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ON THE POLYNOMIAL EIGENVALUE PROBLEM WITH POSITIVE OPERATORS AND LOCATION OF THE SPECTRAL RADIUS

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Introduction. The purpose of this article is to give some estimates for the spectral radius of the polynomial eigenvalue problem, i.e. to derive some estimates for the singularity of the operator-function F,

$$F(\lambda) = \lambda^m A_0 - \sum_{k=1}^{m} \lambda^{m-k} A_k \tag{0.1}$$

with the maximal absolute value. It is assumed that $A_1, \ldots, A_m, A_0^{-1}$ are bounded linear operators mapping a Banach space $\mathscr{Y}$ into itself. Further, it is assumed that the operators B_j, where $B_j = A_0^{-1} A_j$, $j = 1, \ldots, m$, leave a cone $\mathscr{K} \subset \mathscr{Y}$ invariant. This problem was investigated by K. P. HADELER in his paper [1] in the case $\mathscr{Y}$ is either a Euclidean space $\mathscr{E}_l$ or $\mathscr{C}(\Omega)$, the space of continuous functions on a compact set $\Omega \subset \mathscr{E}_l$.

Besides some generalizations of Hadeler's results [1] some of his assertions will be given in a more precise form and further estimates based on Collatz's "Quotienten-satz" (see [1] and [7]) will be derived.

Two methods are used; Müller's method [10], which transfers the initial polynomial eigenvalue problem into an ordinary eigenvalue problem in a Cartesian product and second, a method using the analytic properties of the spectral radius of a positive operator-function depending on a parameter. Using this second method the investigations need not be done in the Cartesian product and so some assumptions on the spectral properties of the operators $A_0, \ldots, A_m$, especially, the usual assumption of compactness of $B_1, \ldots, B_m$, can be weakened.

1. Definitions and notation. Let $\mathscr{Y}$ be a real Banach space with a generating or reproducing and normal cone $\mathscr{K}$ [3], which is assumed to be closed. By $\mathscr{X}$ we denote the complexification $\mathscr{Y} + i\mathscr{Y}$ of the space $\mathscr{Y}$. The dual space to $\mathscr{Y}$ is denoted by $\mathscr{Y}'$ and the space of bounded linear transformations of $\mathscr{Y}$ into $\mathscr{Y}$, and $\mathscr{X}$ into $\mathscr{X}$, will be denoted by $[\mathscr{Y}]$ and $[\mathscr{X}]$ respectively. It is assumed that $\mathscr{Y}'$, $[\mathscr{Y}]$, and $[\mathscr{X}]$ are normed in the usual manner, so that they are Banach spaces.

Assuming that $\mathscr{K}$ is a generating and normal cone we shall denote the dual cone [3] by $\mathscr{K}'$. By definition $\mathscr{K}' = \{y' \in \mathscr{Y}' \mid \langle x, y' \rangle \geqq 0 \text{ if } x \in \mathscr{K}\}$, where $\langle x, y' \rangle$ means $y'(x)$.

A subset $\mathscr{H}' \subset \mathscr{K}'$ is called $\mathscr{K}$-total [7], if and only if $\langle x, x' \rangle \geqq 0$ for all $x' \in \mathscr{H}'$ implies $x \in \mathscr{K}$.

Evidently, $T \in [\mathscr{Y}]$ implies $\tilde{T} \in [\mathscr{X}]$, where $\tilde{T}$ is the complex extension of T, i.e. $\tilde{T}z = Tx + iTy$, for $z = x + iy$, $x, y \in \mathscr{Y}$. We define $\sigma(T) = \sigma(\tilde{T})$ if $T \in [\mathscr{Y}]$ and similarly $r(T) = r(\tilde{T})$, where $\sigma(A)$ is the spectrum and $r(A)$ the spectral radius of the operator $A \in [\mathscr{X}]$.

An operator $T \in [\mathscr{Y}]$ is called positive, more precisely $\mathscr{K}$-positive [3], if and only if $T\mathscr{K} \subset \mathscr{K}$. A $\mathscr{K}$-positive operator T is termed semi-non-supporting [11], if and only if for every pair $x \in \mathscr{K}$, $x \neq o$, $x' \in \mathscr{K}'$, $x' \neq o$ (where o denotes the zero element in both spaces $\mathscr{Y}$ and $\mathscr{Y}'$) there is a positive integer $p = p(x, x')$ such that $\langle T^p x, x' \rangle \neq 0$. A $\mathscr{K}$-positive operator T is called u_0-positive [5, p. 60], if and only if there exists an element $u_0 \in \mathscr{K}$, $\|u_0\| = 1$, such that for every $x \in \mathscr{K}$, $x \neq o$, there are positive numbers $\alpha = \alpha(x)$ and $\beta = \beta(x)$ and a positive integer $p = p(x)$ such that the relations

$$\alpha u_0 < T^p x < \beta u_0$$

are valid, where $u < v$ or $v > u$ means $(v - u) \in \mathscr{K}$.

Remark. It can be shown that a u_0-positive operator T can be treated as a semi-non-supporting operator with respect to another cone $\mathscr{K}_1 = T\mathscr{K}$ in the space Y_1, the linear hull of the manifold $\mathscr{K}_1 - \mathscr{K}_1$. Nevertheless with respect to direct applications it will be meaningful to formulate separately assertions in two versions, for semi-non-supporting and for u_0-positive operators.

A vector $\hat{x} \in \mathscr{K}$ is called quasi-interior [13] or non-supporting [11], if and only if $\langle \hat{x}, x' \rangle \neq 0$ for arbitrary $x' \in \mathscr{K}'$, $x' \neq o$. A vector $\tilde{x} \in \mathscr{Y}$ is called u_0-positive, if and only if there is a positive number τ such that $\tau u_0 < \tilde{x}$. A linear form $x' \in \mathscr{K}'$ is called strictly positive, if and only if $\langle x, x' \rangle \neq 0$ for every $x \in \mathscr{K}$, $x \neq o$ [3].

Let sup and inf exist for every pair of elements in a partially ordered space $\mathscr{Y}$, where the order is given by using a generating and normal cone $\mathscr{K}$. Then and only then $\mathscr{Y}$ is called a Riesz space.

An operator $A \in [\mathscr{X}]$ or $A \in [\mathscr{Y}]$ has property (S), if and only if from the relations $\lambda \in \sigma(A)$, $|\lambda| = r(A)$, it follows that λ is a pole of the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ or $R(\lambda, A) = (\lambda I - \tilde{A})^{-1}$ respectively (I denotes the identity operator).

Let us assume that $A_j \in [\mathscr{X}]$, $j = 0, 1, \ldots$ and let the function G be defined by $G(\lambda) = \sum_{k=0}^{\infty} \lambda^k A_k$ for λ in a complex domain. The value λ will be termed a singularity of the operator-function G, if and only if the operator inverse to $G(\lambda)$ either does not exist or does not belong to $[\mathscr{X}]$. The set of all singularities of the operator-function G

will be denoted by $\Sigma(G)$. The value $\tilde{r}(G) = \sup_{\lambda \in \Sigma(G)} |\lambda|$ is called the spectral radius of G. In case G is a polynomial in λ, $\tilde{r}(G)$ is also called the spectral radius of the corresponding polynomial eigenvalue problem $G(\lambda)\, u = o$.

We shall use the following notation. Suppose that at least one of two following assumptions

(a) operator $T \in [\mathscr{Y}]$ is semi-non-supporting,

(b) operator $T \in [\mathscr{Y}]$ is u_0-positive,

is satisfied. Then we shall shortly say that the operator T fulfills the condition (a, b) or the assumption (a, b).

Suppose $T \in [\mathscr{Y}]$ and $x \in \mathscr{K}$, $x \neq o$. Then we define the functionals

$$r_x(T) = \sup_{(Tx - \nu x) \in \mathscr{K}} \nu \,, \quad r^x(T) = \inf_{(\mu x - Tx) \in \mathscr{K}} \mu \,.$$

Evidently, if $\mathscr{H}' \subset \mathscr{K}'$ is a $\mathscr{K}$-total set in $\mathscr{Y}'$, then

$$r_x(T) = \inf_{\substack{x' \in \mathscr{H}' \\ \varkappa(x')\langle x, x'\rangle \neq 0}} \frac{\langle Tx, x'\rangle}{\langle x, x'\rangle}\,, \quad r^x(T) = \sup_{\substack{x' \in \mathscr{H}' \\ \varkappa(x') \neq 0}} \frac{\langle Tx, x'\rangle}{\langle x, x'\rangle}\,,$$

where $\varkappa(x')$ is the value of a certain functional on $\mathscr{K}'$ the form of which will be given whenever necessary.

Let $\mathscr{H}'$ be a $\mathscr{K}$-total set in $\mathscr{Y}'$ and $\hat{x} \in \mathscr{K}$ a quasi-interior element. Then $\hat{\mathscr{H}}'$, where

$$\hat{\mathscr{H}}' = \left\{ y' = \frac{x'}{\langle \hat{x}\, x'\rangle} \mid x' \in \mathscr{H}', \langle \hat{x}, x'\rangle \neq 0 \right\}$$

is also a $\mathscr{K}$-total set.

Further, we can define the following functionals

$$s(T) = \inf_{y' \in \hat{\mathscr{H}}'} \langle T\hat{x}, y'\rangle \,, \quad S(T) = \sup_{y' \in \hat{\mathscr{H}}'} \langle T\hat{x}, y'\rangle \,.$$

In case $\mathscr{K}$ is a reproducing and normal cone in $\mathscr{Y}$, the set $\mathbf{K} = \mathscr{K} \times \ldots \times \mathscr{K}$ is also a generating and normal cone in $\mathbf{Z} = \mathscr{Y} \times \ldots \times \mathscr{Y} = \mathscr{Y}^m$ where the elements of the space $\mathbf{Z}$ are written as $\mathbf{z} = (y_1, \ldots, y_m)$, $y_j \in \mathscr{Y}$. The cone adjoint to $\mathbf{K}$ will be denoted by $\mathbf{K}'$.

Clearly,

$$\mathbf{H}' = \{\mathbf{x}'_j = (\underbrace{o, \ldots, x'}_{j}, o, \ldots, o) \mid x' \in \mathscr{H}', j = 1, \ldots, m\}$$

is a $\mathbf{K}$-total set in $\mathbf{Z}'$, if $\mathscr{H}'$ is a $\mathscr{K}$-total set in $\mathscr{Y}'$. Similarly one can define another $\mathbf{K}$-total set in $\mathbf{Z}'$ as follows

$$\hat{\mathbf{H}}' = \left\{ \mathbf{y}'_j = \frac{\mathbf{x}'_j}{\langle \hat{x}, x'_j\rangle} \mid x'_j \in \mathscr{H}', \langle \hat{x}, x'_j\rangle \neq 0, j = 1, \ldots, m \right\}$$

where $\hat{x} \in \mathscr{K}$ is a quasi-interior element.

Remark. The definition of the functionals r_x and r^x in which the existence of any $\mathscr{K}$-total set does not occur is of course more simple; however, in some assertions it is more suitable and useful to use the definition in which this concept does explicitly occur because it is more illustrating.

Linear bounded transformations in the space $\mathscr{Y}^m$ will be denoted by bold face italics. In particular, for $B_1, \ldots, B_m \in [\mathscr{Y}]$ we shall write

$$\mathbf{C} = \begin{bmatrix} \Theta & I & \Theta & \ldots & \Theta \\ \Theta & \Theta & I & \ldots & \Theta \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ B_m & B_{m-1} & B_{m-2} & \ldots & B_1 \end{bmatrix},$$

where Θ is the zero-operator and I the identity operator in $\mathscr{Y}$.

2. Existence of positive proper elements. In the following text we shall assume that $\mathscr{K}$ is a normal and generating cone, closed in the norm topology of $\mathscr{Y}$.

The purpose of the following theorem is to guarantee the existence of a positive proper vector x of the polynomial eigenvalue problem

$$(2.1) \qquad \lambda^m A_0 x = \sum_{k=1}^{m} \lambda^{m-k} A_k x$$

assuming the operators $A_0, A_1, \ldots, A_m$ are such that $B_j = A_0^{-1} A_j$, A_0^{-1}, $A_j \in [\mathscr{Y}]$, $j = 1, \ldots, m$, leave the cone $\mathscr{K}$ invariant and that the operator

$$P(\lambda) = \sum_{k=1}^{m} \lambda^{m-k} B_k$$

has property (S) for positive λ's. This assumption is weaker than the normally required compactness of all the operators $B_1, \ldots, B_m$ or of all of them but one which is supposed to have a compact iterate [10], [1]. In this connection special attention should be paid to the case when the operators $B_1, \ldots, B_m$ are Radon-Nikolski operators [6].

Theorem 2.1. *Suppose the operator* $P(\lambda) = \sum_{k=1}^{m} \lambda^{m-k} B_k$, *where* $B_k \in [\mathscr{Y}]$, $B_k \mathscr{K} \subset \mathscr{K}$, *has property* (S) *for* $\lambda \geqq 0$, *the operator* B_m *fulfills condition* (a, b) *and moreover* $B_j u_0 \neq o$ *for at least one index* $j \neq m$ *in case* (b).

Then there exists precisely one eigenvalue $\hat{\lambda} = r(P(\hat{\lambda}))$ *and precisely one proper vector* $x_0 \in \mathscr{K}$, $\|x_0\| = 1$, *of the equation* (2.1). *Furthermore, the relations* $P(\lambda)\, y = \lambda^m y$, $y \in \mathscr{K}$, *for some* λ, *imply* $y = c x_0$, *where* $c \geqq 0$ *and* $\lambda = \hat{\lambda}$. *The vector* x_0 *is a quasi-interior element of the cone* $\mathscr{K}$ *in case* (a) *and a* u_0*-positive element in case* (b).

Proof. Put $p(\lambda) = \lambda^m$, $t(\lambda) = r(P(\lambda))$. Then $p(0) = 0$ and $t(0) = r(B_m)$. It is well known that $r(B_m) > 0$ [11], [2], [5]. Clearly, $\sum_{k=1}^{m} B_k > B_s$ for $1 \leqq s \leqq m$, where

$C < D$ or $D > C$ means $(D - C)\mathscr{K} \subset \mathscr{K}$. Evidently one has $m\lambda^{m-1} \sum_{k=1}^{m} B_k > P(\lambda)$ if $\lambda \geqq 1$. Furthermore, there exists a $\Lambda > 0$ such that

$$P(\lambda) = \lambda^m > m\lambda^{m-1} r(\sum_{k=1}^{m} B_k) \geqq r(P(\lambda)) = t(\lambda)$$

for $\lambda \geqq \Lambda$.

The existence of at least one eigenvalue $\hat{\lambda} > 0$ and at least one proper vector $x_0 \in \mathscr{K}$ as described in Theorem **2.1** follows from the continuity of the functions p and t in $(0, +\infty)$ and from the assumption (a, b) [11] and [7]. The uniqueness is a corollary of the following lemma.

Lemma. *Under the assumptions of Theorem* **2.1** *the function* $t = t(\lambda)$ *is strictly increasing in* $(0, +\infty)$.

Proof of the lemma. Let $0 < \lambda' < \lambda < +\infty$. Clearly $P(\lambda') < P(\lambda)$, $P(\lambda') \neq P(\lambda)$ and moreover $[P(\lambda) - P(\lambda')] u_0 \neq o$ in case (b). As a consequence of these relations we get

$$r(P(\lambda')) < r(P(\lambda)) . \tag{2.2}$$

In case (b), this inequality is proved in [9]. The proof of this inequality in case (a) is the same. Thus the lemma is proved.

We shall prove the inequality (2.2) without the explicit use of any $\mathscr{K}$-total set. For this purpose we shall formulate and prove two autonomous theorems.

Theorem 2.2. *Let* $T \in [\mathscr{Y}]$ *be a positive transformation having property* (S) *and* $r(T) > 0$. *Then*

$$r(T) = \min_{\substack{x \in \mathscr{K} \\ P_q x \neq 0}} r^x(T) = \max_{\substack{x \in \mathscr{K} \\ P_q x \neq 0}} r_x(T) ,$$

where q *is the multiplicity of the pole* $\lambda_0 = r(T)$ *of the resolvent* $R(\lambda, T) = (\lambda I - T)^{-1}$ and [6]

$$P_q = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k^{-q+1} \lambda_0^{-k} T^k .$$

Proof. Let $\lambda_1, \ldots, \lambda_s$ be the part of $\sigma(T)$ lying on the circle $|\lambda| = r(T)$ and let $q_1, \ldots, q_s$ be corresponding multiplicities of $\lambda_1, \ldots, \lambda_s$ as poles of $R(\lambda, T)$. It is well known [12] that $\lambda_1 = r(T)$ and $q_j \leqq q_1 = q$ for $j = 1, \ldots, s$. From this fact it follows [6] that in the norm of $[\mathscr{X}]$

$$S^q_{(n)} = \frac{1}{n} \sum_{k=1}^{n} k^{-q+1} \lambda_1^{-k} T^k \to \frac{\lambda_1^{-q+1}}{(q-1)!} P_q ,$$

where P_q is the leading member of the principal part of the Laurent expansion

$$R(\lambda, T) = \sum_{k=0}^{\infty} A_k(\lambda - \lambda_1)^k + \sum_{k=1}^{q} (\lambda - \lambda_1)^{-k} P_k$$

and where $A_k \in [\mathscr{X}]$ and

$$P_1 = \frac{1}{2\pi i} \int_{C_0} R(\lambda, T)\, d\lambda\,, \quad P_{k+1} = (T - \lambda_1 I)\, P_k\,, \quad k = 1, 2, \ldots$$

with $C_0 = \{\lambda \mid |\lambda - \lambda_1| = \varrho_0\}$, $K_0 = \{\lambda \mid |\lambda - \lambda_1| \leqq \varrho_0\}$, $K_0 \cap \sigma(T) = \{\lambda_1\}$. Specifically, if T is positive, $r(T) > 0$ and T has property (S), then $\lambda_1 = r(T)$ has the required properties, so that the operator

$$P_q = \frac{(q - 1)!}{[r(T)]^{1-q}} \lim_{n \to \infty} S_n^{(q)}$$

is positive.

Furthermore, we evidently have

$$r_x(T) \leqq r_{Bx}(T)\,, \quad r^{Bx}(T) \leqq r^x(T)$$

for every positive operator B commuting with T. Indeed, if $B\mathscr{K} \subset \mathscr{K}$, then from

$$r_x(T)\, x < Tx\,, \quad r^x(T)\, x > Tx$$

it follows that

$$r_x(T)\, Bx < BTx = TBx\,, \quad r^x(T)\, Bx > TBx$$

and the required inequalities follow. Thus we have

$$r_{S_n^{(q)}x}(T) \to r_{P_q x}(T) = r(T)\,, \quad r^{S_n^{(q)}x}(T) \to r_{P_q x}(T) = r(T)$$

for every x for which $P_q x \neq o$. Theorem **2.2** is thus proved.

Remark. With small changes in the proof the statement of theorem **2.2** remains valid also without the assumption that $r(T) > 0$.

Theorem 2.3. *Let T and T_1, both in $[\mathscr{Y}]$, have property* (S) *and fulfill the assumption* (a, b); *let $T_1 > T$, $T_1 \neq T$ in case* (a) *and* $(T_1 - T)\, u_0 \neq 0$ *in case* (b). *Then $r(T_1) > r(T)$.*

Proof. Clearly $r(T) \leqq r(T_1)$. Suppose $x \in \mathscr{K}$, $x \neq o$. Then we have [7] $P_1 x \neq o$, where $P_1 = \lim_{n \to \infty} S_n^{(1)}$, and thus $P_1 x = x_0$ is an eigenvector of the operator T. Assume $r(T_1) = r(T)$. Then $T_1 x_0 = T x_0 + (T_1 - T)\, x_0 = r(T)\, x_0 + (T_1 - T)\, x_0$. Further let x_0' be a linear form, for which $x_0' = [r(T)]^{-1}\, T' x_0'$, $x_0' \in \mathscr{K}'$, $x_0' \neq o$. It is known [7] that x_0' is strictly positive. We have

$$\langle T_1 x_0, x_0' \rangle = r(T) \langle x_0, x_0' \rangle + \langle (T_1 - T)\, x_0, x_0' \rangle$$

and hence

$$\frac{\langle T_1 x_0, x_0' \rangle}{\langle x_0, x_0' \rangle} = r(T) + \frac{\langle (T_1 - T)\, x_0, x_0' \rangle}{\langle x_0, x_0' \rangle}\,.$$

That means

$$r_{x_0}(T_1) > r(T)$$

which contradicts the assumption $r(T_1) = r(T)$. Theorem **2.3** is thus proved.

Now, the validity of (2.2) follows directly from Theorem **2.3**. To finish the proof of Theorem **2.1** it remains to verify the uniqueness of the eigenvector $x_0 \in \mathscr{K}$, $\|x_0\| = 1$. This is a consequence of the assumption (a, b), because all possible positive eigenvectors of $P(\lambda)$ must correspond to the eigenvalue $r(P(\lambda))$. But, there is only one possibility $\lambda = \hat{\lambda}$ as shown above and the required result follows from the simplicity of $r(P(\hat{\lambda}))$. This completes the proof of Theorem **2.1**.

Theorem 2.4. *Suppose the operators $B_1, \ldots, B_m \in [\mathscr{Y}]$ are positive. Then the spectral radius $\tilde{r}(F)$ of the polynomial problem $F(\lambda)\, x = o$ is in the spectrum $\Sigma(F)$.*

Proof. As usual we shall associate the operator **C** defined at the end of Section **1** with the problem $F(\lambda)\, x = o$. Evidently, $\mathbf{C} \in [\mathscr{Y}^m]$ and $\lambda \in \Sigma(F)$ if and only if $\lambda \in \sigma(\mathbf{C})$. It follows that $\tilde{r}(F) = r(C)$. Since $\mathbf{CK} \subset K$, we have $r(\mathbf{C}) \in \sigma(\mathbf{C})$ by Schaefer's theorem [14].

Remark. It cannot be asserted that the inequalities

$$\frac{1}{\lambda^{m-1}}\, r_x(P(\lambda)) \leqq \hat{\lambda} \leqq \frac{1}{\lambda^{m-1}}\, r^x(P(\lambda))\,, \tag{2.3}$$

where $\lambda > 0$, $x \in \mathscr{K}$, $x \neq o$, are true in general.

To show this, let us suppose that the operator-function $P = P(\lambda)$ fulfills the assumptions of Theorem **2.1**, where we take the vector $x = x(\lambda)$ for which

$$P(\lambda)\, x(\lambda) = r(P(\lambda))\, x(\lambda)\,, \quad x(\lambda) \in \mathscr{K}\,, \quad x(\lambda) \neq o\,,$$

according to the assumptions (S) and (a, b) of the theorem [7, 9, 11]. Then we have

$$r_{x(\lambda)}(P(\lambda)) = r^{x(\lambda)}(P(\lambda)) = r(P(\lambda))\,.$$

Moreover let $r(B_m) > \tau > 1$, where τ is sufficiently large and $r(B_1) < \nu < 1$ with ν sufficiently small. Then for arbitrary $\lambda > 0$ the inequalities

$$r(P(\lambda)) \geqq r(B_m) > \tau > 1$$

are true and thus $\hat{\lambda} > 1$.

On the other hand, for $\lambda > \tau > 1$,

$$\frac{r(P(\lambda))}{\lambda^{m-1}} = r\left(\sum_{k=1}^{m} \lambda^{1-k} B_k\right) \leqq r(B_1) + \chi(\lambda),$$

where $0 \leqq \chi(\lambda)$ and $\lim_{\lambda \to +\infty} \chi(\lambda) = 0$. It is easy to see that the estimate

$$\frac{r(P(\hat{\lambda}))}{\hat{\lambda}^{m-1}} = \hat{\lambda} \leqq \frac{1}{\lambda^{m-1}} r^{x(\lambda)}(P(\lambda))$$

cannot be true, because in this case we would have

$$1 < \hat{\lambda} \leqq \frac{r(P(\lambda))}{\lambda^{m-1}} < r(B_1) + \chi(\lambda) < 1$$

for sufficiently large λ. Similarly for the estimate from below.

The estimates (2.3) are given by the author of [1] for the case of finite square matrices and mappings in the space $\mathscr{C}(\Omega)$ of continuous functions on a compact set Ω of a Euclidean space $\mathscr{E}_m$. His assertions in this direction are not precise. The aim of the next section is to derive some estimates for the spectral radius of the polynomial eigenvalue problem $F(\lambda)\, x = o$ in general real Banach spaces.

3. Location of the spectral radius of the polynomial eigenvalue problem. As in the preceding section we shall assume that $\mathscr{K}$ is a normal generating and closed cone in a Banach space $\mathscr{Y}$. Furthermore, let $\mathscr{K}'$ be a $\mathscr{K}$-total set in $\mathscr{Y}'$. Let $\hat{x}_1, \ldots, \hat{x}_m$ be elements of $\mathscr{K}$. Then we define

$$\hat{x}_{j+1} = \sum_{k=1}^{m} B_{m-k+1} \hat{x}_{j-m+k}, \quad j \geqq m, \tag{3.1}$$

where $B_k \in [\mathscr{Y}]$, $B_k \mathscr{K} \subset \mathscr{K}$, $k = 1, \ldots, m$.

Theorem 3.1. *Suppose that the interior* int $\mathscr{K}$ *of the cone* $\mathscr{K}$ *is non-void and that the space* $\mathscr{Y}$ *is a Riesz space relative to the ordering given by the cone* $\mathscr{K}$. *Let* $\hat{x}_1, \ldots, \hat{x}_m$ *be in* $\mathscr{K}$. *Furthermore, let* $A_0^{-1}, A_1, \ldots, A_m \in [\mathscr{Y}]$, $B_j = A_0^{-1} A_j$, $B_j \mathscr{K} \subset \subset \mathscr{K}$, $j = 1, \ldots, m$. Finally, let $\varkappa(x') = 1$. Then

$$\begin{aligned} \min_{j=1,\ldots,m} \left[1, \inf_{\substack{x' \in \mathscr{K}' \\ \langle \hat{x}_j, x' \rangle \neq 0}} \frac{\langle \hat{x}_{j+1}, x' \rangle}{\langle \hat{x}_j, x' \rangle}\right] &\leqq \ldots \leqq r_{\mathbf{C}^p \hat{\mathbf{x}}}(\mathbf{C}) \leqq \\ &\leqq \ldots \leqq r(\mathbf{C}) = \tilde{r}(F) \leqq \ldots \leqq r^{\mathbf{C}^p \hat{x}}(\mathbf{C}) \leqq \ldots \leqq \\ &\leqq \max_{j=1,\ldots,m} \left[1, \sup_{\substack{x' \in \mathscr{K}' \\ \langle \hat{x}_j, x' \rangle \neq 0}} \frac{\langle \hat{x}_{j+1}, x' \rangle}{\langle \hat{x}_j, x' \rangle}\right], \end{aligned} \tag{3.2}$$

where $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_m)$ *and the operator* $\mathbf{C}$ *is as defined at the end of Section* **1**.

Proof. The assumption that the space $\mathscr{Y}$ is a Riesz space guarantees that $\mathscr{Y}^m$ is also a Riesz space relative to the ordering given by the cone $\mathbf{K} = \mathscr{K}^m$. According to the relation $\tilde{r}(F) = r(\mathbf{C})$ it is sufficient to estimate $r(\mathbf{C})$. To this end one may apply Theorem 4.2 in [7]. According to this theorem, we have

$$r_{\hat{\mathbf{x}}}(C) \leqq \ldots \leqq r_{\mathbf{C}^p\hat{\mathbf{x}}}(\mathbf{C}) \leqq \ldots \leqq r(\mathbf{C}) \leqq \ldots \leqq r^{\mathbf{C}^p\hat{\mathbf{x}}}(\mathbf{C}) \leqq \ldots \leqq r^{\hat{\mathbf{x}}}(\mathbf{C}).$$

The proof will be complete if we notice that

$$r_{\hat{\mathbf{x}}}(\mathbf{C}) = \min_{j=1,\ldots,m} \left[1, \inf_{\substack{x' \in \mathscr{K}' \\ \langle \hat{x}_j, x' \rangle \neq 0}} \frac{\langle \hat{x}_{j+1}, x' \rangle}{\langle \hat{x}_j, x' \rangle}\right]$$

and

$$r^{\hat{\mathbf{x}}}(\mathbf{C}) = \max_{j=1,\ldots,m} \left[1, \sup_{\substack{x' \in \mathscr{K}' \\ \langle \hat{x}_j, x' \rangle \neq 0}} \frac{\langle \hat{x}_{j+1}, x' \rangle}{\langle \hat{x}_j, x' \rangle}\right].$$

Let us note that the assumptions of Theorem **3.1** which guarantee the validity of bilateral bounds for the spectral radius $\tilde{r}(F)$ concern mainly the properties of the space $\mathscr{Y}$, its generating cone $\mathscr{K}$, and that the only restriction on the operators is their positivity. The converse situation is described in the next Theorems **3.2** and **3.3**. In the following, these two approaches will be treated concurrently.

Theorem 3.2. *Suppose that* $B_1, \ldots, B_m \in [\mathscr{Y}]$, *where* $B_j = A_0^{-1} A_j$, $A_0^{-1}, A_1, \ldots, \ldots, A_m \in [\mathscr{Y}]$, *are positive and compact transformations, and that* B_m *is a semi-non-supporting operator. Let* $\hat{x}_1, \ldots, \hat{x}_m$ *be positive elements, at least one of which is a non-zero-element. For* $j \geqq m$ *let* $\hat{x}_{j+1}$ *be defined by* (3.1).

Then the estimates (3.2) *hold with* $\varkappa(x') = 1$. *If, moreover, in the spectrum of* B_m *there is a dominant point, i.e. if the relations* $\lambda \in \sigma(B_m)$, $\lambda \neq r(B_m)$, *imply* $|\lambda| < r(B_m)$, *then we have*

$$\lim_{p \to \infty} r_{\mathbf{C}^p\hat{\mathbf{x}}}(\mathbf{C}) = \lim_{p \to \infty} r^{\mathbf{C}^p\hat{\mathbf{x}}}(\mathbf{C}) = r(\mathbf{C}) = \tilde{r}(F). \tag{3.3}$$

Proof. We shall determine bounds for the spectral radius $r(C)$ of the operator $\mathbf{C}$ associated with our polynomial eigenvalue problem. We shall use Theorem **4.1** and the main theorem of [7], according to which the relations (3.2) and (3.3) are valid. It remains to show that the assumptions of these theorems are fulfilled. According to the assumptions of Theorem **3.2**, the operator $\mathbf{C}$ is compact and hence it has property (S). It is also semi-non-supporting according to the same property of B_m. Theorem **3.2** is proved.

Remark. It is known [11] that a semi-non-supporting operator with property (S) has a dominant point in its spectrum if it is non-supporting [11]. It is also known [7] that a strongly positive operator [3] is semi-non-supporting, and if it has property (S), then there is a dominant point in its spectrum. Similarly, an absolutely $\mathscr{K}$-positive operator [8] is semi-non-supporting and if it has property (S) then there is a dominant point in its spectrum.

Theorem 3.3. *Let the operators* $B_1, \ldots, B_m \in [\mathscr{Y}]$ *be defined as in Theorem* **3.2** *and let* $B_1, \ldots, B_m$ *be compact. Moreover, let* B_m *be* u_0*-positive, where* $u_0 \in \mathscr{K}$, $\|u_0\| = 1$, *and let there exist positive constants* $\tau_1, \ldots, \tau_m$ *such that* $B_j u_0 > \tau_j u_0$, $j = 1, \ldots, m$. *Furthermore, for every* $v > u_0$ *let there exist a positive* $\tau = \tau(v)$ *such that* $\tau u_0 > v$. *Finally, let* $\varkappa(x') = \langle u_0, x' \rangle$ *for* $x' \in \mathscr{H}'$. *Then the estimates* (3.2) *and the relations* (3.3) *hold.*

Proof. Theorem **3.3** can be proved in the same way as Theorem **3.2**, on using the same theorems from [7] using $\hat{\mathbf{u}}_0$-positiveness of the operator $\mathbf{C}$ associated to the polynomial eigenvalue problem $F(\lambda)\, x = o$, where $\hat{\mathbf{u}}_0 = (u_0, \ldots, u_0)$ and the fact that $\mathbf{C}$ has property (S).

In the previous theorems the derivation of bilateral bounds for the spectral radius $\tilde{r}(F)$ was based on the so-called "quotient principle" ("Quotientensatz" [1], [7]). In the following theorem we shall apply another principle, by T. Yamamoto, for obtaining bilateral bounds for the spectral radius of a non-negative irreducible matrix [14]. The infinite dimensional analogue of this is contained in [4], the results of which will now be used.

Let $\mathscr{H}'$ be a $\mathscr{K}$-total set, and $\hat{x}_1, \ldots, \hat{x}_m$ quasi-interior elements of the cone $\mathscr{K}$. Put

$$\hat{\mathscr{H}}'_j = \left\{ y'_j = \frac{x'}{\langle \hat{x}_j, x' \rangle} \mid x' \in \mathscr{H}', \langle \hat{x}_j, x' \rangle \neq 0 \right\},$$

$j = 1, \ldots, m$.

It is clear that the functionals s and S defined in Section **1** can be expressed as

$$s(\mathbf{C}) = \min_{j=2,\ldots,m+1} \left[\inf_{y' \in \hat{\mathscr{H}}'_{j-1}} \langle \hat{x}_j, y' \rangle \right]$$

and

$$S(\mathbf{C}) = \max_{j=2,\ldots,m+1} \left[\sup_{y' \in \hat{\mathscr{H}}'_{j-1}} \langle \hat{x}_j, y' \rangle \right],$$

where

$$\hat{x}_{m+1} = \sum_{k=1}^{m} B_k \hat{x}_{m-k+1}$$

and $\mathbf{C}$ is the operator associated to the polynomial eigenvalue problem $F(\lambda)\, x = o$.

Theorem 3.4. *Suppose that space* $\mathscr{Y}$, *cone* $\mathscr{K}$, *and the operators* $B_1, \ldots, B_m \in [\mathscr{Y}]$ *fulfill the conditions of Theorem* **3.1.** *Then*

$$(3.4) \quad s(\mathbf{C}) \leqq \ldots \leqq [s(\mathbf{C}^{2^p})]^{2^{-p}} \leqq \ldots \leqq \tilde{r}(F) \leqq \ldots \leqq [S(\mathbf{C}^{2^p})]^{2^{-p}} \leqq \ldots \leqq S(\mathbf{C}).$$

Furthermore, if the operators $B_1, \ldots, B_m$ *fulfill the conditions of either Theorem* **3.2** *or Theorem* **3.3**, *then*

$$\lim_{p\to\infty} [s(\mathbf{C}^{2^p})]^{2^{-p}} = \lim_{p\to\infty} [S(\mathbf{C}^{2^p})]^{2^{-p}} = r(\mathbf{C}) = \tilde{r}(F). \tag{3.5}$$

Proof. This follows directly from Theorem *2* of [4].

Up to this point we have explicitly used the fact that our polynomial eigenvalue problem $F(\lambda)\,x = o$ is equivalent with the usual eigenvalue problem $\mathbf{C}\mathbf{x} = \lambda\mathbf{x}$ with the operator $\mathbf{C}$ in the Cartesian product $\mathscr{Y}^m$. This made it possible to exploit the greater freedom in the Cartesian product to choose the initial elements, the number of which might equal the number of the components of $\mathscr{Y}^m$, i.e. m. On the other hand we were forced to restrict the admissible class of operators $B_j = A_0^{-1}A_j$, $j = 1, \ldots, m$, by assuming their compactness, because of the presence of the identity operators in the matrix of $\mathbf{C}$. In the following we shall try to avoid the compactness assumptions, paying for this by having less freedom in the choice of the initial elements.

Let $y \in \mathscr{K}$, $y \neq o$, and let ξ be an arbitrary positive number. In accordance with K. P. HADELER [1], set

$$P_y(\xi) = \sum_{k=1}^{m} \xi^{m-k} B_k y\,.$$

Theorem 3.5. *Under the assumptions of Theorem* **3.1** *we have*

$$\min\left[\xi, \frac{1}{\xi^{m-1}}\, r_y(P(\xi))\right] \leqq \tilde{r}(F) \leqq \max\left[\xi, \frac{1}{\xi^{m-1}}\, r^y(P(\xi))\right], \tag{3.6}$$

where $\varkappa(x') = 1$ *in the formulas for* r_y *and* r^y.

Proof. Theorem **3.5** is a consequence of Theorem **3.1** in the special case $\hat{x}_1 = \ldots = \hat{x}_m = y$. Thus Theorem **3.5** is a special case of Theorem **3.1.**

The improvement can be obtained, if we assume that the operator B_m fulfills the assumption (a, b). Then the compactness of $B_1, \ldots, B_m$ can be omitted in Theorems **3.2** and **3.3**.

Theorem 3.6. *Suppose that the assumptions of Theorem* **2.1** *are fulfilled, where* $\varkappa(x') = 1$ *in case* (a) *and* $\varkappa(x') = \langle u_0, x' \rangle$, $x' \in \mathscr{K}'$, *in case* (b). *Let* ξ *be any positive number. Then*

$$\max_{\substack{x\in\mathscr{K}\\ x\neq o}} r_x(P(\xi)) = \min_{\substack{x\in\mathscr{K}\\ x\neq o}} r^x(P(\xi)) = r(P(\xi))\,, \tag{3.7}$$

where

$$r_z(P(\xi)) = r^z(P(\xi)) = r(P(\xi)) \tag{3.8}$$

if and only if

$$z = x_0(\xi) = \frac{1}{r(P(\xi))} P(\xi)\, x_0(\xi)\,,$$

$x_0(\xi) \in \mathscr{K}$, $x_0(\xi) \neq o$.

Furthermore, for every $\xi > 0$ *we have*

(3.9) $$\min\left[\xi, \frac{1}{\xi^{m-1}} r_y(P(\xi))\right] \leqq \tilde{r}(F) = \hat{\lambda} \leqq \max\left[\xi, \frac{1}{\xi^{m-1}} r^y(P(\xi))\right],$$

where $\hat{\lambda}$ *is such that* $\hat{\lambda}^m = r(P(\hat{\lambda}))$ (*see Theorem* **2.1**).

Proof. The validity of (3.7) and the assertion concerning the equality (3.8) is guaranteed by the main theorem of [7], the assumptions of which are fulfilled. It remains to prove (3.9).

Let $\xi > \hat{\lambda}$. Then

$$\frac{1}{\xi^{m-1}} r_y(P(\xi)) = r_y\Big(\sum_{k=1}^{m} \xi^{1-k} B_k\Big) \leqq r_y\Big(\sum_{k=1}^{m} \hat{\lambda}^{1-k} B_k\Big) = \frac{1}{\hat{\lambda}^{m-1}} r_y(P(\hat{\lambda})) \leqq \hat{\lambda}\,.$$

The remaining part of (3.9) can be proved similarly. Theorem **3.6** is thus proved.

Remark. Specifically one has the external bounds in (3.2) if $\hat{x}_1 = \ldots = \hat{x}_m = y$. Let $y \in \mathscr{K}$ be a quasi-interior element, let $\xi > 0$ and let $x' \in \mathscr{K}'$, $x' \neq o$. Set

$$v_{x'}(\xi) = \xi^m \langle y, x' \rangle - \langle p_y(\xi), x' \rangle$$

and denote by $\xi(x')$ the unique root of the equation

$$v_{x'}(\xi) = 0\,.$$

Theorem 3.7. *Let* $\mathscr{H}'$ *be a* $\mathscr{K}$*-total set such that* $o \notin \mathscr{H}'$. *Then, under the assumptions either of Theorems* **2.1** *or* **3.1**,

(3.10) $$\inf_{x' \in \mathscr{H}'} \xi(x') \leqq \tilde{r}(F) \leqq \sup_{x' \in \mathscr{H}'} \xi(x')\,.$$

Proof. Let $\xi_0 = \inf_{x' \in \mathscr{H}'} \xi(x')$. If $\xi_0 = 0$, then evidently $\xi_0 \leqq \tilde{r}(F)$. Suppose $\xi_0 > 0$. Then for every linear form $x' \in \mathscr{H}'$ we have

$$\frac{\langle p_y(\xi_0), x' \rangle}{\xi_0^{m-1} \langle y, x' \rangle} \geqq \frac{\langle p_y(\xi(x')), x' \rangle}{[\xi(x')]^{m-1} \langle y, x' \rangle}$$

and thus

$$\inf_{x' \in \mathscr{H}'} \frac{\langle p_y(\xi_0), x' \rangle}{\xi_0^{m-1} \langle y, x' \rangle} \geqq \inf_{x' \in \mathscr{H}'} \frac{\langle p_y(\xi(x')), x' \rangle}{[\xi(x')]^{m-1} \langle y, x' \rangle} = \xi_0\,.$$

Similarly,

$$\xi_s = \sup_{x' \in \mathcal{K}'} \frac{\langle p_y(\xi(x')), x' \rangle}{[\xi(x')]^{m-1} \langle y, x' \rangle} \geqq \sup_{x' \in \mathcal{K}'} \frac{\langle p_y(\xi_s), x' \rangle}{\xi_s^{m-1} \langle y, x' \rangle},$$

where $\xi_s = \sup_{x \in' \mathcal{K}'} \xi(x')$.

To complete the proof of Theorem **3.7** it suffices to apply Theorem **3.5** or Theorem **3.6** respectively according to which

$$\xi_0 = \min \left[\xi_0, \inf_{x' \in \mathcal{K}'} \frac{\langle p_y(\xi_0), x' \rangle}{\xi_0^{m-1} \langle y, x' \rangle} \right] \leqq \tilde{r}(F) \leqq \max \left[\xi_s, \sup_{x' \in \mathcal{K}'} \frac{\langle p_y(\xi_s), x' \rangle}{\xi_s^{m-1} \langle y, x' \rangle} \right] = \xi_s ;$$

hence we conclude (3.10).

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Výtah

LOKALISACE SPEKTRÁLNÍHO POLOMĚRU POLYNOMIÁLNÍHO PROBLÉMU VLASTNÍCH HODNOT S KLADNÝMI OPERÁTORY

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Cílem práce je odvodit některé odhady pro spektrální poloměr polynomiální úlohy, tj. odhady maximální co do modulu singularity operátorové funkce F definované vztahem (0.1) a to za předpokladu, že $A_1, \ldots, A_m, A_0^{-1}$ jsou ohraničená zobrazení Banachova prostoru $\mathscr{Y}$ do sebe a operátory B_j, $B_j = A_0^{-1}A_j$, $j = 1, \ldots, m$ reprodukují kužel $\mathscr{K} \subset \mathscr{Y}$. Touto úlohou se zabýval v práci [1] K. P. Hadeler pro případ zobrazení v Eukleidově prostoru E_l a v prostoru $C(\Omega)$ funkcí spojitých na kompaktu $\Omega \subset E_l$.

Kromě zobecnění Hadelerových výsledků obsahuje naše pojednání též některá zpřesnění Hadelerových tvrzení a mimo to ještě další odhady založené nikoliv na principu Collatzově (viz [7]), nýbrž na principu Brauer-Yamamotově.

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