N_{ls}\xi = \sum_{i,j} n_{lsij} \xi_i \xi_j .$$

Let us choose three triplets of $N_l \boldsymbol{v}$ which correspond in (5.5) to $(v_{1,11}, v_{(1,2)1}, v_{(1,3)1})$, $(v_{(1,2)2}, v_{2,22}, v_{(3.2)2})$ and $(v_{(1,3)3}, v_{(2,3)3}, v_{3,33})$. The corresponding three determinants of the matrix $N_{ls}\xi$ equal to $\frac{1}{4}\xi_1^6$, $\frac{1}{4}\xi_2^6$ and $\frac{1}{4}\xi_3^6$. If $\xi \neq 0$, $\xi \in \boldsymbol{C}_3$, at least one of them does not vanish and therefore, the rank of $N_{ls}\xi$ is $3 = m$. For $\boldsymbol{v} \in \mathscr{P}$ there holds

$$\varepsilon_{ij}(\boldsymbol{v}) = \varkappa_{ijk}(v) = 0$$

almost everywhere in Ω and therefore

$$\mathscr{P} = \{\boldsymbol{v} = \{v_1, v_2\, v_3\} \in V,\ \ v_k = a_k + b_{kj}x_j,\ \ \ a_k = \text{const.},\ \ \ b_{kj} = -b_{jk} = \text{const.}\}\ .$$

Theorem I.1.2 yields: The *necessary and sufficient condition for the existence of a weak solution is*

$$p \in \mathscr{P} \ \ \Rightarrow\ \ f(p) + g(p) = 0\ . \tag{5.6}$$

The solution is determined except for $p \in \mathscr{P}$. *The inequality* (I.1.7) *implies the continuous dependence of the solution on* $\bar{u}_i$, $\bar{P}_i$, $\bar{R}_i$, $\bar{Q}_i$.

Let us investigate at least two important boundary-value problems:

(1) Let Γ_u be empty. Then (5.6) is equivalent to the following system of equilibrium conditions

$$\int_\Omega X_i\,\mathrm{d}X + \int_\Gamma \bar{P}_i\,\mathrm{d}\Gamma + \int_B \bar{Q}_i\,\mathrm{d}B = 0\,,$$

$$\int_\Omega X_{[i}x_{j]}\,\mathrm{d}X + \int_\Gamma (\bar{P}_{[i}x_{j]} + \bar{R}_{[i}x_{j]})\,\mathrm{d}\Gamma + \int_B \bar{Q}_{[i}x_{j]}\,\mathrm{d}B = 0\,,$$

where

$$A_{[i}b_{j]} = \tfrac{1}{2}(A_ib_j - A_jb_i)\,.$$

(2) Let Γ_u contain a non-empty set open in Γ. Then $\mathscr{P} = \{0\}$ and (5.6) is satisfied. There exists one and only one weak solution.

On the basis of Theorem I.1.3, it is possible to formulate the boundary-value problems so that the solution will be unique even in case (1). We could easily prove that for the linear independent functionals $p_i(\boldsymbol{v})$ introduced in Theorem I.1.3, for example systems (a) or (c) of (2.8) may be taken.

6. THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY AND THE PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY

Using the results of Section I.1 and defining $A(\boldsymbol{v}, \boldsymbol{u})$, $f(\boldsymbol{v})$, $g(\boldsymbol{v})$ by (5.1)–(5.3), we establish the principle of minimum potential energy in the following form:

The quadratic functional $\mathscr{L}(\boldsymbol{u})$ *defined for* $\boldsymbol{u} = \{u_1, u_2, u_3\} \in W$ *by*

$$\mathscr{L}(\boldsymbol{u}) = \int_\Omega [\mathscr{A}(\varepsilon_{ij}(\boldsymbol{u}), \varkappa_{ijk}(\boldsymbol{u})) - X_iu_i]\,\mathrm{d}X - \tag{6.1}$$

$$- \int_{\Gamma_T} [\bar{P}_iu_i + \bar{R}_iu_{i,l}n_l]\,\mathrm{d}\Gamma - \int_B \bar{Q}_iu_i\,\mathrm{d}B$$

attains the minimum on the set

$$\bar{\boldsymbol{u}} \oplus V ,$$

if and only if

$$\boldsymbol{u} = \mathring{\boldsymbol{u}} + \boldsymbol{p} ,$$

where $\mathring{\boldsymbol{u}}$ is the weak solution, $\boldsymbol{p} \in \mathscr{P}$.

The formulation of the principle with subspaces V_p is obvious. Similarly to Section 3, we could derive the principle of minimum complementary energy. We restrict ourselves only to presenting the principle:

The quadratic functional

$$\tilde{\mathscr{S}}(\boldsymbol{T}) = \int_\Omega [\tilde{\mathscr{A}}(\tau_{ij}, \mu_{ijk}) - \{\tau_{ij}\, \varepsilon_{ij}(\bar{\boldsymbol{u}}) + \mu_{ijk}\, \varkappa_{ijk}(\bar{\boldsymbol{u}})\}] \, \mathrm{d}X \tag{6.2}$$

attains the minimum on the set of $\boldsymbol{T} \in \mathscr{T}$ which satisfy the equations of equilibrium (4.2) *and the statical boundary conditions* (4.6), (4.8) *in the weak sense, i.e. for each $\boldsymbol{v} = \{v_1, v_2, v_3\} \in V$ there holds*

$$\int_\Omega [\tau_{ij}\, \varepsilon_{ij}(\boldsymbol{v}) + \mu_{ijk}\, \varkappa_{ijk}(\boldsymbol{v})] \, \mathrm{d}X = \int_\Omega X_i v_i \, \mathrm{d}X + \\ + \int_{\Gamma_T} (\bar{P}_i v_i + \bar{R}_i v_{i,l} n_l) \, \mathrm{d}\Gamma + \int_B \bar{Q}_i v_i \, \mathrm{d}B . \tag{6.3}$$

The minimum of the functional is realized just for $\boldsymbol{T} = \boldsymbol{T}(\mathring{\boldsymbol{u}})$ where $\mathring{\boldsymbol{u}}$ is the weak solution.

Here $\mathscr{T}$ is the Banach space of the stress fields $\boldsymbol{T}$

$$\boldsymbol{T} = \{\tau_{ij}, \mu_{ijk}\} , \quad \tau_{ij} = \tau_{ji} \in L_2(\Omega) , \quad \mu_{ijk} \in L_2(\Omega)$$

with the norm

$$|\boldsymbol{T}|^2_{\mathscr{T}} = \sum_{i,j=1}^{3} |\tau_{ij}|^2_{L_2(\Omega)} + \sum_{i,j,k=1}^{3} |\mu_{ijk}|^2_{L_2(\Omega)} ,$$

$\tilde{\mathscr{A}}(\tau_{ij}, \mu_{ijk})$ is the elastic energy per unit volume expressed by stresses. Further,

$$\varepsilon_{ij}(\bar{\boldsymbol{u}}) = \bar{u}_{(i,j)} , \quad \varkappa_{ijk}(\bar{\boldsymbol{u}}) = \bar{u}_{(j,k)i}$$

and $\bar{P}_i$, $\bar{R}_i$, $\bar{Q}_i$ are given boundary values.

If moreover the weak solution $\mathring{\boldsymbol{u}}$ is such that $\boldsymbol{T}(\mathring{\boldsymbol{u}})$ is a statically admissible stress field, we can state the following alternative of the principle:

The quadratic functional

$$\mathscr{S}(\boldsymbol{T}) = \int_\Omega \tilde{\mathscr{A}}(\tau_{ij}, \mu_{ijk}) \, \mathrm{d}X - \int_{\Gamma u} (P_i \bar{u}_i + R_i \bar{u}_{i,l} n_l) \, \mathrm{d}\Gamma$$

where P_i, R_i are defined by (4.6) *attains the minimum on the set of statically admis-*

sible stress fields, if and only if

$$\left|T - T(\mathring{u})\right|_{\mathscr{T}} = 0 .$$

Remark 1. For both cases investigated in the present paper, i.e. for Mindlin's elasticity with microstructure and for the first strain-gradient theory, it is possible to estimate errors of approximate solution based on the principle of minimum potential energy and the principle of minimum complementary energy, following the procedure in [1].

Remark 2. The couple-stress elasticity with constrained rotations is a particular case of the first strain-gradient theory, if we assume that the elastic energy per unit volume has the form

$$\mathscr{A}(\varepsilon_{ij}; \varepsilon_{i[j,k]}) ,$$

i.e. $\mathscr{A}$ does not depend on $\varepsilon_{i(j,k)}$ (see [3]). However, if we take ε_{ij} and $\varepsilon_{i[j,k]}$ for $N_l v$, then all the third-order determinants of the matrix $N_{ls}\xi$ vanish identically and therefore its rank is lower then $3 = m$ for each $\xi \in C_3$. Hence, such a choice of $N_l v$ fails to be coercive. Unfortunately we have not found any other suitable coercive system. Thus the approach used in the previous sections seems to be hardly applicable to the couple-stress elasticity with constrained rotations.

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Výtah

EXISTENCE A JEDNOZNAČNOST ŘEŠENÍ A NĚKTERÉ VARIAČNÍ PRINCIPY V LINEÁRNÍCH TEORIÍCH PRUŽNOSTI S MOMENTOVÝMI NAPĚTÍMI

IVAN HLAVÁČEK, MIROSLAV HLAVÁČEK

Část 2: MINDLINOVA TEORIE PRUŽNOSTI S MIKROSTRUKTUROU A TEORIE PRUŽNOSTI S PRVNÍM GRADIENTEM DEFORMACE

V druhé části jsou vyšetřovány Mindlinova teorie mikrostruktury a teorie pružnosti uvažující vliv gradientu deformace. Pro statický případ je formulováno zobecněné řešení okrajových úloh pro omezená, anisotropní, nehomogenní tělesa, jsou dokázány existence, jednoznačnost a spojitá závislost zobecněného řešení na daných zatíženích. Jsou uvedeny princip minima potenciální energie a minima doplňkové energie.

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