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ON GENERAL NONLINEAR AND QUASILINEAR UNANTICIPATIVE FEEDBACK SYSTEMS

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0. As known, a general feedback system is obtained by interconnecting a system $\mathfrak{A}$, which has two inputs u and Ψ and two outputs y and Φ, with a system $\mathfrak{X}$ having an input $\tilde{\Phi}$ and output $\tilde{\Psi}$. (See Fig. 1.) Since the interconnection imposes constraints $\Phi = \tilde{\Phi}$ and $\Psi = \tilde{\Psi}$, the formed system has a single input u and a single output y.

Suppose that F is a nonempty set, whose elements are interpreted as possible signals and responses. Then the external behavior of $\mathfrak{A}$ can be described by equations $\Phi = A(u, \Psi)$ and $y = B(u, \Psi)$, where A and B are operators mapping $F \times F$ into F; similarly, $\mathfrak{X}$ can be described by the equation $\Psi = X\Phi$, where X is an operator mapping F into itself.

Let an operator W mapping F into itself exist such that for every $u \in F$ there exist uniquely determined elements Φ, Ψ and y which satisfy the equations

$$\begin{aligned} \Phi &= A(u, \Psi), \quad & y &= B(u, \Psi), \\ \Psi &= X\Phi, \quad & y &= Wu; \end{aligned} \tag{1}$$

then W will be called the over-all transfer operator of the feedback system.

Obviously, the operator W relates the signal u to the corresponding response y of the entire feedback system.

If the response y is bounded in a certain sense whenever the signal u is bounded, we will say that we have the input-output boundedness.

If the operator W is continuous in a certain sense, we will say that the feedback system is input-output stable.

A more thorough discussion of the physical meaning of these concepts may be found in [1].

1. Let us now turn to the exact treatment.

Let Ω be a fixed nonempty set of real numbers; if $T \in \Omega$, we will denote $[T] = (-\infty, T] \cap \Omega$.

Next, let $\mathfrak{F}$ be a nonempty linear set, and let $\tilde{F}$ be the system of all mappings from Ω into $\mathfrak{F}$. The system $\tilde{F}$ is a linear set with customary operations of addition and multiplication by a constant.

Furthermore, let F and F^* be nonempty linear subsets of $\tilde{F}$ such that $\tilde{F} \supset F \supset F^*$.

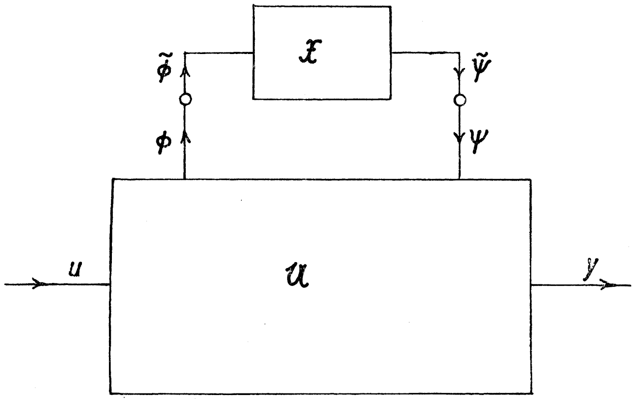

Fig. 1.

For each $T \in \Omega$ let us have a linear mapping S_T from $\tilde{F}$ into itself which has the following properties:

a) If $x \in F$, then $S_T x \in F^*$ for any $T \in \Omega$.
b) If $x, y \in \tilde{F}$ and $T \in \Omega$, then $x(t) = y(t)$ on $[T]$ iff $S_T x = S_T y$.
c) $S_{T_1} S_{T_2} = S_{T_1}$ for any $T_1 \leqq T_2$, $T_1, T_2 \in \Omega$.

For the sake of brevity, we will also use the notation $S_T x = (x)_T = x_T$.

Observe that c) implies $S_T^k = S_T$, $k = 1, 2, \ldots$, and a), b) imply: If $x \in \tilde{F}$, then $x(t) = x_T(t)$ on $[T]$.

Finally, we introduce the following axiom:

AI. If, for an $x \in \tilde{F}$, we have $x_T \in F^*$ for any $T \in \Omega$, then $x \in F$.

Before proceeding further, let us present few examples of particular sets $\tilde{F}$, F and F^*:

Let $\Omega = [0, \infty)$ and $\mathfrak{F} = E^n$, i.e. the set of all n-tuples of numbers; then $\tilde{F}$ is the set of all n-vector-valued functions defined on $[0, \infty)$.

1. Let $1 \leqq p < \infty$ and put

$$F = \bar{L}_p = \{x : x \in \tilde{F}, x \text{ measurable}, \textstyle\int_0^\tau |x(t)|^p \, \mathrm{d}t < \infty \text{ for any } 0 < \tau < \infty\},$$
$$F^* = L_p = \{x : x \in \tilde{F}, x \text{ measurable}, \textstyle\int_0^\infty |x(t)|^p \, \mathrm{d}t < \infty\}.$$

If $p = \infty$, let

$$F \;= \bar{L}_\infty = \{x : x \in \tilde{F},\, x \text{ measurable, } \operatorname*{ess\,sup}_{[0,\tau]} |x(t)| < \infty \text{ for any } 0 < \tau < \infty\}\,,$$
$$F^* = L_\infty = \{x : x \in \tilde{F},\, x \text{ measurable, } \operatorname*{ess\,sup}_{[0,\infty)} |x(t)| < \infty\}\,.$$

Putting $x_T(t) = x(t)$ for $0 \leqq t \leqq T$, $x_T(t) = \theta$ for $t > T$, (θ signifies the zero vector), then we can readily verify that the requirements a), b), c) and AI are satisfied.

2. Let

$$F \;= \bar{C} = \{x : x \in \tilde{F},\, x \text{ continuous on } [0, \infty)\}\,,$$
$$F^* = C = \{x : x \in \tilde{F},\, x \text{ continuous on } [0, \infty) \text{ and } |x(t)| \text{ bounded on } [0, \infty)\}\,.$$

Putting $x_T(t) = x(t)$ for $0 \leqq t \leqq T$, $x(t) = x(T)$ for $t > T$, then again the above requirements are satisfied.

3. Let $F = \bar{C}$ as in 2. and let $\lambda > 0$; put

$$F^* = C_\lambda = \{x : x \in \bar{C} \text{ and } e^{\lambda t}|x(t)| \text{ is bounded on } [0, \infty)\}\,,$$

and $x_T(t) = x(t)$ for $0 \leqq t \leqq T$, $x_T(t) = x(T)\, e^{\lambda(T-t)}$ for $t > T$. A little thought will persuade us that a), b), c) and AI are fulfilled.

Analogously, let $\Omega = \{1, 2, 3, \ldots\}$ and $\mathfrak{F} = E$; then $\tilde{F}$ is the set of all infinite sequences of numbers. Putting $\tilde{F} = F$, $F^* = l_p$ with $1 \leqq p < \infty$, i.e.

$$l_p = \{x : x \in \tilde{F},\, \sum_{i=1}^{\infty} |x(i)|^p < \infty\}\,,$$

and $x_T(t) = x(t)$ for $t = 1, 2, \ldots, T$, $x_T(t) = 0$ for $t > T$, we can again verify that a), b), c) and AI are satisfied.

Let us now carry out some preliminary considerations.

If $T \in \Omega$ is a fixed number, let

$$F_T = S_T F = \{x : x = S_T y,\; y \in F\}\,.$$

By a) it follows that $F_T \subset F^*$; moreover, by c) $x_T = x$ for any $x \in F_T$, i.e., S_T is the identity operator on F_T.

Let A be an operator mapping F into itself; A will be called unanticipative, if

$$S_T A = S_T A S_T \tag{2}$$

for any $T \in \Omega$, i.e., if $(Ax)_T = (Ax_T)_T$ for any $T \in \Omega$ and $x \in F$.

Remark 1. Observe that unanticipativity of A is equivalent to the following requirement:

$$v_1, v_2 \in F\,, \quad v_1(t) = v_2(t) \quad \text{on} \quad [T] \Rightarrow (Av_1)(t) = (Av_2)(t) \quad \text{on} \quad [T]\,. \tag{2$'$}$$

Lemma 1. *Let A, B be unanticipative operators mapping $F \to F$; then $A + B$ and AB are also unanticipative.*

The proof is obvious.

Lemma 2. *Let $T \in \Omega$ be fixed; if A is one-to-one from F onto F and both A and the inverse A^{-1} are unanticipative, then the operator $S_T A$ is one-to-one from F_T onto F_T and has the inverse $S_T A^{-1}$.*

Proof. Observe first that $S_T A$ and $S_T A^{-1}$ map F_T into itself and S_T is the identity operator on F_T. Putting $B = S_T A^{-1}$, we have $B(S_T A) = S_T A^{-1} S_T A = S_T A^{-1} A = S_T$; hence, $S_T A$ is one-to-one. Furthermore, $(S_T A) B = S_T A S_T A^{-1} = S_T A A^{-1} = S_T$; hence, $S_T A$ is onto and consequently, B is the inverse $(S_T A)^{-1}$.

Remark 2. The assumption that A is one-to-one from F onto F and unanticipative need not necessarily imply that A^{-1} is also unanticipative. This can be demonstrated by the operator A defined by $(Ax)(t) = x(t/2)$, $x \in F$, $t \geqq 0$.

Lemma 3. *Let the axiom* AI *be satisfied, and let A be an unanticipative operator mapping F into F. If, for every $T \in \Omega$, the operator $S_T A$ has a unique fixed point in F_T, i.e. a unique $\Phi^{(T)} \in F_T$ exists such that*

$$\Phi^{(T)} = S_T A \Phi^{(T)}, \tag{3}$$

then there exists a unique element $\Phi \in F$ satisfying the equation

$$\Phi = A\Phi. \tag{4}$$

Before turning to the proof let us make the following comment. First, observe that $\Phi^{(T)}$ is also a unique fixed point of $S_T A$ in F, because if a $\Phi^{(T)} \in F$ satisfies (3), we have $\Phi^{(T)} \in F_T$ due to the definition of F_T. Second, Lemma 3 obviously reduces the problem of solving (4) in F to solving (3) in a narrower set $F_T \subset F^*$.

Proof of Lemma 3. Let $T, T' \in \Omega$ and $T \leqq T'$; we are going to show that $\Phi_T^{(T')} = \Phi^{(T)}$. Actually, we have by (3),

$$\Phi^{(T)} = S_T A \Phi^{(T)}, \quad \Phi^{(T')} = S_{T'} A \Phi^{(T')}. \tag{5}$$

Applying S_T to the second equation (5) and using c) we get

$$\Phi_T^{(T')} = S_T(S_{T'} A \Phi^{(T')}) = S_T A \Phi^{(T')} = S_T A S_T \Phi^{(T')} = S_T A \Phi_T^{(T')}.$$

Since $\Phi_T^{(T')} \in F_T$, it follows due to the uniqueness that $\Phi_T^{(T')} = \Phi^{(T)}$.

Next, define the element $\Phi \in F$ as follows: If $t \in \Omega$, put $\Phi(t) = \Phi^{(T)}(t)$, where $T \in \Omega$ and $T \geqq t$. The definition is clearly meaningful, because taking $T' \in \Omega$, $T' \geqq T$, we have by the above result, $\Phi^{(T)} = \Phi_T^{(T')}$, i.e. $\Phi_T^{(T)} = \Phi_T^{(T')}$, so that $\Phi^{(T)}(\tau) = \Phi^{(T')}(\tau)$ for $\tau \in [T]$ by b). Hence, $\Phi^{(T)}(t) = \Phi^{(T')}(t)$.

On the other hand, if $T \in \Omega$ is any number, then we have $\Phi(t) = \Phi^{(T)}(t)$ on $[T]$, i.e. by b), $\Phi_T = \Phi_T^{(T)}$. Because $\Phi_T^{(T)} \in F_T \subset F^*$, it follows by AI that $\Phi \in F$. Thus, the value $A\Phi$ is defined.

As a next step, we are going to show that Φ satisfies (4). Actually, let $t \in \Omega$ and choose $T \geqq t$, $T \in \Omega$. Then $\Phi_T = \Phi_T^{(T)} = \Phi^{(T)}$. Moreover, $(A\Phi)_T = (A\Phi_T)_T = (A\Phi^{(T)})_T = \Phi^{(T)} = \Phi_T$, i.e. by b), $(A\Phi)(\tau) = \Phi(\tau)$ for $\tau \in [T]$, and consequently, $(A\Phi)(t) = \Phi(t)$.

For proving the uniqueness assume that a $\Psi \in F$ exists such that $\Psi = A\Psi$. Choosing a $T \in \Omega$ we have $\Psi_T \in F_T$ and $\Psi_T = (A\Psi)_T = (A\Psi_T)_T$. Consequently, by the uniqueness of $\Phi^{(T)}$, $\Psi_T = \Phi^{(T)}$, and by the above, $\Psi_T = \Phi_T$; hence, in view of b), $\Psi(t) = \Phi(t)$ on $[T]$ and the lemma is proved.

Now we are ready for stating a proposition concerning the feedback system.

Theorem 1. *Let the axiom* AI *be satisfied; let X be an unanticipative operator mapping $F \to F$, let A and B map $F \times F \to F$ and A be such that*

$$\{A(u, v_T)\}_T = \{A(u, v)\}_T \tag{6}$$

for any $T \in \Omega$ and $u, v \in F$.

If for any $u \in F$ and $T \in \Omega$ the operator $S_T A(u, X.)$ has a unique fixed point in F_T, then the over-all transfer operator $W: F \to F$ exists, i.e., for any $u \in F$ there exist uniquely determined elements Φ, Ψ, $y \in F$ such that equations (1) *are satisfied.*

If, in addition, A and B satisfy the conditions

$$\{A(u_T, v)\}_T = \{A(u, v)\}_T\,, \tag{7}$$

$$\{B(u_T, v_T)\}_T = \{B(u, v)\}_T \tag{8}$$

then W is unanticipative.

Proof. For a fixed $u \in F$ denote $G_u = A(u, X.)$, and $\tilde{A}_u = A(u, .)$. Then $\tilde{A}_u$ is unanticipative by (6), and consequently, $G_u = \tilde{A}_u X$ is also unanticipative by Lemma 1. Thus, due to the hypothesis of the theorem and Lemma 3, there exists a unique $\Phi \in F$ satisfying the equation

$$\Phi = G_u \Phi = A(u, X\Phi)\,. \tag{9}$$

Hence, (9) defines an operator $Q : F \to F$ by $\Phi = Qu$. From (1) it follows that if we set

$$Wu = B(u, XQu) \tag{10}$$

for any $u \in F$, the sought operator W is established.

Next, if (7) holds it turns out that Q is unanticipative. Actually, choose $u \in F$ and let $\Phi = Qu$, $\tilde{\Phi} = Qu_T$, i.e.,

$$\Phi = A(u, X\Phi)\,, \quad \tilde{\Phi} = A(u_T, X\tilde{\Phi})\,. \tag{11}$$

Then we have by (7),

$$\Phi_T = \{A(u, X\Phi)\}_T\,, \quad \tilde{\Phi} = \{A(u, X\tilde{\Phi})\}_T\,.$$

Thus, by (6)

$$\Phi_T = \{A(u, (X\Phi)_T)\}_T = \{A(u, (X\Phi_T)_T)\}_T = \{A(u, X\Phi_T)\}_T\,,$$

and similarly,

$$\tilde{\Phi}_T = \{A(u, (X\tilde{\Phi})_T)\}_T = \{A(u, (X\tilde{\Phi}_T)_T)\}_T = \{A(u, X\tilde{\Phi}_T)\}_T\,.$$

Hence, by the assumption on uniqueness of the fixed point in F_T, $\tilde{\Phi}_T = \Phi_T$, i.e., $(Qu)_T = (Qu_T)_T$.

Finally, by (10) and (8),

$$(Wu)_T = \{B(u, XQu)\}_T = \{B(u_T, (XQu)_T\}_T\,.$$

However, since both X and Q are unanticipative, we have $(XQu)_T = (XQu_T)_T$, and consequently,

$$(Wu)_T = \{B((u_T)_T, (XQu_T)_T\}_T = \{B(u_T, XQu_T)\}_T = (Wu_T)_T\,.$$

This concludes the proof.

In the sequel we will assume that F^* is a Banach space with norm $\|\,.\,\|$. Moreover, let us introduce the following axioms concerning the norm in F^*.

A2: AI is satisfied; if $x \in F^*$, then $\|x_T\| \leqq \|x\|$ for any $T \in \Omega$.

A3: A2 is satisfied; if $x \in F$ and a constant $\Lambda > 0$ exists such that $\|x_T\| \leqq \Lambda$ for all $T \in \Omega$, then $x \in F^*$ and $\|x\| \leqq \Lambda$.

The reader can easily verify that the particular spaces $\bar{L}_p$, L_p, $\bar{L}_\infty$, L_∞, $\bar{C}$, C and C_λ, and l_p mentioned above satisfy the axiom A3 with the corresponding mapping S_T and customary norm. Note that in C_λ we define the norm by $\|x\| = \sup\limits_{[0,\infty)} e^{\lambda t}|x(t)|$.

Lemma 4. *Let* A2 *hold; then, for any* $T \in \Omega$, F_T *with the norm of* F^* *is a Banach space.*

Proof. F_T is clearly a linear normed space. Let $\{x_i\}$, $x_i \in F_T$, $i = 1, 2, \ldots$ be a Cauchy sequence, i.e. $\|x_i - x_k\|$ can be made arbitrarily small by taking i and k sufficiently large. Since also $x_i \in F^*$ and F^* is complete by assumption, there exists an $x \in F^*$ such that $x_i \to x$ in F^*, i.e. $\|x_i - x\| \to 0$ as $i \to \infty$. However, due to A2, $\|(x_i)_T - x_T\| = \|x_i - x_T\| = \|(x_i - x)_T\| \leqq \|x_i - x\| \to 0$; hence, it follows that $x_i \to x_T$ in F^*, and consequently, $x = x_T$. Thus, $x \in F_T$ and the lemma is proven.

For our purposes we will slightly generalize the concept of continuity of an operator.

Let A map $F \to F$; A is called continuous at $x \in F$, if for every $\varepsilon > 0$ a number $\delta > 0$ exists such that for any $\tilde{x} \in F$ with $x - \tilde{x} \in F^*$ and $\|x - \tilde{x}\| < \delta$ we have

$Ax - A\tilde{x} \in F^*$ and $\|Ax - A\tilde{x}\| < \varepsilon$. If A is continuous at every point $x \in F$, then A is called the continuous operator.

Similarly, if B maps $F \times F \to F$, then B is called continuous at $(x, y) \in F \times F$, if for every $\varepsilon > 0$ a $\delta > 0$ exists such that for any $(\tilde{x}, \tilde{y}) \in F \times F$ with $(x - \tilde{x}, y - \tilde{y}) \in F^* \times F^*$ and $\|x - \tilde{x}\| < \delta$, $\|y - \tilde{y}\| < \delta$ we have $B(x, y) - B(\tilde{x}, \tilde{y}) \in F^*$ and $\|B(x, y) - B(\tilde{x}, \tilde{y})\| < \varepsilon$. The operator B is called continuous, if it is continuous at every point $(x, y) \in F \times F$.

It is clear that if A_1, A_2 are continuous operators mapping $F \to F$, then $A_1 + A_2$ and $A_1 A_2$ are also continuous.

Theorem 2. *Let the axiom* A2 *hold, let the operators A, B map $F \times F \to F$, let A satisfy condition* (6), *and let X be an unanticipative operator mapping $F \to F$. Furthermore, for every $T \in \Omega$ let an integer $m_T \geqq 1$ and a number $\lambda_T < 1$ exist such that*

$$\|S_T(\tilde{A}_u X)^{m_T} x - S_T(\tilde{A}_u X)^{m_T} y\| \leqq \lambda_T \|x - y\| \tag{12}$$

for all $u \in F$, $x, y \in F_T$, where $\tilde{A}_u = A(u, .)$. Then the over-all transfer operator W given by (10) *exists.*

Moreover, if

(i) *the axiom* A3 *holds,*
(ii) *a fixed integer $m \geqq 1$ and number $\lambda < 1$ exist such that* (12) *is true for every $T \in \Omega$ and $x, y \in F_T$, $u \in F$,*
(iii) *the operator $(\tilde{A}_u X)^m x$ is continuous at every $(x, u) \in F \times F$,*
(iv) *the operators X and B are continuous,*

then W is continuous.

If, in addition, $(\tilde{A}_u X)^m \theta \in F^*$ for any $u \in F^*$, X maps $F^* \to F^*$ and B maps $F^* \times F^* \to F^*$, then W maps F^* into itself.

Proof. Choose a $u \in F$ and denote $G_u = \tilde{A}_u X = A(u, X .)$. Then for any integer $k \geqq 1$,

$$S_T G_u^k = (S_T G_u)^k . \tag{13}$$

Actually, (13) is clearly true for $k = 1$. Above we have shown that G_u is unanticipative; consequently, by Lemma 1, G_u^k is also unanticipative. Supposing that (13) is true for some $k \geqq 1$, we easily conclude by induction that it holds for $k + 1$, too.

Next, condition (12) shows that, for any $T \in \Omega$, $S_T G_u^{m_T}$ is a contraction on F_T, which is a Banach space by Lemma 4. Hence, in view of (13), there exists a unique $x^T \in F_T$ such that

$$x^T = (S_T G_u)^{m_T} x^T . \tag{14}$$

However, then x^T is a unique solution of

$$x^T = S_T G_u x^T \tag{15}$$

in F_T. Indeed, from (14) it follows that $S_T G_u x^T = (S_T G_u)^{m_T+1} x^T = (S_T G_u)^{m_T} \,.\, (S_T G_u x^T)$; consequently, due to the uniqueness of x_T, we have necessarily $x^T = S_T G_u x^T$. Moreover, assuming that $y^T \in F_T$ satisfies the equation $y^T = S_T G_u y^T$, it follows that $y^T = (S_T G_u)^2 y^T = \ldots = (S_T G_u)^{m_T} y^T$; hence, $y^T = x^T$.

Thus, the assumptions of Theorem 1 are met and consequently, the operator W exists; the first part of Theorem 2 is proved.

Assume now that the conditions (i) through (iv) are fulfilled, and let $u \in F$, $\varepsilon > 0$. Choosing a $T \in \Omega$ and $\tilde{u} \in F$ such that $u - \tilde{u} \in F^*$, let

$$x^T = (S_T G_u)^m x^T, \quad \tilde{x}^T = (S_T G_{\tilde{u}})^m \tilde{x}^T. \tag{16}$$

Then we have by (12) and (13),

$$\|\tilde{x}^T - x^T\| \leqq \|(S_T G_{\tilde{u}})^m \tilde{x}^T - (S_T G_{\tilde{u}})^m x^T\| + \|(S_T G_{\tilde{u}})^m x^T - (S_T G_u)^m x^T\| \leqq$$
$$\leqq \lambda \|\tilde{x}^T - x^T\| + \|S_T\{G_{\tilde{u}}^m x^T - G_u^m x^T\}\| ;$$

hence

$$\|\tilde{x}^T - x^T\| \leqq (1 - \lambda)^{-1} \|S_T\{G_{\tilde{u}}^m x^T - G_u^m x^T\}\| . \tag{17}$$

On the other hand, as shown above $x^T = x_T$ and $\tilde{x}^T = \tilde{x}_T$, where $x = A(u, Xx)$ and $\tilde{x} = A(\tilde{u}, X\tilde{x})$, i.e. $x = Qu$ and $\tilde{x} = Q\tilde{u}$. Since $G_{\tilde{u}}^m$ and G_u^m are unanticipative, it follows that $S_T\{G_{\tilde{u}}^m x^T - G_u^m x^T\} = S_T\{G_{\tilde{u}}^m x - G_u^m x\}$.

However, according to continuity of $G_u^m x$, there exists a $\delta > 0$ such that for $\|\tilde{u} - u\| < \delta$ we have $G_{\tilde{u}}^m x - G_u^m x \in F^*$ and $\|G_{\tilde{u}}^m x - G_u^m x\| < (1 - \lambda)\,\varepsilon$. Introducing this into (17) and using the axiom A3, we obtain

$$\|S_T(\tilde{x} - x)\| < \varepsilon . \tag{18}$$

Consequently, by the axiom A3, $Q\tilde{u} - Qu \in F^*$ and $\|Q\tilde{u} - Qu\| \leqq \varepsilon$ whenever $\tilde{u} - u \in F^*$ and $\|\tilde{u} - u\| < \delta$, i.e. the operator $Q : F \to F$ defined by the above equations is continuous.

Since then XQ is continuous by (iv), it follows that the operator $W = B(.\,, XQ.)$ is continuous, too. Hence, the proof.

As for the last assertion, let $u \in F^*$ and choose $T \in \Omega$. Then (14), (13) and (12) yield

$$\|x^T\| \leqq \|S_T G_u^m x^T - S_T G_u^m \theta\| + \|S_T G_u^m \theta\| \leqq \lambda \|x^T - \theta\| + \|S_T G_u^m \theta\| .$$

Hence,

$$\|x_T\| \leqq (1 - \lambda)^{-1} \|S_T G_u^m \theta\| \leqq (1 - \lambda)^{-1} \|G_u^m \theta\|$$

by A3. Using again A3 it follows that $x \in F^*$ and $\|x\| \leqq (1 - \lambda)^{-1} \|G_u^m \theta\|$, i.e. the operator Q maps F^* into itself. The employment of assumptions on ranges of X and B concludes the proof.

The theorem just proved clearly gives sufficient conditions for the input-output boundedness and stability of a feedback system; observe that these conditions are given in terms of "local behavior" of the operator $A(u, X.)$.

Let us now present two simple examples clarifying the application of the above results.

Example 1. Let the sets L_p and $\bar{L}_p$ have the same meaning as in the beginning of the paper. Let the n-vector valued functions G, V, H and U have the following properties:

1. $G(t, u)$ and $V(t, u)$ are defined for $0 \leqq t < \infty$ and $u \in E^n$, and are such that $G(t, u(t))$, $V(t, u(t)) \in \bar{L}_p$ whenever $u(t) \in \bar{L}_p$.
2. $H(t, \tau, u, v)$ is defined for $0 \leqq \tau \leqq t < \infty$, $u, v \in E^n$ and is such that

$$\int_0^t H(t, \tau, u(\tau), v(\tau))\, d\tau \in \bar{L}_p$$

   whenever $u(t), v(t) \in \bar{L}_p$.
3. $U(t, \tau, u)$ is defined for $0 \leqq \tau \leqq t < \infty$, $u \in E^n$ and is such that $\int_0^t U(t, \tau, u(\tau))\, d\tau \in$ $\in \bar{L}_p$ whenever $u(t) \in \bar{L}_p$.

Furthermore, for each $T > 0$ let constants $\varkappa_T, \mu_T, \lambda_T > 0$ exist such that

$$|H(t, \tau, u, v_1) - H(t, \tau, u, v_2)| \leqq \varkappa_T |v_1 - v_2| \quad \text{for} \quad 0 \leqq \tau \leqq t < \infty\,, \quad u, v_1, v_2 \in E^n\,, \tag{19}$$

$$|U(t, \tau, v_1) - U(t, \tau, v_2)| \leqq \mu_T |v_1 - v_2| \quad \text{for} \quad 0 \leqq \tau \leqq t < \infty\,, \quad v_1, v_2 \in E^n\,, \tag{20}$$

$$|V(t, v_1) - V(t, v_2)| \leqq \lambda_T |v_1 - v_2| \quad \text{for} \quad t \geqq 0\,, \quad v_1, v_2 \in E^n\,. \tag{21}$$

Let the operator $A : \bar{L}_p \times \bar{L}_p \to \bar{L}_p$ and $X : \bar{L}_p \to \bar{L}_p$ be defined by

$$\{A(u, v)\}(t) = G(t, u(t)) + \int_0^t H(t, \tau, u(\tau), v(\tau))\, d\tau\,, \tag{22}$$

$$\{Xv\}(t) = V(t, v(t)) + \int_0^t U(t, \tau, v(\tau))\, d\tau\,, \tag{23}$$

and let $B = A$. Then the over-all transfer operator $W: \bar{L}_p \to \bar{L}_p$ exists and W is unanticipative.

Actually, A clearly satisfies conditions (6), (7), and X is unanticipative; thus, all what remains to do is to show that (12) holds. Choosing a fixed $u \in \bar{L}_p$ and $T > 0$, assume that $p > 1$. Then for any $v_1, v_2 \in (\bar{L}_p)_T$ and integer $m \geqq 1$ we have

$$\|S_T G_u^m v_1 - S_T G_u^m v_2\| \leqq \frac{T(T s_T)^m}{(m-1)!} \|v_1 - v_2\| , \tag{24}$$

where $s_T = \varkappa_T(\lambda_T + \mu_T T)$ and $G_u = A(u, X.)$. For proving (24) show that, for any $t \in [0, T]$,

$$|\{G_u^m v_1\}(t) - \{G_u^m v_2\}(t)| \leqq \frac{t^{(1/q)+m-1}}{(m-1)!} s_T^m \|v_1 - v_2\| . \tag{25}$$

By equations (19) through (23) it follows that, for $t \in [0, T]$,

$$a = |\{G_u v_1\}(t) - \{G_u v_2\}(t)| \leqq \int_0^t \varkappa_T |\{X v_1 - X v_2\}(\tau)| \, d\tau ,$$

and

$$|\{X v_1 - X v_2\}(t)| \leqq \lambda_T |v_1(t) - v_2(t)| + \int_0^t \mu_T |v_1(\tau) - v_2(\tau)| \, d\tau \leqq$$
$$\leqq \lambda_T |v_1(t) - v_2(t)| + \mu_T t^{1/q} \|v_1 - v_2\| .$$

Hence,

$$a \leqq \varkappa_T \int_0^t (\lambda_T |v_1(\tau) - v_2(\tau)| + \mu_T \tau^{1/q} \|v_1 - v_2\|) \, d\tau \leqq$$
$$\leqq t^{1/q} \varkappa_T(\lambda_T + \mu_T T) \|v_1 - v_2\| .$$

Thus, (25) holds for $m = 1$. Assuming the validity of (25) for some $m \geqq 1$, we obtain,

$$b = |\{G_u^{m+1} v_1\}(t) - \{G_u^{m+1} v_2\}(t)| \leqq \varkappa_T \int_0^t |\{X G_u^m v_1 - X G_u^m v_2\}(\tau)| \, d\tau$$

and

$$|\{X G_u^m v_1 - X G_u^m v_2\}(t)| \leqq \lambda_T |\{G_u^m v_1 - G_u^m v_2\}(t)| +$$
$$+ \mu_T \int_0^t |\{G_u^m v_1 - G_u^m v_2\}(\tau)| \, d\tau \leqq \frac{1}{\varkappa_T} \frac{t^{(1/q)+m-1}}{(m-1)!} s_T^{m+1} \|v_1 - v_2\| .$$

Consequently,

$$b \leqq \frac{t^{(1/q)+m}}{(m-1)! \left(\frac{1}{q} + m\right)} s_T^{m+1} \|v_1 - v_2\| \leqq \frac{t^{(1/q)+m}}{m!} s_T^{m+1} \|v_1 - v_2\| ,$$

and (25) is proved.

Realizing that $\{S_T w\}(t) = 0$ for $t > T$, we obtain immediately (24) from (25).

However, for any $T > 0$, the number $T(Ts_T)^m/(m - 1)!$ can be made less than one by taking m sufficiently large, i.e. (12) holds and our assertion is proved.

The cases $p = 1$ and $p = \infty$ can be treated in the same way.

Example 2. Let the signals and responses be interpreted as elements of the sets $\bar{C}$ and C mentioned above. Let $G(t, u, v)$, $V(t, v)$, $H(t, \tau, u, v)$ and $U(t, \tau, v)$ be continuous n-vector valued functions of all arguments $0 \leqq \tau \leqq t < \infty$, $u, v \in E^n$. Furthermore, assume that the following conditions are satisfied:

1. Constants $c_1, c_2, c_3 > 0$ exist such that

$$|G(t, u_1, v_1) - G(t, u_2, v_2)| \leqq c_1|u_1 - u_2| + c_2|v_1 - v_2| \tag{26}$$

and

$$|V(t, u_1) - V(t, u_2)| \leqq c_3|u_1 - u_2| \tag{27}$$

for all $t \geqq 0$, $u_1, u_2, v_1, v_2 \in E^n$.

2. Nonnegative functions $h_1(t, \tau)$, $h_2(t, \tau)$, $p(t, \tau)$ with

$$\sup_{[0,\infty)} \int_0^t h_i(t, \tau)\, d\tau = k_i < \infty\,, \quad i = 1, 2, \quad \sup_{[0,\infty)} \int_0^t p(t, \tau)\, d\tau = l < \infty$$

exist such that

$$|H(t, \tau, u_1, v_1) - H(t, \tau, u_2, v_2)| \leqq h_1(t, \tau)\,|u_1 - u_2| + h_2(t, \tau)\,|v_1 - v_2| \tag{28}$$

and

$$|U(t, \tau, u_1) - U(t, \tau, u_2)| \leqq p(t, \tau)\,|u_1 - u_2|$$

for any $0 \leqq \tau \leqq t < \infty$ and $u_1, u_2, v_1, v_2 \in E^n$.

Let, for $u, v \in \bar{C}$ and $t \geqq 0$,

$$\{A(u, v)\}\,(t) = G(t, u(t), v(t)) + \int_0^t H(t, \tau, u(\tau), v(\tau))\, d\tau\,, \tag{29}$$

$$\{Xv\}\,(t) = V(t, v(t)) + \int_0^t U(t, \tau, v(\tau))\, d\tau\,,$$

and let $B = A$.

First, it is clear that A maps $\bar{C} \times \bar{C} \to \bar{C}$, X maps $\bar{C} \to \bar{C}$, and that A satisfies conditions (6), (7). Moreover, it is a matter of a simple routine to verify that, for any $u_1, u_2, v_1, v_2 \in \bar{C}$ and $T > 0$,

$$\begin{aligned} \|S_T\{A(u_1, Xv_1) - A(u_2, Xv_2)\}\| &\leqq (c_1 + k_1)\| S_T(u_1 - u_2)\| + \\ &+ (c_2 + k_2)(c_3 + l)\,\|S_T(v_1 - v_2)\|\,. \end{aligned} \tag{30}$$

If $\lambda = (c_2 + k_2)(c_3 + l) < 1$, then (30) shows that the condition (12) in Theorem 2 is satisfied for any $T > 0$. (Note that $S_T v_1, S_T v_2 \in \bar{C}_T$); hence, the unanticipative over-all transfer operator $W : \bar{C} \to \bar{C}$ exists.

Moreover, (30) shows that the operator $A(., X.)$ is continuous at any point $(u, v) \in \in \bar{C} \times \bar{C}$; actually, let $u_1, u_2, v_1, v_2 \in \bar{C}$ be such that $u_1 - u_2 \in C$ and $v_1 - v_2 \in C$. Then, due to the axiom A3, the right-hand side of (30) is less than $(c_1 + k_1) \|u_1 - - u_2\| + \lambda \|v_1 - v_2\|$; consequently, again by A3, $A(u_1, Xv_1) - A(u_2, Xv_2) \in C$ and $\|A(u_1, Xv_1) - A(u_2, Xv_2)\| \leqq (c_1 + k_1) \|u_1 - u_2\| + \lambda \|v_1 - v_2\|$. Hence, condition (iii) is satisfied.

Similarly, from (27) and (28) we obtain easily that

$$\|S_T\{Xv_1 - Xv_2\}\| \leqq (c_3 + l) \| S_T(v_1 - v_2)\| \tag{31}$$

for $v_1, v_2 \in \bar{C}$, and from (26), (28),

$$\begin{aligned}\|S_T\{A(u_1, v_1) - A(u_2, v_2)\}\| \leqq (c_1 + k_1) \|S_T(u_1 - u_2)\| + \\ + (c_2 + k_2) \|S_T(v_1 - v_2)\|\end{aligned} \tag{32}$$

for $u_1, u_2, v_1, v_2 \in \bar{C}$. By the same argument as above we conclude that X and $A = B$ are continuous operators. Thus, condition (iv) is fulfilled, and consequently, W is a continuous operator.

Finally, if we assume that

$$\{X\theta\}(t) = V(t, \theta) + \int_0^t U(t, \tau, \theta) \, d\tau \in C$$

and

$$\{A(\theta, \theta)\}(t) = G(t, \theta, \theta) + \int_0^t H(t, \tau, \theta, \theta) \, d\tau \in C ,$$

then we have by (31) and (32) for $u, v \in C$,

$$\|S_T X v\| \leqq (c_3 + l) \|S_T v\| + \|S_T X \theta\|$$

and

$$\|S_T A(u, v)\| \leqq (c_1 + k_1) \|S_T u\| + (c_2 + k_2) \|S_T v\| + \|S_T A(\theta, \theta)\| ;$$

consequently, by A3, $Xv \in C$ and $A(u, v) \in C$. Hence, according to the last assertion of Theorem 2, W maps C into itself, i.e. we have the input-output boundedness.

Let us now consider the quasi-linear case of a feedback system.

Lemma 5. *Let C be a linear unanticipative operator mapping $F \to F$.*

a) *If* A2 *holds, C maps $F^* \to F^*$ and C is bounded on F^*, then $\|S_T C\| \leqq \|C\|$ for any $T \in \Omega$.*

b) *Let* A3 *hold; if, for every* $T \in \Omega$, *the operator* $S_T C$ *is bounded on* F_T *and a constant* $\Lambda > 0$ *exists such that* $\|S_T C\| \leqq \Lambda$, *then* C *is bounded on* F^* *and* $\|C\| \leqq \Lambda$.

Proof. a) is obvious. As for b), choose $T \in \Omega$ and $x \in F^*$; then $\|S_T(Cx)\| \leqq \leqq \|S_T C\| \cdot \|x\| \leqq \Lambda \|x\|$. Thus, by A3, $Cx \in F^*$ and $\|Cx\| \leqq \Lambda \|x\|$; consequently, $\|C\| \leqq \Lambda$.

Theorem 3. *Let* A2 *hold; let the operators* $A_1, \tilde{X}, C_1, C_2$ *map* F *into itself, let* $\tilde{X}, C_1, C_2$ *be unanticipative and* C_1, C_2 *be linear and such that* $I - C_1 C_2$ *is one-to-one from* F *onto* F *and* $(I - C_1 C_2)^{-1}$ *is unanticipative.* (*I is the identity operator.*) *Let the operator* $\tilde{A}$ *map* $F \times F \to F$ *and satisfy the condition* (6), *and let* B *map* $F \times F \to F$. *Furthermore, for every* $T \in \Omega$ *let*

(i) $S_T C_1$ *and* $S_T C_2$ *be bounded on* F_T,

(ii) *numbers* $d_1^T, d_2^T > 0$ *exist such that*

$$\|S_T \tilde{A}(u, v_1) - S_T \tilde{A}(u, v_2)\| \leqq d_1^T \|v_1 - v_2\| \tag{33}$$

and

$$\|S_T \tilde{X} v_1 - S_T \tilde{X} v_2\| \leqq d_2^T \|v_1 - v_2\| \tag{34}$$

for all $u \in F$, $v_1, v_2 \in F_T$ *and*

$$\|S_T(I - C_1 C_2)^{-1}\| \, \{\|S_T C_1\| \, d_2^T + \|S_T C_2\| \, d_1^T + d_1^T d_2^T\} < 1 \,. \tag{35}$$

If, for all $u, v \in F$,

$$A(u, v) = A_1 u + C_1 v + \tilde{A}(u, v) \,, \quad Xv = C_2 v + \tilde{X} v \,, \tag{36}$$

then the over-all transfer operator $W: F \to F$ *exists.*

Moreover, if

(iii) A3 *holds,*

(iv) C_1, C_2 *map* $F^* \to F^*$ *and are bounded on* F^*, *and* $\|S_T(I - C_1 C_2)^{-1}\| \leqq \mu$ *for all* $T \in \Omega$ *with a fixed* $\mu > 0$,

(v) *fixed numbers* $d_1, d_2 > 0$ *exist such that* (33) *and* (34) *are true for any* $u \in F$, $v_1, v_2 \in F^*$,

(vi) $\mu(\|C_1\| \, d_2 + \|C_2\| \, d_1 + d_1 d_2) < 1$,

(vii) $A_1, \tilde{A}$ *and* B *are continuous,*

then W *is continuous.*

If, in addition, $A_1, \tilde{X}$ *map* $F^* \to F^*$ *and* $\tilde{A}, B$ *map* $F^* \times F^* \to F^*$, *then* W *maps* F^* *into itself.*

Proof. First, observe the following facts. Since $I - C_1 C_2 = K$ is one-to-one from F onto F and K, K^{-1} are unanticipative, then, due to Lemma 2, $S_T K$ is one-to-one from F_T onto F_T and $(S_T K)^{-1} = S_T K^{-1}$. Since by (i) $S_T K$ is clearly bounded on F_T, and F_T is a Banach space by Lemma 4, it follows by Banach theorem (see [2], p. 123) that $S_T K^{-1}$ is also bounded on F_T; hence, $\|S_T K^{-1}\| < \infty$.

Moreover, if conditions (iii) and (iv) are satisfied, then K is one-to-one from F^* onto F^*; actually, since K is one-to-one, all we have to show is that K is onto. Thus, choose $y \in F^*$; then $K^{-1} y \in F$. However, for any $T \in \Omega$ we have $\|S_T K^{-1} y\| \leqq \leqq \|S_T K^{-1}\| \, . \, \|y\| \leqq \mu \|y\|$; hence, by A3, $K^{-1} y \in F^*$ and $\|K^{-1} y\| \leqq \mu \|y\|$.

Next, recalling Theorem 1 choose $u \in F$ and consider the equation

$$(37) \qquad \Phi^{(T)} = S_T A(u, X\Phi^{(T)}) = = S_T\{A_1 u + C_1(C_2 \Phi^{(T)} + \tilde{X}\Phi^{(T)}) + \tilde{A}(u, C_2 \Phi^{(T)} + \tilde{X}\Phi^{(T)})\}$$

on the space F_T, i.e. the equation

$$(38) \qquad S_T(I - C_1 C_2)\, \Phi^{(T)} = S_T\{A_1 u + C_1 \tilde{X}\Phi^{(T)} + \tilde{A}(u, C_2 \Phi^{(T)} + \tilde{X}\Phi^{(T)})\}\,.$$

However, in view of the above considerations, (38) is equivalent to

$$(39) \qquad \Phi^{(T)} = S_T R_u \Phi^{(T)}\,,$$

where

$$(40) \qquad R_u \Phi^{(T)} = K^{-1}\{A_1 u + C_1 \tilde{X}\Phi^{(T)} + \tilde{A}(u, C_2 \Phi^{(T)} + \tilde{X}\Phi^{(T)})\}\,.$$

As a next step we are going to show that $S_T R_u$ is a contraction on F_T. Actually, for $\Phi_1, \Phi_2 \in F_T$ we have by (40),

$$(41) \qquad \varkappa = \|S_T R_u \Phi_1 - S_T R_u \Phi_2\| \leqq \|S_T K^{-1} C_1(\tilde{X}\Phi_1 - \tilde{X}\Phi_2)\| + + \|S_T K^{-1}\{\tilde{A}(u, C_2 \Phi_1 + \tilde{X}\Phi_1) - \tilde{A}(u, C_2 \Phi_2 + \tilde{X}\Phi_2)\}\|\,.$$

However, since $S_T K^{-1} C_1 = S_T K^{-1} S_T C_1 = (S_T K^{-1})(S_T C_1)\, S_T$, we have by (34),

$$\|S_T K^{-1} C_1(\tilde{X}\Phi_1 - \tilde{X}\Phi_2)\| \leqq \|S_T K^{-1}\| \, . \, \|S_T C_1\| \, . \, d_2^T \|\Phi_1 - \Phi_2\|\,.$$

On the other hand, by (6) we have

$$S_T \tilde{A}(u, C_2 \Phi_i + \tilde{X}\Phi_i) = S_T \tilde{A}(u, S_T(C_2 \Phi_i + \tilde{X}\Phi_i)) = = S_T \tilde{A}(u, S_T C_2 \Phi_i + S_T \tilde{X}\Phi_i)\,; \quad i = 1, 2\,;$$

hence, (33) and (34) yield

$$\|S_T K^{-1}\{\tilde{A}(u, C_2 \Phi_1 + \tilde{X}\Phi_1) - \tilde{A}(u, C_2 \Phi_2 + \tilde{X}\Phi_2)\| \leqq \leqq \|S_T K^{-1}\| \, \{d_1^T \|S_T C_2\| + d_1^T d_2^T\} \, \|\Phi_1 - \Phi_2\|\,.$$

Introducing this into (41), we obtain finally,

$$(42) \qquad \|S_T R_u \Phi_1 - S_T R_u \Phi_2\| \leqq \lambda_T \|\Phi_1 - \Phi_2\|$$

with

$$\lambda_T = \|S_T K^{-1}\| \{\|S_T C_1\| d_2^T + \|S_T C_2\| d_1^T + d_1^T d_2^T\} ;$$

hence, by (35), $S_T R_u$ is a contraction on F_T. Consequently, (39) and also (37) has a unique solution in F_T, i.e. the operator W exists by Theorem 1.

Assume now that the conditions (iii) through (vii) are satisfied and let $u \in F$, $\varepsilon > 0$; choosing a $\tilde{u} \in F$ such that $\tilde{u} - u \in F^*$ and $T \in \Omega$, let

$$\Phi^{(T)} = S_T A(u, X\Phi^{(T)}) , \quad \tilde{\Phi}^{(T)} = S_T A(\tilde{u}, X\tilde{\Phi}^{(T)}) , \tag{43}$$

i.e.

$$\Phi^{(T)} = S_T R_u \Phi^{(T)} , \quad \tilde{\Phi}^{(T)} = S_T R_u \tilde{\Phi}^{(T)} . \tag{44}$$

From Lemma 5 it follows that $\|S_T C_i\| \leqq \|C_i\|$, $i = 1, 2$, and consequently, $\lambda_T \leqq \leqq \mu(\|C_1\| d_2 + \|C_2\| d_1 + d_1 d_2) = \lambda < 1$. Thus, we can write by (42) and (44),

$$\begin{aligned} \|\tilde{\Phi}^{(T)} - \Phi^{(T)}\| &= \|S_T R_{\tilde{u}} \tilde{\Phi}^{(T)} - S_T R_u \Phi^{(T)}\| \leqq \\ &\leqq \|S_T R_{\tilde{u}} \tilde{\Phi}^{(T)} - S_T R_{\tilde{u}} \Phi^{(T)}\| + \|S_T R_{\tilde{u}} \Phi^{(T)} - S_T R_u \Phi^{(T)}\| \leqq \\ &\leqq \lambda \|\tilde{\Phi}^{(T)} - \Phi^{(T)}\| + \|S_T R_{\tilde{u}} \Phi^{(T)} - S_T R_u \Phi^{(T)}\| , \end{aligned}$$

i.e.

$$\|\tilde{\Phi}^{(T)} - \Phi^{(T)}\| \leqq (1 - \lambda)^{-1} \|S_T R_{\tilde{u}} \Phi^{(T)} - S_T R_u \Phi^{(T)}\| . \tag{45}$$

Next, recall the fact that $\tilde{\Phi}^{(T)} = \tilde{\Phi}_T$ and $\Phi^{(T)} = \Phi_T$, where $\tilde{\Phi}$ and Φ is the solution of $\tilde{\Phi} = A(u, X\tilde{\Phi})$ and $\Phi = A(u, X\Phi)$, respectively, and put for brevity $\varrho = C_2\Phi + \tilde{X}\Phi$. Then $S_T\varrho = S_T C_2 \Phi + S_T \tilde{X}\Phi = S_T(C_2\Phi^{(T)} + \tilde{X}\Phi^{(T)})$.

However, since both operators A_1 and $\tilde{A}$ are continuous, there exists a $\delta > 0$ such that for $\|\tilde{u} - u\| < \delta$ we will have

$$\begin{aligned} &A_1\tilde{u} - A_1 u \in F^* , \quad \tilde{A}(\tilde{u}, \varrho) - \tilde{A}(u, \varrho) \in F^* \quad \text{and} \quad \|A_1\tilde{u} - A_1 u\| < \\ &< \frac{\varepsilon}{2} \mu^{-1}(1 - \lambda) , \quad \|\tilde{A}(\tilde{u}, \varrho) - \tilde{A}(u, \varrho)\| < \frac{\varepsilon}{2} \mu^{-1}(1 - \lambda) . \end{aligned}$$

Then (45) and (40) yield with the aid of (6) and A3,

$$\begin{aligned} \|\tilde{\Phi}^{(T)} - \Phi^{(T)}\| &\leqq (1 - \lambda)^{-1} \|S_T K^{-1}\{(A_1\tilde{u} - A_1 u) + \\ &+ (\tilde{A}(\tilde{u}, C_2\Phi^{(T)} + \tilde{X}\Phi^{(T)}) - \tilde{A}(u, C_2\Phi^{(T)} + \tilde{X}\Phi^{(T)}))\}\| \leqq \\ &\leqq (1 - \lambda)^{-1} \|S_T K^{-1}\| . \frac{\varepsilon}{2} \mu^{-1}(1 - \lambda) + (1 - \lambda)^{-1} \|S_T K^{-1}\{S_T\tilde{A}(\tilde{u}, S_T\varrho) - \\ &- S_T\tilde{A}(u, S_T\varrho)\}\| < \frac{\varepsilon}{2} + (1 - \lambda)^{-1} \|S_T K^{-1} S_T\{\tilde{A}(\tilde{u}, \varrho) - \tilde{A}(u, \varrho)\}\| < \\ &< \frac{\varepsilon}{2} + (1 - \lambda)^{-1} \|S_T K^{-1}\| . \frac{\varepsilon}{2} \mu^{-1}(1 - \lambda) < \varepsilon . \end{aligned}$$

Hence, by A3, $\tilde{\Phi} - \Phi \in F^*$ and $\|\tilde{\Phi} - \Phi\| \leqq \varepsilon$, i.e. the operator $Q : F \to F$ defined by the equations $Qu = \Phi$, $\Phi = A(u, X\Phi)$ is continuous.

Finally, the inequality (34) together with A3 show that $\tilde{X}$ is continuous; consequently, X is a continuous operator. Since B is continuous, the continuity of W follows immediately.

As for the last assertion of Theorem 3, let $u \in F^*$ and $\Phi^{(T)} = S_T R_u \Phi^{(T)}$. Then (42) yields

$$\|\Phi^{(T)}\| \leqq \lambda \|\Phi^{(T)}\| + \|S_T R_u \theta\| ,$$

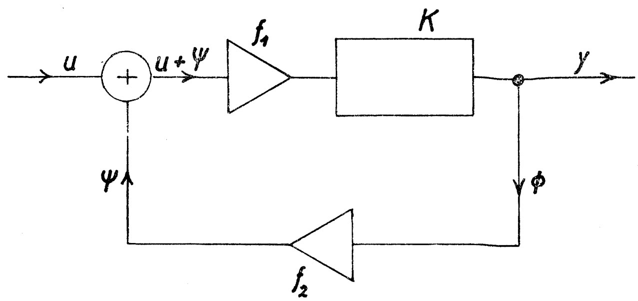

Fig. 2.

i.e.

$$(46) \qquad \|\Phi^{(T)}\| \leqq (1 - \lambda)^{-1} \|S_T R_u \theta\| .$$

However, the assumptions and (40) imply that $R_u \theta \in F^*$; then (46) with A3 show that $\Phi \in F^*$ and $\|\Phi\| \leqq (1 - \lambda)^{-1} \|R_u \theta\|$, i.e. Q maps F^* into itself. The assumption on the range of B concludes the proof.

The physical interpretation of Theorem 3 is straightforward; if the linearized system described by operators $A'(u, v) = A_1 u + C_1 v$, $X'v = C_2 v$, and $B'(u, v) = B(u, v)$ is well-behaved, then the feedback system itself is well-behaved provided the nonlinearities are not too large. (Witness (35) and (vi).)

Let us now present two simple examples.

Example 3. Consider the classical feedback configuration portrayed in Fig. 2, where f_1, f_2 signify pure memoryless gains and K is a linear system governed by the equation

$$(Kx)(t) = \int_0^t k(t, \tau)\, x(\tau)\, d\tau + a\, x(t)$$

(a is a constant). For the system of signals and responses we shall take the set $\bar{C}$. Furthermore, we will assume that

1. $k(t, \tau)$ is continuous for $0 \leqq \tau \leqq t < \infty$ and

$$M = \sup_{[0,\infty)} \int_0^t |k(t, \tau)| \, d\tau < \infty ,$$

2. constants μ_i and $\alpha_i > 0$ exist such that

$$\mu_i - \alpha_i \leqq \frac{f_i(\xi_1) - f_i(\xi_2)}{\xi_1 - \xi_2} \leqq \mu_i + \alpha_i \tag{47}$$

for any $\xi_1, \xi_2 \in E$, $\xi_1 \neq \xi_2$, $i = 1, 2$, and $1 - \mu_1\mu_2 a \neq 0$.

Our task is to find constraints for α_1, α_2 which guarantee the input-output stability and boundedness of the system, provided the linearized system is assumed to be well-behaved (see below).

From Fig. 2 it follows that here

$$\{A(u, v)\}(t) = \{B(u, v)\}(t) = af_1(u(t) + v(t)) + \\ + \int_0^t k(t, \tau) f_1(u(\tau) + v(\tau)) \, d\tau , \tag{48}$$

$\{Xv\}(t) = f_2(v(t))$.

It is clear that A actually maps $\bar{C} \times \bar{C} \to \bar{C}$ and X maps $\bar{C} \to \bar{C}$. Moreover, (47) and 1. show that A maps $C \times C \to C$ and X maps $C \to C$.

Using the language of Theorem 3, put

$$(A_1 u)(t) = (C_1 u)(t) = a\mu_1 u(t) + \mu_1 \int_0^t k(t, \tau) u(\tau) \, d\tau , \tag{49}$$

$$C_2 v = \mu_2 v ;$$

consequently,

$$\{\tilde{A}(u, v)\}(t) = a\{f_1(u(t) + v(t)) - \mu_1(u(t) + v(t))\} + \\ + \int_0^t k(t, \tau) \{f_1(u(\tau) + v(\tau)) - \mu_1(u(\tau) + v(\tau))\} \, d\tau , \tag{50}$$

$$\tilde{X}v = f_2(v) - \mu_2 v .$$

It can be easily verified by using (47) that

$$\|S_T(\tilde{X}v_1 - \tilde{X}v_2)\| \leqq \alpha_2 \|S_T(v_1 - v_2)\| \tag{51}$$

and

$$\|S_T(\tilde{A}(u_1, v_1) - \tilde{A}(u_2, v_2))\| \leqq \alpha_1(M + |a|) \{\|S_T(u_1 - u_2)\| + \\ + \|S_T(v_1 - v_2)\|\} \tag{52}$$

for any $u_1, u_2, v_1, v_2 \in F$ and $T > 0$. Since A3 holds, (51) and (52) show that $\tilde{X}$ and $\tilde{A}$ are continuous operators.

Furthermore, we obtain

$$\|C_1\| \leqq |\mu_1| (|a| + M), \quad \|C_2\| = |\mu_2|, \tag{53}$$

and from (51), (52),

$$d_1 = \alpha_1(|a| + M), \quad d_2 = \alpha_2 .$$

Next,

$$(I - C_1 C_2)\, x = (1 - \mu_1 \mu_2 a) \left\{ x - \frac{\mu_1 \mu_2}{1 - \mu_1 \mu_2 a} \int_0^t k(t, \tau)\, x(\tau)\, \mathrm{d}\tau \right\};$$

consequently, for $x \in C$,

$$(I - C_1 C_2)^{-1}\, x = (1 - \mu_1 \mu_2 a)^{-1} \left\{ x + \int_0^t h(t, \tau)\, x(\tau)\, \mathrm{d}\tau \right\}, \tag{54}$$

where $h(t, \tau)$ is given by

$$h(t, \tau) = \sum_{i=1}^{\infty} \left(\frac{\mu_1 \mu_2}{1 - \mu_1 \mu_2 a} \right)^i k^{\times i}(t, \tau) .$$

Equation (54) shows that $(I - C_1 C_2)^{-1}$ is an unanticipative operator.

Assume now that $N = \sup_{[0,\infty)} \int_0^t |h(t, \tau)|\, \mathrm{d}\tau < \infty$, i.e. that the linearized system is well-behaved as indicated above. Then $\|(I - C_1 C_2)^{-1}\| \leqq |1 - \mu_1 \mu_2 a|^{-1} (1 + N)$, and, by (vi) in Theorem 3, the sought constraint for α_1, α_2 reads

$$(1 + N)(M + |a|)\, |1 - \mu_1 \mu_2 a|^{-1} (|\mu_1|\, \alpha_2 + |\mu_2|\, \alpha_1 + \alpha_1 \alpha_2) < 1 . \tag{55}$$

Hence, if (55) holds, then the considered feedback system has the over-all transfer operator and is input-output stable and bounded.

Example 4. Consider the same system as in Example 3, but now let K be time-invariant and set $F = C$, $F^* = C_\lambda$, where C_λ is the set introduced in the beginning of the paper. It can be easily verified that C_λ with the norm $\|x\| = \sup_{[0,\infty)} e^{\lambda t} |x(t)|$ is a Banach space, and that the axiom A3 is satisfied with the mapping S_T defined above.

Here, let

$$\{A(u, v)\}(t) = \{B(u, v)\}(t) = \int_0^t k(t - \tau)\, f_1(u(\tau) + v(\tau))\, \mathrm{d}\tau , \tag{56}$$

$$\{Xv\}(t) = f_2(v(t)) ,$$

$u, v \in C$.

Assume that the following conditions are met:

1. Positive constants $\mu_1, \mu_2, \alpha_1, \alpha_2$ exist such that (47) holds for any $\xi_1, \xi_2 \in E$, $\xi_1 \neq \xi_2$, and $f_1(0) = f_2(0) = 0$.
2. $k(t) \geqq 0$ and is continuous for $t \geqq 0$, and the Laplace integral $K(p) = \int_0^\infty k(t)\, e^{-pt}\, dt$ converges for $\operatorname{Re} p > -\lambda - \varepsilon$, $\varepsilon > 0$, λ fixed.
3. The function $K(p)/(1 - \mu_1\mu_2 K(p))$ is analytic for $\operatorname{Re} p > -\lambda - \varepsilon$.

Our task is to establish conditions for α_1 and α_2, under which the over-all transfer operator W exists and maps C_λ into itself.

As before, let

$$(57)\qquad (A_1 u)(t) = (C_1 u)(t) = \int_0^t k(t-\tau)\, \mu_1\, u(\tau)\, d\tau\,, \quad C_2 v = \mu_2 v\,,$$

$$\tilde{X} v = f_2(v) - \mu_2 v\,,$$

$$\{\tilde{A}(u, v)\}(t) = \int_0^t k(t-\tau)\, \{f_1(u+v) - \mu_1(u+v)\}\, d\tau\,.$$

First of all, we are going to show that C_1 maps C_λ into itself and $\|C_1\| = \mu_1 K(-\lambda)$. Actually, let $x \in C_\lambda$; then for any $t \geqq 0$, $|x(t)| \leqq \|x\|\, e^{-\lambda t}$, and consequently,

$$|(C_1 x)(t)\, e^{\lambda t}| \leqq \mu_1 \int_0^t |k(t-\tau)|\, e^{\lambda t} \,.\, \|x\|\, e^{-\lambda\tau}\, d\tau =$$

$$= \mu_1 \|x\| \int_0^t |k(t-\tau)|\, e^{\lambda(t-\tau)}\, d\tau = \mu_1 \|x\| \int_0^t k(\sigma)\, e^{\lambda\sigma}\, d\sigma = \mu_1 \|x\|\, K(-\lambda)\,.$$

Hence, $\|C_1 x\| \leqq \mu_1 K(-\lambda)\, \|x\|$.

On the other hand, letting $x_0 = e^{-\lambda t}$, we have $\|x_0\| = 1$ and

$$(C_1 x_0)(t)\, e^{\lambda t} = \mu_1 \int_0^t k(t-\tau)\, e^{\lambda t}\, d\tau = \mu_1 \int_0^t k(\sigma)\, e^{\lambda\sigma}\, d\sigma\,,$$

so that $\sup\limits_{[0,\infty)} |(C_1 x_0)(t)\, e^{\lambda t}| = \mu_1 K(-\lambda)$; consequently, $\|C_1\| = \mu_1 K(-\lambda)$.

Next, we have $\|C_2\| = \mu_2$, and

$$(58)\qquad (I - C_1 C_2)\, x = x - \mu_1\mu_2 \int_0^t k(t-\tau)\, x(\tau)\, d\tau\,, \quad x \in C\,.$$

Then

$$(I - C_1 C_2)^{-1} x = x + \int_0^t h(t - \tau)\,\mathrm{d}\tau\,, \quad x \in C\,,$$

where $h(t)$ is given by $h(t) = \sum_{i=1}^{\infty} (\mu_1 \mu_2)^i\, k^{*i}(t)$, (here, k^{*i} signifies the i-times iterated convolution of k) and satisfies the equation

$$h(t) - \mu_1\mu_2\, k(t) - \mu_1\mu_2\, h(t) * k(t) = 0\,. \tag{59}$$

Hence, $(I - C_1 C_2)^{-1}$ is unanticipative, and due to $\mu_1\mu_2 > 0$ and $k(t) \geqq 0$ we have $h(t) \geqq 0$. Since assumption 2. implies that $h(t)$ is Laplace transformable, we have by (59) in some half-plane $\operatorname{Re} p > \xi$,

$$H(p) - \mu_1\mu_2\, K(p) - \mu_1\mu_2\, H(p)\, K(p) = 0\,. \tag{60}$$

However, since by 3. $H(p)$ from (60) is analytic for $\operatorname{Re} p > -\lambda - \varepsilon$, we have

$$H(-\lambda) = \frac{\mu_1\mu_2\, K(-\lambda)}{1 - \mu_1\mu_2\, K(-\lambda)} = \int_0^{\infty} h(t)\, e^{\lambda t}\,\mathrm{d}t < \infty\,. \tag{61}$$

Hence, using the same argument as in considering the above operator C_1 we conclude that $(I - C_1 C_2)^{-1}$ maps $C_\lambda \to C_\lambda$ and $\|(I - C_1 C_2)^{-1}\| \leqq 1 + H(-\lambda)$. (Observe that we have also $1 - \mu_1\mu_2\, K(-\lambda) > 0$.) Thus, condition (iv) in Theorem 3 is satisfied.

Now, let $u_1, u_2, v_1, v_2 \in C$, $T > 0$; then we have by (57), (47),

$$\left|e^{\lambda t}\{\tilde{A}(u_1, v_1) - \tilde{A}(u_2, v_2)\}(t)\right| \leqq \int_0^t |k(t - \tau)|\, e^{\lambda t} \alpha_1 (|v_1 - v_2| + |u_1 - u_2|)\,\mathrm{d}\tau \leqq$$
$$\leqq \alpha_1 K(-\lambda) \left\{\|S_T(u_1 - u_2)\| + \|S_T(v_1 - v_2)\|\right\}, \quad 0 \leqq t \leqq T;$$

consequently,

$$\|S_T(\tilde{A}(u_1, v_1) - \tilde{A}(u_2, v_2))\| \leqq \alpha_1 K(-\lambda)\,(\|S_T(u_1 - u_2)\| + \|S_T(v_1 - v_2)\|)\,. \tag{62}$$

Similarly, we get

$$\|S_T(\tilde{X} v_1 - \tilde{X} v_2)\| \leqq \alpha_2 \|S_T(v_1 - v_2)\|\,. \tag{63}$$

Inequalities (62) and (63) show that the operators $\tilde{A}$ and $\tilde{X}$ are continuous, and that we may set $d_1 = \alpha_1 K(-\lambda)$, $d_2 = \alpha_2$. Thus, condition (vii) holds, too.

Using the fact that $f_1(0) = f_2(0) = 0$ and (62), (63), we easily conclude in an obvious way that $\tilde{A}$ maps $C_\lambda \times C_\lambda \to C_\lambda$ and $\tilde{X}$ maps $C_\lambda \to C_\lambda$. Hence, the last assumption in Theorem 3 is also satisfied.

Finally, the condition (vi) reads

$$(1 + H(-\lambda)) \{\mu_1 K(-\lambda) \alpha_2 + \mu_2 \alpha_1 K(-\lambda) + \alpha_1 \alpha_2 K(-\lambda)\} < 1 ; \tag{64}$$

substituting (61) into (64), we obtain the sought condition for α_1, α_2,

$$K(-\lambda) (\alpha_1 + \mu_1) (\alpha_2 + \mu_2) < 1 . \tag{65}$$

Hence, if (65) is satisfied, the over-all transfer operator W maps C_λ into itself, i.e. any exponentially decreasing signal produces an exponentially decreasing response.

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Souhrn

O OBECNÝCH NELINEÁRNÍCH A KVAZILINEÁRNÍCH KAUZÁLNÍCH ZPĚTNOVAZEBNÍCH SOUSTAVÁCH

VÁCLAV DOLEŽAL

V článku jsou vyšetřeny obecné nelineární a kvazilineární kauzální zpětnovazební soustavy. Použitím metod teorie abstraktních prostorů jsou dokázány věty o existenci operátoru přenosu a o ohraničenosti a stabilitě typu vstup-výstup. Užití výsledků je ilustrováno na několika konkrétních příkladech.

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