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ALGORITMY

18. sfbesj

ALGORITHMUS FOR EVALUATION OF SPHERICAL BESSEL FUNCTIONS

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Presented procedure, *sfbesj*, is an algorithmus for evaluation of the spherical Bessel functions, defined as

$$j_l(z) = \sqrt{\left(\frac{\pi}{2z}\right)}\, J_{l+1/2}(z) \tag{1}$$

where l is a positive integer or zero and $J_\nu(z)$ denotes the ("cylindrical") Bessel function (of the 1st kind) of the order of ν. It is to note that on the basis of eq. (1), the sfbesj procedure can be used for calculation of the Bessel function of the half-integer order, too.

The evaluation of our function is based on well-known relations[1]) namely

$$j_{i-1}(z) + j_{i+1}(z) = \frac{2i+1}{z}\, j_i(z)\,, \quad i > 0 \tag{2}$$

(recurrence is started with $j_0(z) = \sin z/z$, $j_1(z) = \sin z/z^2 - \cos z/z$) and

$$J_\nu(z) = \sum_{i=0}^{\infty} (-1)^i \frac{(z/2)^{\nu+2i}}{i!\, \Gamma(\nu + i + 1)} \tag{3}$$

The last one is valid for the Bessel functions in general. Taking into account eqs. (1) and (3) as well as the relations $\Gamma(x+1) = x\,\Gamma(x)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ ($\Gamma(x)$ is the Euler gamma function) and setting ν for $l + \frac{1}{2}$ we obtain

$$j_l(z) = \frac{(z/2)^l}{\frac{3}{2} \cdot \frac{5}{2} \ldots \nu} \sum_{i=0}^{\infty} (-1)^i \frac{(z/2)^{2i}}{i!\,(\nu+1)(\nu+2)\ldots(\nu+i)} \tag{4}$$

[1]) See any standard text on the theory of Bessel equation, e.g. G. N. Watson, "Theory of Bessel Functions", 2nd edition, Camb. Univ. Press, New York 1945.

For the reason of obtaining high accuracy for any relation of the argument and the order of the $j_l(z)$ it seems to be convenient to calculate it according to the reccurence formula (2) for $l \leqq z$ and by summing of the series (4) for $l > z$. In the alternating series two successive terms are taken together in each turn, the summation is continued until the current value of the added term is less than $_{10}{-7}$ times the value of the sum.

```
real procedure sfbesj(l, z); value l, z; integer l; real z;
comment sfbesj evaluates the spherical Bessel function j_l(z) = √(π/2z) . J_{l+1/2}(z)
    (l ≧ 0, z ≧ 0) by using of the reccurence formula
```

$$j_{i-1}(z) + j_{i+1}(z) = \frac{2i+1}{z} j_i(z)\,, \quad i > 0$$

$$\left(j_0(z) = \frac{\sin z}{z}\,, j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}\right)$$

for $l \leqq z$, if $l > z$ the truncated summation of the series

$$j_l(z) = \sqrt{\left(\frac{\pi}{2z}\right)} \sum_{i=0}^{\infty} (-1)^i \frac{(z/2)^{l+1/2+2i}}{i!\,\Gamma(l+\frac{1}{2}+i+1)}$$

is employed;

```
begin real p, koef, sum; integer i;
  if l ≦ entier(z) then
    begin
      if z ≦ ₁₀−8 then sum := 1 else begin sum := sin (z)/z; p := cos (z)/z end;
      l := l + l;
      for i := 2 step 2 until l do
      begin koef := sum; sum := koef × (i −1)/z − p ; p := koef end
    end
  else
    begin
      p := l + .5; z := z/2; koef := 1;
      for i := 1 step 1 until l do koef := koef × z/(i + .5);
      z := z × z; sum := koef × (1 − z/(p + 1));
      for i := 2, i + 2 while koef > abs(sum) × ₁₀−7 do
      begin p := p + 2; koef := koef × z↑2/(i − 1)/i/(p − 1)/p;
        sum := sum + koef × (1 − z/(i + 1)/(p + 1))
      end
    end;
  sfbesj := sum
end of sfbesj;
```

The procedure was tested by GIER computer (30 bits for mantissa, 10 bits for exponent, 0·12 msec for floating point addition) for $l \leqq 24$, $z \leqq 24$. In this range

average execution time	~25 msec	
accuracy	8 significant digits usually (7 at least)	
typical results	calculated	tabulated[2]
$j_4(3)$	$5{\cdot}61497144_{10}-2$	$5{\cdot}614971433_{10}-2$
$j_4(20)$	$5{\cdot}04761492_{10}-2$	$5{\cdot}047615_{10}-2$
$j_{20}(3)$	$2{\cdot}39422493_{10}-16$	$2{\cdot}394224927_{10}-16$
$j_{20}(20)$	$3{\cdot}83248497_{10}-2$	$3{\cdot}832485_{10}-2$

[2]) Tables of Spherical Bessel Functions, vol. I and II, Camb. Univ. Press, New York 1947.