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PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR WAVE EQUATION IN E_3 IN A SPHERICALLY SYMMETRICAL CASE

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In [1] the existence of periodic solutions of a linear and weakly nonlinear wave equation in one spacial dimension was studied. The spherically symmetrical case of a linear or weakly nonlinear wave equation in three dimensions may be treated analogously. Therefore we shall concentrate our attention to those points in which the two problems differ.

§ 1. THE LINEAR CASE.

As well known, supposing that the right-hand side and the solution depends only on $r^2 = x^2 + y^2 + z^2$ the wave equation in E_3 has the form

$$u_{tt} - u_{rr} - \frac{2}{r} u_r = f(t, r) . \tag{1.1}$$

We shall study the problem $(\mathscr{P}_0)$ given by (1.1) and

$$|u(t, 0)| < +\infty , \quad u(t, \pi) = 0 , \tag{1.2}$$

$$u(t + 2\pi, r) - u(t, r) = 0 . \tag{1.3}$$

Together with this we shall investigate the problem $(\mathscr{M}_0)$ given by (1.1), (1.2) and

$$u(0, r) = \varphi(r) , \quad u_t(0, r) = \psi(r) . \tag{1.4}$$

Note, that making use of the substitution $u(t, r) = v(t, r)/r$ the equations (1.1) to (1.4) take the form:

$$v_{tt} - v_{rr} = r f(t, r) ; \tag{1.1'}$$

$$\lim_{r \to 0} \left| \frac{v(t, r)}{r} \right| < +\infty , \quad v(t, \pi) = 0 ; \tag{1.2'}$$

$$(1.3') \qquad v(t + 2\pi, r) - v(t, r) = 0 ;$$

$$(1.4') \qquad v(0, r) = r\,\varphi(r), \quad v_t(0, r) = r\,\psi(r) .$$

(Supposing that $v(t, r)$ is continuously differentiable with respect to r in the neighbourhood of $r = 0$ the first condition in (1.2′) is equivalent to $v(t, 0) = 0$.)

Consulting the results of § 1 in [1] this leads us to formulate the following assumptions:

($\mathscr{A}_1$) The function $r\,f(t, r)$ together with its derivative with respect to r is continuous in t and r for $0 \leqq r \leqq \pi$, $0 \leqq t < +\infty$,

$$(1.5) \qquad f(t + 2\pi, r) - f(t, r) = 0 \quad \text{and} \quad [r\,f(t, r)]_{r=0} = [r\,f(t, r)]_{r=\pi} = 0 .$$

($\mathscr{A}_2$) The functions $r\,\varphi(r)$ and $r\,\psi(r)$ are of class C^2 or C^1, respectively, and it holds

$$(1.6) \qquad \begin{aligned} &[r\,\varphi(r)]_{r=0} = [r\,\varphi(r)]''_{r=0} = [r\,\psi(r)]_{r=0} = 0 , \\ &[r\,\varphi(r)]_{r=\pi} = [r\,\varphi(r)]''_{r=\pi} = [r\,\psi(r)]_{r=\pi} = 0 . \end{aligned}$$

Let us continue the functions φ, ψ and f in r onto $(-\infty, +\infty)$ by the relations

$$(1.7) \qquad \begin{aligned} r\,\varphi(r) &= r\,\varphi(-r) &&= (r + 2\pi)\,\varphi(r + 2\pi) , \\ r\,\psi(r) &= r\,\psi(-r) &&= (r + 2\pi)\,\psi(r + 2\pi) , \\ r\,f(t, r) &= r\,f(t, -r) &&= (r + 2\pi)\,f(t, r + 2\pi) . \end{aligned}$$

Let us denote the continued functions by the same letters as before. According to (1.5), (1.6) it may be easily verified that the continued functions have the same degree of smoothness as the original ones.

Let us introduce the function $s(r)$ by the relation

$$(1.8) \qquad s(r) = \tfrac{1}{2}\left[r\,\varphi(r) + \int_0^r \sigma\,\psi(\sigma)\,\mathrm{d}\sigma + c\right],$$

c being an arbitrary constant. The function s being given the functions φ and ψ are uniquely determined as

$$(1.9) \qquad \begin{aligned} r\,\varphi(r) &= s(r) - s(-r) , \\ r\,\psi(r) &= s'(r) - s'(-r) . \end{aligned}$$

The functions $r\,\varphi(r)$ and $r\,\psi(r)$ satisfy the assumption ($\mathscr{A}_2$) if and only if the function $s(r)$ satisfies the assumption:

($\mathscr{A}_3$) The function $s(r)$ is 2π-periodic and of class C^2 for $-\infty < r < +\infty$.

We shall denote $\mathfrak{S}$ the B-space of functions $s(r)$ satisfying the assumption ($\mathscr{A}_3$) provided with the norm

$$\|s\| = \sup_{0 \leqq r \leqq 2\pi} [|s(r)|, |s'(r)|, |s''(r)|] .$$

A solution of the problem $(\mathscr{P}_0)$ or $(\mathscr{M}_0)$, respectively, will be sought in a B-space $\mathfrak{A}$ of functions $u(t, r)$ which have first derivatives with respect to t and r continuous for $0 < r \leqq \pi, 0 \leqq t < +\infty$ and such that $u(t, r)$, $u_t(t, r)$, $r\,u_r(t, r)$, $r\,u_{tr}(t, r)$, $r\,u_{tt}(t, r)$, $r^2u_{rr}(t, r)$ have finite limits for $r \to 0$ and any $t \in \langle 0, +\infty)$ which define the values of these functions for $r = 0$. The norm in the space $\mathfrak{A}$ is defined by

$$\|u\| = \max_{\substack{0 \leqq r \leqq \pi \\ 0 \leqq t < +\infty}} [|u|, |u_t|, |ru_r|, |ru_{tt}|, |ru_{tr}|, |r^2u_{rr}|] .$$

The following lemma holds:

Lemma 1.1. *Let the problem $(\mathscr{M}_0)$ be given. Let the assumptions $(\mathscr{A}_1)$ and $(\mathscr{A}_2)$ be fulfilled.*

Then the problem $(\mathscr{M}_0)$ has a unique solution in $\mathfrak{A}$. It is given by

$$(1.10) \qquad u(t, r) = \frac{1}{r}\left[s(r + t) - s(-r + t) + \frac{1}{2}\int_0^t \int_{r-t+\vartheta}^{r+t-\vartheta} \varrho f(\vartheta, \varrho)\,d\varrho\,d\vartheta\right].$$

Proof. It may be easily verified that the function $u(t, r)$ given by (1.10) belongs to $\mathfrak{A}$ and satisfies the equations (1.1), (1.2) and (1.4) in the usual way. The uniqueness of the solution follows readily from the energetic inequality. Indeed, u_1 and u_2 being two solutions of $(\mathscr{M}_0)$ the function $v = u_1 - u_2$ satisfies the relation

$$0 = \int_0^\pi \int_0^t \varrho^2\, v_t(\vartheta, \varrho)\left[v_{tt}(\vartheta, \varrho) - v_{rr}(\vartheta, \varrho) - \frac{2}{\varrho}\, v_r(\vartheta, \varrho)\right] d\vartheta\, d\varrho =$$
$$= \int_0^\pi \tfrac{1}{2}\,\varrho^2[v_t^2(t, \varrho) + v_r^2(t, \varrho)]\,d\varrho .$$

Whence in virtue of $v(0, r) \equiv 0$ the assertion follows.

Now let the problem $(\mathscr{P}_0)$ be given with f satisfying the assumption $(\mathscr{A}_1)$. Then the condition (1.3) is equivalent to the conditions

$$(1.11) \qquad u(2\pi, r) - u(0, r) = 0\,, \quad u_t(2\pi, r) - u_t(0, r) = 0\,, \quad 0 \leqq r \leqq \pi\,.$$

Inserting (1.10) into (1.11), differentiating the first of them with respect to r (the differentiate equation is equivalent to the primitive one since this is satisfied for $r = \pi$) and then adding and subtracting the two equation we obtain (according to (1.7_3))

$$\int_0^{2\pi} (r - \vartheta) f(\vartheta, r - \vartheta)\,d\vartheta = 0\,, \int_0^{2\pi} (r + \vartheta) f(\vartheta, r + \vartheta)\,d\vartheta = 0\,,$$
$$0 \leqq r \leqq \pi\,.$$

Taking again into account (1.7_3) these two conditions may be joined to a single one

$$(1.12) \qquad \int_0^{2\pi} (r - \vartheta) f(\vartheta, r - \vartheta) \, \mathrm{d}\vartheta = 0 \,, \quad 0 \leqq r \leqq 2\pi \,.$$

Thus the following theorem holds.

Theorem 1.1. *Let the problem* $(\mathscr{P}_0)$ *be given, let* f *satisfy the assumption* $(\mathscr{A}_1)$. *Then the problem* $(\mathscr{P}_0)$ *has a solution if and only if* (1.12) *holds. If this condition is satisfied, then the solution* (1.10) *of* $(\mathscr{M}_0)$, *for any* s *satisfying* $(\mathscr{A}_3)$ *is a solution of* $(\mathscr{P}_0)$, *too.*

The necessity of the condition (1.12) may be also found with help of Green's formula. It may be easily found that the problem $(\mathscr{P}_0^*)$ adjoined to $(\mathscr{P}_0)$ reads

$$(1.13) \qquad w_{tt} - w_{rr} + 2 \frac{\partial}{\partial r}\left(\frac{w}{r}\right) = 0 \,;$$

$$(1.14) \qquad |w(t, 0)| < +\infty \,, \quad w(t, \pi) = 0 \,;$$

$$(1.15) \qquad w(t + 2\pi, r) - w(t, r) = 0$$

and its solution is given by

$$(1.16) \qquad w(t, r) = r(\sigma(r + t) - \sigma(-r + t))$$

for any σ satisfying the assumption $(\mathscr{A}_3)$. Then, by the known procedure

$$(1.17) \quad 0 = \int_0^{2\pi} \int_0^{\pi} w(t, r) f(t, r) \, \mathrm{d}r \, \mathrm{d}t = \int_0^{2\pi} \int_0^{\pi} (\sigma(r + t) - \sigma(-r + t)) \, r f(t, r) \, \mathrm{d}r \, \mathrm{d}t$$

for any σ satisfying $(\mathscr{A}_3)$.

Performing similar calculations as in [1] we find that (1.17) has (1.12) as a consequence.

§ 2. WEAKLY NONLINEAR CASE.

Let the problem $(\mathscr{P})$ be given by

$$(2.1) \qquad u_{tt} - u_{rr} - \frac{2}{r} u_r = \varepsilon f(t, r, u, u_t, u_r, \varepsilon)$$

and by (1.2) and (1.3). Analogously let the problem $(\mathscr{M})$ be given by (2.1), (1.2) and (1.4).

Let the following assumption be fulfilled:

($\mathscr{B}_1$) The function $\tilde{f}(t, r, u_0, u_1, u_2, \varepsilon) = f(t, r, u_0, u_1, u_2/r, \varepsilon)$ together with its derivatives $\partial\tilde{f}/\partial r$, $\partial\tilde{f}/\partial u_i$, $\partial^2\tilde{f}/(\partial r\, \partial u_i)$, $\partial^2\tilde{f}/(\partial u_i\, \partial u_j)$ $(i, j = 0, 1, 2)$ is continuous in all its arguments for $0 \leqq t < \infty$, $0 \leqq r \leqq \pi$, $-\infty < u_i < \infty$, $0 \leqq \varepsilon \leqq \varepsilon_0$ $(\varepsilon_0 > 0)$. Besides it is 2π-periodic in t and

$$f(t, \pi, 0, 0, u_2, \varepsilon) = 0\,. \tag{2.2}$$

Let us continue the function f in r onto $(-\infty, +\infty)$ by the relations

$$\begin{aligned} r\, f(t, r, u_0, u_1, u_2, \varepsilon) &= r\, f(t, -r, u_0, u_1, -u_2, \varepsilon) = \\ &= (r + 2\pi)\, f(t, r + 2\pi, u_0, u_1, u_2, \varepsilon)\,. \end{aligned} \tag{2.3}$$

According to (2.2) the function (which we shall denote again by f) continued in this way has the same degree of smoothness as before.

It may be easily verified that every solution $u \in \mathfrak{A}$ of $(\mathscr{M})$ satisfies the integral equation

$$\begin{aligned} P(u)(s)(\varepsilon)(t, r) \equiv -u(t, r) + \frac{1}{r}\Big[s(r + t) - s(-r + t) + \\ + \tfrac{1}{2}\varepsilon \int_0^t \int_{r-t+\vartheta}^{r+t-\vartheta} F(u)(\varepsilon)(\vartheta, \varrho)\, \mathrm{d}\varrho\, \mathrm{d}\vartheta \Big] = 0\,, \end{aligned} \tag{2.4}$$

where s has the same meaning as in § 1 and

$$F(u)(\varepsilon)(t, r) = r\, f(t, r, u(t, r), u_t(t, r), u_r(t, r), \varepsilon)\,. \tag{2.5}$$

On the other hand every solution $u \in \mathfrak{A}$ of (2.4) is a solution of $(\mathscr{M})$. The existence of a solution of (2.4) for $0 \leqq t \leqq T (T > 0)$, $0 \leqq r \leqq \pi$ and ε sufficiently small may be proved with help of the following lemma.

Lemma 2.1. *Let the equation*

$$P(u)(s)(\varepsilon) \equiv -u + L(s) + \varepsilon\, R(u)(\varepsilon) = 0 \tag{2.6}$$

be given, where $P(u)(s)(\varepsilon)$ maps the direct product $\mathfrak{A} \times \mathfrak{S}$ into $\mathfrak{A}$ for every value of the numerical parameter ε from $\mathfrak{E} = \langle 0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$.

Let $L \in [\mathfrak{S} \to \mathfrak{A}]$. Let $R(u)(\varepsilon)$ be continuous in u and ε and have a $\mathfrak{G}$-derivative $R'_u(u)(\varepsilon)$ continuous in u and ε for any $u \in \mathfrak{A}$ and $\varepsilon \in \mathfrak{E}$. Then to every $\tilde{s} \in \mathfrak{S}$ there exist numbers δ and ε^*, $\delta > 0$, $0 < \varepsilon^* \leqq \varepsilon_0$ such that the equation (2.6) has a unique solution $U(s)(\varepsilon) \in \mathfrak{A}$ for each $s \in S(\tilde{s}; \delta)$ and $\varepsilon \in \langle 0, \varepsilon^* \rangle$. This solution has a $\mathfrak{G}$-derivative $U'_s(s)(\varepsilon)$ continuous in s and ε.

(Notation. $[\mathfrak{S} \to \mathfrak{A}]$ is the space of all linear operators mapping $\mathfrak{S}$ into $\mathfrak{A}$; $S(\tilde{s}; \delta)$ is the sphere with the center $\tilde{s}$ and the radius δ. See the theorem 2.1 in [1] where also the proof is indicated.)

For the spaces $\mathfrak{A}$ and $\mathfrak{S}$ of Lemma 2.1 let us choose the spaces $\mathfrak{A}$ and $\mathfrak{S}$ defined in § 1.

Let

$$L(s)(t, r) = \frac{1}{r}\left[s(r + t) - s(-r + t)\right],$$

$$R(u)(\varepsilon)(t, r) = \frac{1}{2r}\int_0^t \int_{r-t+\vartheta}^{r+t-\vartheta} F(u(\vartheta, \varrho))(\varepsilon)(\vartheta, \varrho)\,\mathrm{d}\varrho\,\mathrm{d}\vartheta\,.$$

It may be easily verified that under the conditions on φ, ψ and f as stated above all assumptions of Lemma 2.1 are fulfilled and the following theorem follows.

Theorem 2.1. *Let the problem $(\mathscr{M})$ be given. Let the assumptions $(\mathscr{A}_2)$ and $(\mathscr{B}_1)$ be fulfilled.*

Then a function $\tilde{s} \in \mathfrak{S}$ and a number $T > 0$ being given, there exist numbers $\delta > 0$ and ε^*, $0 < \varepsilon^* \leqq \varepsilon_0$ such that the problem $(\mathscr{M})$ for $0 \leqq \varepsilon \leqq \varepsilon^*$ and for all $s \in S(\tilde{s}; \delta)$ has a unique solution $u^*(\varepsilon)(t, r) = U(s)(\varepsilon)(t, r) \in \mathfrak{A}$. The operator U is together with its $\mathfrak{G}$-derivative $U'_s(s)(\varepsilon)$ continuous in s and ε, while

$$u^*(0)(t, r) = U(s)(0)(t, r) = s(r + t) - s(-r + t)\,.$$

Now let us write down that the solution $U(s)(\varepsilon)(t, r)$ of $(\mathscr{M})$ is a solution of the problem $(\mathscr{P})$ i.e. that it satisfies the conditions (1.11). Inserting $u(t, r) = U(s)(\varepsilon)(t, r)$, into (1.11) making use of the fact that $U(s)(\varepsilon)$ satisfies the equation (2.4) and performing the same arrangements as in § 1 we find that $U(s)(\varepsilon)$ is a solution of the problem $(\mathscr{P})$ if and only if

$$G(s)(\varepsilon)(r) \equiv \int_0^{2\pi} (r - \vartheta)\, F(U(s)(\varepsilon)(\vartheta, r - \vartheta))(\varepsilon)(\vartheta, r - \vartheta)\,\mathrm{d}\vartheta = 0\,, \quad 0 \leqq r \leqq 2\pi\,. \tag{2.7}$$

To bring this condition to a more practical form we make use of the following lemma.

Lemma 2.2. *Let the equation*

$$G(p)(\varepsilon) = 0 \tag{2.8}$$

be given, where $G(p)(\varepsilon)$ maps a B-space $\mathfrak{P}$ into a B-space $\mathfrak{Q}$ for all $\varepsilon \in \mathfrak{E} = \langle 0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$. Let the following assumptions be fulfilled.

(i) *The equation*

$$G(p_0)(0) = 0 \tag{2.9}$$

has a solution $p_0 = p_0^* \in \mathfrak{P}$.

(ii) *The operator $G(p)(\varepsilon)$ is continuous in p and ε and has a $\mathfrak{G}$-derivative $G'_p(p)(\varepsilon)$ continuous in p and ε for $p \in S(p_0^*; \delta)$ ($\delta > 0$ being a suitable chosen number such that $S(p_0^*; \delta) \subset \mathfrak{P}$) and $\varepsilon \in \mathfrak{E}$.*

(iii) *There exists*

$$H = [G'_p(p_0^*)(0)]^{-1} \in [\mathfrak{Q} \to \mathfrak{P}] .$$

Then there exists $\varepsilon^* > 0$ such that the equation (2.8) has for $0 \leqq \varepsilon \leqq \varepsilon^*$ a unique solution $p = p^*(\varepsilon) \in \mathfrak{P}$, continuous in ε such that $p^*(0) = p_0^*$.

(For the proof cf. [1].)

In our case $p = s$ and the equation, (2.9) reads

$$(2.10) \qquad G(s_0)(0)(r) \equiv$$
$$\equiv \int_0^{2\pi} (r - \vartheta) f(\vartheta, r - \vartheta, s_0(r) - s_0(-r + 2\vartheta), s_0'(r) - s_0'(-r + 2\vartheta), s_0'(r) + $$
$$+ s_0'(-r + 2\vartheta), 0) \, d\vartheta = 0 .$$

By our above assumptions we have to choose for the space $\mathfrak{P}$ a subspace $\tilde{\mathfrak{S}}$ of the space $\mathfrak{S}$. The choice of the space $\mathfrak{Q}$ depends upon the form of the function f. If $|\partial f/\partial u_t| + |\partial f/\partial u_r| \neq 0$ it is natural to take for $\mathfrak{Q}$ a subspace $\tilde{\mathfrak{S}}_1$ of the space of 2π-periodic functions of class C^1 with the usual norm which we shall denote $\mathfrak{S}_1$. On the other hand if $f = f(t, x, u, \varepsilon)$ it is more natural to suppose that in (2.7) no loss of smoothness occurs and to take for $\mathfrak{Q}$ a subspace $\tilde{\tilde{\mathfrak{S}}}$ of $\mathfrak{S}$. Further, we find easily that in the first case with regard to the continuity of $U'_s(s)(\varepsilon)$ in s and ε the assumption $(\mathscr{B}_1)$ ensures the existence of the derivative $G'_s(s)(\varepsilon)$ continuous in s and ε. On the contrary to guarantee the existence of $G'_s(s)(\varepsilon)$ continuous in s and ε in the second case the assumption on f must be strengthened as follows:

$(\mathscr{B}_2)$ The function $\tilde{f}(t, r, u, \varepsilon) = r f(t, r, u, \varepsilon)$ together with its partial derivatives $\partial\tilde{f}/\partial r$, $\partial\tilde{f}/\partial u$, $\partial^2\tilde{f}/\partial r^2$, $\partial^2\tilde{f}/(\partial u\, \partial r)$, $\partial^2\tilde{f}/\partial u^2$, $\partial^3\tilde{f}/(\partial r^2\, \partial u)$, $\partial^3\tilde{f}/(\partial r\, \partial u^2)$, $\partial^3\tilde{f}/\partial u^3$ is continuous in all its arguments for $0 \leqq t < \infty$, $0 \leqq r \leqq \pi$, $-\infty < u < +\infty$ $0 \leqq \varepsilon \leqq \varepsilon_0$. Besides it is 2π-periodic in t and $f(t, \pi, 0, \varepsilon) = 0$.

Then the following two theorems may be proved easily.

Theorem 2.2. *Let the problem $(\mathscr{P})$ be given. Let besides the assumption $(\mathscr{B}_1)$ the following assumptions be fulfilled:*

(i) *The equation (2.10) has a solution $s_0 = s_0^*(r) \in \mathfrak{S}$.*

(ii) *There exists the operator*

$$H = [G'_s(s_0^*)(0)]^{-1} \in [\tilde{\mathfrak{S}}_1 \to \tilde{\mathfrak{S}}] ,$$

where $\tilde{\mathfrak{S}}_1 \supset G(\tilde{\mathfrak{S}})(\varepsilon)$.

Then there exists a number $\varepsilon^* > 0$ such that the problem $(\mathscr{P})$ has for any $\varepsilon \in \langle 0, \varepsilon^* \rangle$ a unique solution $U(s^*(\varepsilon))(\varepsilon)(t, r) \in \mathfrak{A}$ such that $s^*(0)(r) = s_0^*(r)$, while the function $s^*(\varepsilon)(r) \in \tilde{\mathfrak{S}}$ is continuous in ε.

Theorem 2.3. *Let the problem* $(\mathscr{P})$ *be given with* $f = f(t, r, u, \varepsilon)$. *Let besides the assumption* $(\mathscr{B}_2)$ *the following two conditions be fulfilled*:

(i) *The equation* (2.10) *has a solution* $s_0 = s_0^*(r) \in \tilde{\mathfrak{S}}$.

(ii) *There exists the operator* $H = [G_s'(s_0^*)(0)]^{-1} \in [\tilde{\tilde{\mathfrak{S}}} \to \tilde{\mathfrak{S}}]$, *where* $\tilde{\tilde{\mathfrak{S}}} \supset G(\tilde{\mathfrak{S}})(\varepsilon)$.

Then there exists $\varepsilon^* > 0$, *such that the problem* $(\mathscr{P})$ *has for any* $\varepsilon \in \langle 0, \varepsilon^* \rangle$ a uni*que solution* $u^*(\varepsilon)(t, r) = U(s^*(\varepsilon))(\varepsilon)(t, r) \in \mathfrak{A}$ *such that* $s^*(0)(r) = s_0^*(r)$, *while the function* $s^*(\varepsilon) \in \tilde{\mathfrak{S}}$ *is continuous in* ε.

References

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Výtah

PERIODICKÁ ŘEŠENÍ SLABĚ NELINEÁRNÍ VLNOVÉ ROVNICE V E_3 V KULOVĚ SYMETRICKÉM PŘÍPADĚ

OTTO VEJVODA

V článku se vyšetřují podmínky existence 2π-periodického řešení v t úlohy (2,1), (1,2) za předpokladu, že funkce f je dostatečně hladká a 2π-periodická v t.

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