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PERIODIC SOLUTIONS OF THE FIRST BOUNDARY VALUE PROBLEM FOR A LINEAR AND WEAKLY NONLINEAR HEAT EQUATION

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The existence of periodic solutions for a heat equation (even strongly nonlinear) with some boundary conditions has been studied recently in many papers, cf. [1]–[5]. Here, we shall investigate the existence of ω-periodic solutions of the problem

$$u_t = u_{xx} + cu + g(t, x) + \varepsilon f(t, x, u, u_x, \varepsilon), \tag{0.1}$$

$$u(t, 0) = h_0(t) + \varepsilon\chi_0(t, u(t, 0), u(t, \pi), \varepsilon), \tag{0.2}$$

$$u(t, \pi) = h_1(t) + \varepsilon\chi_1(t, u(t, 0), u(t, \pi), \varepsilon),$$

where $g, f, h_0, h_1, \chi_0, \chi_1$ are ω-periodic in t. Our assumptions differ from those used in the papers quoted in two directions:

(i) c need not be less than 0,

(ii) there is a limitation on the magnitude of the function f but not on its growth with respect to u or u_x.

In the sequel we shall apply two theorems of Functional analysis whose proofs may be found e.g. in [6].

Theorem 0.1. *Let the equation*

$$P(u, s)(\varepsilon) \equiv -u + L(s) + z + \varepsilon R(u)(\varepsilon) = 0 \tag{0.3}$$

be given, where $z \in \mathfrak{U}$ and $P(u, s)(\varepsilon)$ maps the direct product $\mathfrak{U} \times \mathfrak{S}$ of B-spaces $\mathfrak{U}$ and $\mathfrak{S}$ into $\mathfrak{U}$ for every value of the numerical parameter ε from $\mathfrak{E} \equiv \langle 0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$. Let $L \in [\mathfrak{S} \to \mathfrak{U}]$. Let $R(u)(\varepsilon)$ have a $\mathscr{G}$-derivative $R'_u(u)(\varepsilon)$ and $R(u)(\varepsilon)$ together with $R'_u(u)(\varepsilon)$ be continuous in u and ε for any $u \in \mathfrak{U}$ and $\varepsilon \in \mathfrak{E}$.

Then to every $\tilde{s} \in \mathfrak{S}$ there exist numbers $\delta > 0$ and ε^*, $0 < \varepsilon^* \leqq \varepsilon_0$ such that the equation (0.3) has a unique solution $U(s)(\varepsilon) \in \mathfrak{U}$ for each $s \in S(\tilde{s}; \delta)$ ($S(\tilde{s}; \delta)$ denotes the sphere with the center in $\tilde{s}$ and of the radius δ) and $\varepsilon \in \langle 0, \varepsilon^* \rangle$. This solution has a $\mathscr{G}$-derivative $U'_s(s)(\varepsilon)$ continuous in s and ε.

Theorem 0.2. *Let the equation*

$$G(p)(\varepsilon) = 0 \tag{0.4}$$

be given, where $G(p)(\varepsilon)$ maps a B-space $\mathfrak{P}$ into a B-space $\mathfrak{Q}$ for all $\varepsilon \in \mathfrak{E}$. Let the following assumptions be fulfilled.

(i) The equation $G(p_0)(0) = 0$ has a solution $p_0 = p_0^* \in \mathfrak{P}$.

(ii) The operator $G(p)(\varepsilon)$ is continuous in p and ε and has a $\mathcal{G}$-derivative $G'_p(p)(\varepsilon)$ continuous in p and ε for $p \in S(p_0^*, \delta)$, $\varepsilon \in \mathfrak{E}$.

(iii) There exists

$$H = [G'_p(p_0^*)(0)]^{-1} \in [\mathfrak{Q} \to \mathfrak{P}] .$$

Then there exists $\varepsilon^* > 0$ such that the equation (0.4) has for $0 \leqq \varepsilon \leqq \varepsilon^*$ a unique solution $p = p^*(\varepsilon) \in \mathfrak{P}$ continuous in ε such that $p^*(0) = p_0^*$.

1. THE LINEAR CASE

Denote $\mathfrak{X} = (0, \pi)$ and $\mathfrak{T} = (0, \omega\rangle$.

Let the problem $(\mathscr{P}_0)$ be given by

$$u_t = u_{xx} + cu + g(t, x) , \tag{1.1}$$

$$u(t, 0) = h_0(t) , \quad u(t, \pi) = h_1(t) , \tag{1.2}$$

$$u(0, x) = u(\omega, x) , \tag{1.3}$$

while the following assumptions are fulfilled:

$(\mathscr{A}_1)$ $g(t, x)$, $g_x(t, x)$, $h_0(t)$ and $h_1(t)$ are continuous and bounded for $x \in \mathfrak{X}$ and $t \in \mathfrak{T}$;

$(\mathscr{A}_2)$ $g(t, x)$, $h_0(t)$ and $h_1(t)$ are ω-periodic in t for $x \in \mathfrak{X}$.

First, let us introduce the associated initial-boundary value problem $(\mathscr{M}_0)$ given by (1.1), (1.2) and

$$u(0, x) = \varphi(x) \tag{1.4}$$

where $\varphi(x)$ satisfies the assumption

$(\mathscr{A}_3)$ $\varphi(x)$ is continuous in $\overline{\mathfrak{X}}$.

We shall seek a classical solution of $(\mathscr{M}_0)$ (or $(\mathscr{P}_0)$, respectively) i.e. a function which is bounded on $\overline{\mathfrak{T} \times \mathfrak{X}}$, continuous on $\overline{\mathfrak{T} \times \mathfrak{X}}$ except for points $[0, 0]$ and $[0, \pi]$, fulfils (1.2) and (1.4) (or (1.3), respectively), has continuous derivatives u_t, u_{xx} on $\mathfrak{T} \times \mathfrak{X}$ and satisfies there the equation (1.1).

Denote

$$(1.5)\qquad \Theta(t, v) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi^2 n^2 t} \cos 2\pi n v = \frac{1}{\sqrt{(\pi t)}} \sum_{-\infty}^{+\infty} e^{-(v+n)^2/t},$$

$$(1.6)\qquad \mathbf{G}(t; v, \zeta) = \frac{1}{2\pi}\left[\Theta\left(\frac{t}{\pi^2}, \frac{v-\zeta}{2\pi}\right) - \Theta\left(\frac{t}{\pi^2}, \frac{v+\zeta}{2\pi}\right)\right].$$

It may be verified by a standard method that the unique solution of the problem $(\mathscr{M}_0)$ is given by

(1.7)

$$u(t, x) = e^{ct}\int_0^{\pi} \mathbf{G}(t; x, \zeta)\, \varphi(\zeta)\, d\zeta + e^{ct}\int_0^t \int_0^{\pi} e^{-c\tau}\, \mathbf{G}(t-\tau; x, \zeta)\, g(\tau, \zeta)\, d\zeta\, d\tau +$$

$$+ \frac{1}{\pi} e^{ct} \int_0^t e^{-c\tau} \frac{\partial}{\partial x}\Theta\left(\frac{t-\tau}{\pi^2}, \frac{\pi - x}{2\pi}\right) h_1(\tau)\, d\tau -$$

$$- \frac{1}{\pi} e^{ct} \int_0^t h_0(\tau)\, e^{-c\tau} \frac{\partial}{\partial x}\Theta\left(\frac{t-\tau}{\pi^2}, \frac{x}{2\pi}\right) d\tau,$$

$$u(0, x) = \varphi(x).$$

(The uniqueness follows readily from the maximum principle.)

Hence, a solution of $(\mathscr{P}_0)$ exists if and only if there exists a function φ (fulfilling $(\mathscr{A}_3)$) such that the solution of $(\mathscr{M}_0)$ satisfies (1.3) i.e. if and only if there exists φ such that the equation

(1.8)

$$\varphi(x) = e^{c\omega}\int_0^{\pi} \mathbf{G}(\omega; x, \zeta)\, \varphi(\zeta)\, d\zeta + \int_0^{\omega} e^{c(\omega-\tau)} \int_0^{\pi} \mathbf{G}(\omega-\tau; x, \zeta)\, g(\tau, \zeta)\, d\zeta\, d\tau -$$

$$- \frac{1}{\pi}\int_0^{\omega} e^{c(\omega-\tau)}\, h_0(\tau) \frac{\partial}{\partial x}\Theta\left(\frac{\omega-\tau}{\pi^2}, \frac{x}{2\pi}\right) d\tau +$$

$$+ \frac{1}{\pi}\int_0^{\omega} e^{c(\omega-\tau)}\, h_1(\tau) \frac{\partial}{\partial x}\Theta\left(\frac{\omega-\tau}{\pi^2}, \frac{\pi - x}{2\pi}\right) d\tau$$

holds.

This equation represents the Fredholm integral equation of the second kind, whose kernel and the right-hand side are of class C^2 for $x, \zeta \in (0, \pi)$ and are bounded for $x, \zeta \in \langle 0, \pi \rangle$. By the third Fredholm's theorem this equation has a solution if and only if the right-hand side is orthogonal to each solution ψ of the adjoint integral equation i.e.

$$(1.9)\qquad \psi(x) - e^{c\omega}\int_0^{\pi} \mathbf{G}(\omega; x, \zeta)\, \psi(\zeta)\, d\zeta = 0.$$

(We write $\mathbf{G}(\omega; x, \zeta)$ instead of $\mathbf{G}(\omega; \zeta, x)$ as the kernel is symmetric.)

Note that the integral in (1.9) tends to 0 for $x \to 0$ and $x \to \pi$. Thus, for every continuous solution it is $\psi(0) = \psi(\pi) = 0$. Further in view of $\mathbf{G}(\omega; x, \zeta) \in C^2$ every continuous solution of (1.9) is of class C^2. Hence, it may be sought in the form of a uniformly convergent series

$$\psi(x) = \sum_{k=1}^{\infty} \psi_k \sin kx \,. \tag{1.10}$$

Because of

$$\frac{1}{2\sqrt{(\pi\omega)}} \int_{-\infty}^{+\infty} e^{-\eta^2/4\omega} \sin k(x + \eta)\,\mathrm{d}\eta = e^{-k^2\omega} \sin kx$$

inserting (1.10) into (1.9) we get

$$\psi_k(1 - e^{(c-k^2)\omega}) = 0\,, \quad k = 1, 2, \ldots. \tag{1.11}$$

We have to distinguish two cases.

(i) $c \neq k^2$ for all integers k,

(ii) $c = l^2$ for some integer l.

First, let us consider the case (i). Then the only solution of (1.9) is the trivial solution and hence the equation (1.8) has a unique solution for any right-hand side.

According to properties of the integrals in (1.8) for $x \to 0$ and $x \to \pi$, this solution fulfils the conditions $\varphi(0) = h_0(0)$, $\varphi(\pi) = h_1(0)$. (Hence, the solution $u(t, x)$ of $(\mathscr{P}_0)$ is even continuous on $\overline{\mathfrak{T} \times \mathfrak{X}}$.) Thus, the following theorem holds.

Theorem 1.1. *Let the problem $(\mathscr{P}_0)$ be given, let the assumptions $(\mathscr{A}_1)$ and $(\mathscr{A}_2)$ are fulfilled. Let $c \neq k^2$, $k = 1, 2, \ldots$ Then the problem $(\mathscr{P}_0)$ has a unique classical solution $u(t, x) = U(\varphi)(t, x)$ defined by (1.7), where φ is determined as the unique solution of (1.8).*

Now let us return to the case (ii), when $c = l^2$. Then (1.9) has precisely one linearly independent solution $\psi(x) = \sin lx$. Making use of it we find after some arrangements the condition of solvability of (1.7) in the form

$$l \int_0^{\omega} [h_0(\tau) + (-1)^{l+1} h_1(\tau)]\,\mathrm{d}\tau + \int_0^{\omega}\int_0^{\pi} g(\tau, \zeta) \sin l\zeta\,\mathrm{d}\zeta\,\mathrm{d}\tau = 0\,. \tag{1.12}$$

This condition being satisfied, all solutions of (1.8) are given by

$$\varphi(x) = d \sin lx + \hat{\varphi}(x)\,, \tag{1.13}$$

where d is an arbitrary real constant and $\hat{\varphi}(x)$ is a particular solution of (1.8). Hence, the following theorem takes place.

Theorem 1.2. *Let the problem $(\mathscr{P}_0)$ be given. Let the assumptions $(\mathscr{A}_1)$ and $(\mathscr{A}_2)$ be fulfilled. Let $c = l^2$, l is an integer. Then there exists a solution of $(\mathscr{P}_0)$ if and only if* (1.12) *holds. This condition being satisfied, the solution $U(\varphi)(t, x)$ of $(\mathscr{P}_0)$ is defined by* (1.7) *where φ is determined by* (1.13).

2. WEAKLY NONLINEAR INITIAL-BOUNDARY VALUE PROBLEM

Let us consider the problem $(\mathscr{M})$ given by

$$u_t = u_{xx} + cu + g(t, x) + \varepsilon f(t, x, u, u_x, \varepsilon)\,, \tag{2.1}$$

$$u(t, 0) = h_0(t) + \varepsilon\, \chi_0(t, u(t, 0), u(t, \pi), \varepsilon)\,, \tag{2.2}$$

$$u(t, \pi) = h_1(t) + \varepsilon\, \chi_1(t, u(t, 0), u(t, \pi), \varepsilon)\,.$$

$$u(0, x) = \varphi(x)\,. \tag{2.3}$$

Let besides the assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_3)$ the assumption

$(\mathscr{A}_4)$ The functions $f(t, x, u, v, \varepsilon)$ and $\chi_j(t, \alpha, \beta, \varepsilon)$ $(j = 0, 1)$ together with their derivatives $\partial f/\partial x$, $\partial f/\partial u$, $\partial f/\partial v$, $\partial^2 f/\partial u\, \partial x$, $\partial^2 f/\partial v\, \partial x$, $\partial \chi_j/\partial \alpha$, $\partial \chi_j/\partial \beta$ are continuous and bounded for $t \in \mathfrak{T}$, $x \in \mathfrak{X}$, $u, v, \alpha, \beta \in \mathfrak{R} = (-\infty, +\infty)$ and $\varepsilon \in \mathfrak{E} = \langle 0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$;

hold.

Theorem 2.1. *Let the problem $(\mathscr{M})$ be given. Let the assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_3)$ and $(\mathscr{A}_4)$ be fulfilled. Then to every $\tilde{\varphi} \in C(\langle 0, \pi \rangle)$ there exist numbers $\delta > 0$ and ε^*, $0 < \varepsilon^* \leqq \varepsilon_0$ such that for all $\varepsilon \in \langle 0, \varepsilon^* \rangle$ and all $\varphi \in S(\tilde{\varphi}, \delta)$ there exists a unique classical solution $u^*(t, x) = U(\varphi)(t, x)(\varepsilon)$. For $\varphi \in S(\tilde{\varphi}, \delta)$ the linear operator $U(\varphi)(\varepsilon)$ has a $\mathscr{G}$-derivative $U'_\varphi(\varphi)(\varepsilon)$, which is together with $U(\varphi)(\varepsilon)$ continuous in φ and ε.*

Proof. It may be easily verified that the problem $(\mathscr{M})$ is equivalent to the integro-differential equation

$$P(u, \varphi)(\varepsilon)(t, x) \equiv -u(t, x) + e^{ct} \int_0^\pi \mathbf{G}(t; x, \zeta)\, \varphi(\zeta)\, \mathrm{d}\zeta + \tag{2.4}$$

$$+ \int_0^t \int_0^\pi e^{c(t-\tau)}\, \mathbf{G}(t - \tau; x, \zeta)\, [g(\tau, \zeta) + \varepsilon f(\tau, \zeta, u(\tau, \zeta), u_x(\tau, \zeta), \varepsilon)]\, \mathrm{d}\zeta\, \mathrm{d}\tau -$$

$$- \frac{1}{\pi} \int_0^t e^{c(t-\tau)} \frac{\partial}{\partial x}\, \Theta\left(\frac{t - \tau}{\pi^2}, \frac{x}{2\pi}\right) [h_0(\tau) + \varepsilon\, \chi_0(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)]\, \mathrm{d}\tau +$$

$$+ \frac{1}{\pi} \int_0^t e^{c(t-\tau)} \frac{\partial}{\partial x}\, \Theta\left(\frac{t - \tau}{\pi^2}, \frac{\pi - x}{2\pi}\right) [h_1(\tau) + \varepsilon\, \chi_1(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)]\, \mathrm{d}\tau = 0\,.$$

It is readily seen that any solution u of (2.4) which is together with u_x and u_{xx} continuous and bounded on $\mathfrak{T} \times \mathfrak{X}$ possesses u_t continuous and bounded on $\mathfrak{T} \times \mathfrak{X}$ at the same time. Therefore it suffices to seek a solution of (2.4) in the B-space $\mathfrak{U}$ of functions $u(t, x)$ continuous and bounded together with u_x and u_{xx} on $\mathfrak{T} \times \mathfrak{X}$ and with the norm

$$\|u\| = \sup_{\mathfrak{T}\times\mathfrak{X}} |u| + \sup_{\mathfrak{T}\times\mathfrak{X}} |u_x| + \sup_{\mathfrak{T}\times\mathfrak{X}} |u_{xx}| .$$

Denote

$$L(\varphi)(t, x) \equiv e^{ct} \int_0^\pi \mathbf{G}(t, x, \zeta)\, \varphi(\zeta)\, \mathrm{d}\zeta ,$$

$$\begin{aligned} R(u)(\varepsilon)(t, x) \equiv & \int_0^t \int_0^\pi e^{c(t-\tau)}\, \mathbf{G}(t - \tau; x, \zeta) f(\tau, \zeta, u(\tau, \zeta), u_x(\tau, \zeta), \varepsilon)\, \mathrm{d}\zeta\, \mathrm{d}\tau - \\ & - \frac{1}{\pi} \int_0^t e^{c(t-\tau)} \frac{\partial}{\partial x} \Theta\left(\frac{t - \tau}{\pi^2}, \frac{x}{2\pi}\right) \chi_0(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)\, \mathrm{d}\tau + \\ & + \frac{1}{\pi} \int_0^t e^{c(t-\tau)} \frac{\partial}{\partial x} \Theta\left(\frac{t - \tau}{\pi^2}, \frac{\pi - x}{2\pi}\right) \chi_1(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)\, \mathrm{d}\tau , \end{aligned}$$

$$\begin{aligned} z(t, x) \equiv & \int_0^t \int_0^\pi e^{c(t-\tau)}\, \mathbf{G}(t - \tau; x, \zeta)\, g(\tau, \zeta)\, \mathrm{d}\zeta\, \mathrm{d}\tau - \\ & - \frac{1}{\pi} \int_0^t e^{c(t-\tau)} h_0(\tau) \frac{\partial}{\partial x} \Theta\left(\frac{t - \tau}{\pi^2}, \frac{x}{2\pi}\right) \mathrm{d}\tau + \\ & + \frac{1}{\pi} \int_0^t e^{c(t-\tau)} h_1(\tau) \frac{\partial}{\partial x} \Theta\left(\frac{t - \tau}{\pi^2}, \frac{\pi - x}{2\pi}\right) \mathrm{d}\tau . \end{aligned}$$

The operator P maps $\mathfrak{U} \times C(\langle 0, \pi \rangle)$ into $\mathfrak{U}$ and L is a linear operator. By $(\mathscr{A}_4)$ there exists a $\mathscr{G}$-derivative

$$\begin{aligned} R'_u(u)(\varepsilon)(\bar{u})(t, x) \equiv & \int_0^t \int_0^\pi e^{c(t-\tau)}\, \mathbf{G}(t - \tau; x, \zeta) \,. \\ & . \left[\frac{\partial f}{\partial u}(\tau, \zeta, u, u_x, \varepsilon)\, \bar{u}(\tau, \zeta) + \frac{\partial f}{\partial u_x}(\tau, \zeta, u, u_x, \varepsilon)\, \bar{u}_x(\tau, \zeta)\right] \mathrm{d}\zeta\, \mathrm{d}\tau - \\ & - \frac{1}{\pi} \int_0^t e^{c(t-\tau)} \frac{\partial}{\partial x} \Theta\left(\frac{t - \tau}{\pi^2}, \frac{x}{2\pi}\right) . \\ & . \left[\frac{\partial \chi_0}{\partial \alpha}(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)\, \bar{u}(\tau, 0) + \frac{\partial \chi_0}{\partial \beta}(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)\, \bar{u}(\tau, \pi)\right] \mathrm{d}\tau + \\ & + \frac{1}{\pi} \int_0^t e^{c(t-\tau)} \frac{\partial}{\partial x} \Theta\left(\frac{t - \tau}{\pi^2}, \frac{\pi - x}{2\pi}\right) . \\ & . \left[\frac{\partial \chi_1}{\partial \alpha}(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)\, \bar{u}(\tau, 0) + \frac{\partial \chi_1}{\partial \beta}(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)\, \bar{u}(\tau, \pi)\right] \mathrm{d}\tau \end{aligned}$$

which is with $R(u)(\varepsilon)$ continuous in u and ε for $u \in \mathfrak{U}$, $\varepsilon \in \mathfrak{E}$. From here by Theorem (0.1) the assertion of Theorem (2.1) follows readily.

Remark 2.1. From the proof of Theorem (2.1) it is readily seen that it suffices to suppose that the functions $f(t, x, u, u_x, \varepsilon)$ and $\chi_j(t, u(t, 0), u(t, \pi), \varepsilon)$ $(j = 0, 1)$ have the stated properties only for $\|u - u_0\| \leqq r$. Then it is only necessary to ensure by a suitable choice of ε^* that the found solution of the problem $(\mathscr{M})$ lies in $S(u_0, r)$.

3. PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR HEAT EQUATION

Let the problem $(\mathscr{P})$ be given by

$$u_t = u_{xx} + cu + g(t, x) + \varepsilon f(t, x, u, u_x, \varepsilon)\,, \tag{3.1}$$

$$u(t, 0) = h_0(t) + \varepsilon \chi_0(t, u(t, 0), u(t, \pi), \varepsilon)\,, \tag{3.2}$$

$$u(t, \pi) = h_1(t) + \varepsilon \chi_1(t, u(t, 0), u(t, \pi), \varepsilon)\,,$$

$$u(\omega, x) - u(0, x) = 0\,. \tag{3.3}$$

Let besides the assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, $(\mathscr{A}_4)$ the assumption

$(\mathscr{A}_5)$ The functions f and χ_j $(j = 0, 1)$ are ω-periodic in t

be fulfilled. By Theorem (2.1) for any function $\varphi \in C(\langle 0, \pi \rangle)$ there exists ε_1, $0 < \varepsilon_1 \leqq \varepsilon_0$ such that for all $\varepsilon \in \langle 0, \varepsilon_1 \rangle$ there exists a classical solution $U(\varphi)(\varepsilon)(t, x)$ of the problem $(\mathscr{M})$.

This function is a solution of the problem $(\mathscr{P})$ if and only if the function φ satisfies the following equation

$$\begin{aligned} G(\varphi)(\varepsilon)(x) \equiv {} & -\varphi(x) + e^{c\omega} \int_0^\pi \mathbf{G}(\omega, x, \zeta) \,.\, \varphi(\zeta)\, d\zeta \, + \\ & + e^{c\omega} \int_0^\omega \int_0^\pi e^{-c\tau}\, \mathbf{G}(\omega - \tau; x, \zeta) \,.\, [g(\tau, \zeta) + \varepsilon f(\tau, \zeta, u, u_x, \varepsilon)]\, d\zeta\, d\tau \, - \\ & - \frac{1}{\pi} \int_0^\omega e^{c(\omega - \tau)} \frac{\partial}{\partial x}\, \Theta\left(\frac{\omega - \tau}{\pi^2}, \frac{x}{2\pi}\right) . \left[h_0(\tau) + \varepsilon \chi_0(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)\right] d\tau \, + \\ & + \frac{1}{\pi} \int_0^\omega e^{c(\omega - \tau)} \frac{\partial}{\partial x}\, \Theta\left(\frac{\omega - \tau}{\pi^2}, \frac{\pi - x}{2\pi}\right) . \left[h_1(\tau) + \varepsilon \chi_1(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)\right] d\tau = 0\,. \end{aligned} \tag{3.4}$$

Let $c \neq k^2$, $k = 1, 2, \ldots$ Then applying Theorem (0.2) the following theorem may be proved.

Theorem 3.1. *Let the problem $(\mathscr{P})$ be given. Let the assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, $(\mathscr{A}_4)$, $(\mathscr{A}_5)$ be fulfilled. Let $c \neq k^2$, $k = 1, 2, \ldots$ Then there exists ε^*, $0 < \varepsilon^* \leqq \varepsilon_0$ such that for all $\varepsilon \in \langle 0, \varepsilon^* \rangle$ the problem $(\mathscr{P})$ has a unique classical solution.*

Proof. The operator $G(\varphi)(\varepsilon)$ maps $C(\langle 0, \pi \rangle)$ into $C(\langle 0, \pi \rangle)$ for all $\varepsilon \in \mathfrak{E}$. For $\varepsilon = 0$ the equation (3.4) has by Theorem (1.1) a unique solution $\varphi_0^* \in C(\langle 0, \pi \rangle)$. By Theorem (2.1) there exist numbers $\delta > 0$ and ε_2, $0 < \varepsilon_2 \leqq \varepsilon_0$ such that for $\varepsilon \in \langle 0, \varepsilon_2 \rangle$ and $\varphi \in S(\varphi_0^*, \delta)$ the solution $U(\varphi)(\varepsilon)$ of $(\mathscr{M})$ is $\mathscr{G}$-differentiable and $U(\varphi)(\varepsilon)$ and $U'_\varphi(\varphi)(\varepsilon)$ are continuous in φ and ε. Hence, the operator $G(\varphi)(\varepsilon)$ is $\mathscr{G}$-differentiable and $G(\varphi)(\varepsilon)$ together with $G'_\varphi(\varphi)(\varepsilon)$ are continuous in φ and ε for $\varphi \in S(\varphi_0^*, \delta)$ and $\varepsilon \in \langle 0, \varepsilon_2 \rangle$. Finally by the proof of Theorem (1.1) the equation

$$G'_\varphi(\varphi_0^*)(0)(\bar{\varphi})(x) \equiv -\bar{\varphi}(x) + \int_0^\pi e^{c\omega} \mathbf{G}(\omega, x, \zeta)\, \bar{\varphi}(\zeta)\, d\zeta = p(x) \tag{3.5}$$

where p is an arbitrary element of $C(\langle 0, \pi \rangle)$, has under the assumption $c \neq k^2$, $k = 1, 2, \ldots$ a unique solution $\bar{\varphi} \in C(\langle 0, \pi \rangle)$ such that $\|\bar{\varphi}\|_C \leqq K\|p\|_C$. Thus, putting $\mathfrak{P} = \mathfrak{Q} = C(\langle 0, \pi \rangle)$ and $\mathfrak{E} = \langle 0, \varepsilon_2 \rangle$, all assumptions of Theorem (0.2) are satisfied and our assertion follows readily.

In the case $c = l^2$, l an integer, a somewhat another procedure has to be applied. The problem $(\mathscr{P})$ is equivalent to the system of equations (2.4) and (3.4). Suppose that this system has a solution for $\varepsilon = 0$. For this it is necessary and sufficient that the condition (1.12) holds. Let us seek the function φ in the form

$$\varphi(x) = \mu \sin lx + \hat{\varphi}(x) + \varrho(x), \tag{3.6}$$

where μ is an arbitrary constant, $\hat{\varphi}(x)$ is a particular solution of (1.8) and $\varrho(x) \in C(\langle 0, \pi \rangle)$ fulfils for the sake of uniqueness the condition

$$\int_0^\pi \varrho(\zeta) \sin l\zeta \, d\zeta = 0. \tag{3.7}$$

Then the system (2.4), (3.4) may be rewritten as

$$\begin{aligned} G_1(u, \varrho, \mu)(\varepsilon)(t, x) \equiv{}& -u(t, x) + \mu \sin lx + \\ &+ e^{l^2 t} \left\{ \int_0^\pi \mathbf{G}(t; x, \zeta)\, \hat{\varphi}(\zeta)\, d\zeta + \int_0^\pi \mathbf{G}(t; x, \zeta)\, \varrho(\zeta)\, d\zeta + \right. \\ &+ \int_0^t \int_0^\pi e^{-l^2\tau} \mathbf{G}(t - \tau; x, \zeta) \left[g(\tau, \zeta) + \varepsilon f(\tau, \zeta, u, u_x, \varepsilon) \right] d\zeta \, d\tau + \\ &+ \frac{1}{\pi} \int_0^t e^{-l^2\tau} \left[-\frac{\partial}{\partial x} \Theta\left(\frac{t - \tau}{\pi^2}, \frac{x}{2\pi} \right) . \left(h_0(\tau) + \varepsilon \chi_0(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon) \right) + \right. \\ &\left.\left. + \frac{\partial}{\partial x} \Theta\left(\frac{t - \tau}{\pi^2}, \frac{\pi - x}{2\pi} \right) \left(h_1(\tau) + \varepsilon \chi_1(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon) \right) \right] d\tau \right\} = 0, \end{aligned} \tag{3.8}$$

$$(3.9)\qquad G_2(u, \varrho, \mu)(\varepsilon)(x) \equiv -\varrho(x) + e^{l^2\omega}\left\{\int_0^\pi \mathbf{G}(\omega, x, \zeta)\,\varrho(\zeta)\,d\zeta + \right.$$

$$+ \varepsilon \int_0^\omega \int_0^\pi e^{-l^2\tau}\,\mathbf{G}(\omega - \tau, x, \zeta)\, f(\tau, \zeta, u, u_x, \varepsilon)\,d\zeta\,d\tau +$$

$$+ \varepsilon \frac{1}{\pi} \int_0^\omega e^{-l^2\tau}\left[-\frac{\partial}{\partial x}\,\Theta\left(\frac{t-\tau}{\pi^2}, \frac{x}{2\pi}\right)\chi_0(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon) + \right.$$

$$\left.\left. + \frac{\partial}{\partial x}\,\Theta\left(\frac{t-\tau}{\pi^2}, \frac{\pi - x}{2\pi}\right)\chi_1(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)\right] d\tau\right\} = 0\,.$$

By the third Fredholm theorem the equation (3.9) has a solution if and only if

$$(3.10)\qquad G_3(u, \varrho, \mu)(\varepsilon) \equiv \int_0^\omega \int_0^\pi \sin l\zeta\, f(\tau, \zeta, u, u_x, \varepsilon)\,d\zeta\,d\tau +$$

$$+ l \int_0^\omega [\chi_0(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon) + (-1)^{l+1}\,\chi_1(\tau, u(\tau, 0), u(\tau, \pi), \varepsilon)]\,d\tau = 0\,.$$

To prove the existence of a solution of the equations (3.8, 3.9, 3.10) we shall make use of Theorem (0.2). Put $p = (u, \varrho, \mu)$, $\mathfrak{P} = (\mathfrak{U}, \mathfrak{A}, \mathfrak{R})$ where $\mathfrak{A}$ is the subspace of functions from $C(\langle 0, \pi\rangle)$ satisfying the condition (3.7). Define the norm in the B-space $\mathfrak{P}$ by

$$\|p\|_{\mathfrak{P}} = \|u\|_{\mathfrak{U}} + \|\varrho\|_C + |\mu|\,.$$

Clearly, the operator $G = (G_1, G_2, G_3)$ maps $\mathfrak{P}$ into $\mathfrak{P}$.

Theorem 3.2. *Let the problem* $(\mathscr{P})$ *be given. Let besides the assumptions* $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, $(\mathscr{A}_4)$, $(\mathscr{A}_5)$, $c = l^2$, *l an integer, the following assumptions be fulfilled.*

(i) *the condition* (1.12) *holds,*

(ii) *the equation*

$$(3.11)\qquad G_3(\mu_0 \sin lx + v_0(t, x), 0, \mu_0)(0) \equiv \int_0^\pi \int_0^\omega \sin l\zeta\, f(\tau, \zeta, \mu_0 \sin l\zeta +$$

$$+ v_0(\tau, \zeta), \mu_0 l \cos l\zeta + v_{0x}(\tau, \zeta), 0)\,d\tau\,d\zeta + l \int_0^\omega [\chi_0(\tau, v_0(\tau, 0)\,,$$

$$v_0(\tau, \pi), 0) + (-1)^{l+1}\,\chi_1(\tau, v_0(\tau, 0), v_0(\tau, \pi), 0)]\,d\tau = 0$$

where

$$v_0(t, x) = e^{l^2 t}\int_0^\pi \mathbf{G}(t, x, \zeta)\,\hat{\varphi}(\zeta)\,d\zeta + \int_0^t \int_0^\pi e^{l^2(t-\tau)}\,\mathbf{G}(t - \tau, x, \zeta)\,.$$

$$. g(\tau, \zeta) \, d\zeta \, d\tau + \frac{1}{\pi} \int_0^t e^{l^2(t-\tau)} \left[- \frac{\partial}{\partial x} \Theta\left(\frac{t-\tau}{\pi^2}, \frac{x}{2\pi}\right) h_0(\tau) + \right.$$

$$\left. + \frac{\partial}{\partial x} \Theta\left(\frac{t-\tau}{\pi^2}, \frac{\pi - x}{2\pi}\right) h_1(\tau) \right] d\tau ,$$

has a real solution $\mu_0 = \mu_0^*$;

(iii) *the condition*

(3.12)
$$a \equiv \int_0^\omega \int_0^\pi \sin l\zeta \left[\frac{\partial f}{\partial u} (\tau, \zeta, \mu_0^* \sin l\zeta + v_0(\tau, \zeta), \mu_0^* l \cos l\zeta + \right.$$

$$+ v_{0x}(\tau, \zeta), 0) \sin l\zeta + \frac{\partial f}{\partial u_x} (\tau, \zeta, \mu_0^* \sin l\zeta + v_0(\tau, \zeta), \mu_0^* l \cos l\zeta +$$

$$\left. + v_{0x}(\tau, \zeta), 0) \cos l\zeta) \right] d\zeta \, d\tau \neq 0$$

is satisfied.

Then there exists ε_3, $0 < \varepsilon_3 \leqq \varepsilon_0$, *such that for all* $\varepsilon \in \langle 0, \varepsilon_3 \rangle$ *the problem* $(\mathscr{P})$ *has a unique classical solution* $u^*(\varepsilon)(t, x)$ *such that* $u^*(0)(t, x) = \mu_0^* \sin lx + v_0(t, x)$.

Proof. The assumption (i) of Theorem (0.2) is satisfied since we may put

$$p_0^* = (\mu_0^* \sin lx + v_0(t, x), 0, \mu_0^*) .$$

We may also verify easily that in every sphere with the center p_0^* in virtue of $(\mathscr{A}_4)$ the assumption (ii) of Theorem (0.2) holds. To prove that the assumption (iii) of Theorem (0.2) is also satisfied let us show that the system

(3.13)
$$G_1'(p_0^*)(0)(\bar{p})(t, x) \equiv$$

$$\equiv -\bar{u}(t, x) + \bar{\mu} \sin lx + e^{l^2 t} \int_0^\pi \mathbf{G}(t; x, \zeta) \, \bar{\varrho}(\zeta) \, d\zeta = q_1(t, x) ,$$

(3.14)
$$G_2'(p_0^*)(0)(\bar{p})(x) \equiv$$

$$\equiv -\bar{\varrho}(x) + \int_0^\pi e^{l^2 \omega} \mathbf{G}(\omega; x, \zeta) \, \bar{\varrho}(\zeta) \, d\zeta = q_2(x) ,$$

(3.15)
$$G_3'(p_0^*)(0)(\bar{p}) \equiv a\bar{\mu} = q_3 ,$$

where $q = (q_1, q_2, q_3)$ is an arbitrary point of $\mathfrak{P}$, has a unique solution $\bar{p} \in \mathfrak{P}$ and it holds $\|\bar{p}\| \leqq K\|q\|$. By (3.12) the equation (3.15) has a solution $\bar{\mu}^* = a^{-1} q_3$. In virtue of $q_2 \in \mathfrak{A}$, the equation (3.14) has a unique solution $\bar{\varrho}^* \in \mathfrak{A}$. Finally the equation (3.13) has a unique solution

$$\bar{u}^*(t, x) = \bar{\mu}^* \sin lx + \int_0^\pi e^{l^2 t} \mathbf{G}(t; x, \zeta) \, \bar{\varrho}^*(\zeta) \, d\zeta - q_1(t, x) .$$

Evidently there exists K_3 such that $|\bar{\mu}^*| \leqq K_3|q_3|$. Since the operator $G_2'(p_0^*)(0)$ is linear, bounded and one-to-one and maps $\mathfrak{A}$ onto $\mathfrak{A}$ by the Banach theorem the inverse operator $[G_2'(p_0^*)(0)]^{-1} \in [\mathfrak{A} \to \mathfrak{A}]$ is also bounded and hence there exists a constant K_2 such that

$$\|\bar{\varrho}^*\| \leqq K_2\|q_2\| .$$

And as well

$$\|\bar{u}^*\| \leqq K_1[|\bar{\mu}^*| + \|\bar{\varrho}^*\| + \|q_1\|]$$

what yields the existence of the constant K and the proof of our theorem is complete.

Example

Let

$$c = l^2 , \quad \omega = 2\pi , \quad g \equiv h_0 \equiv h_1 \equiv \chi_0 \equiv \chi_1 \equiv 0$$

and

$$f(t, x, u, u_x, \varepsilon) = \sin lx \cos^2 lt + \gamma u^2 \sin lx, \gamma \neq 0 .$$

Then the equation (3.11) reads $2 + 3\gamma\mu_0^2 = 0$ and $a = \frac{3}{2}\pi^2\gamma\mu_0^*$. Thus, if $\gamma < 0$ there exist for sufficiently small ε two 2π-periodic solutions of the given problem and

$$u^*(0)(t, x) = \sqrt{\left(\frac{-2}{3\gamma}\right)} \sin lx \quad \text{or} \quad -\sqrt{\left(\frac{-2}{3\gamma}\right)} \sin lx .$$

On the other hand, if $\gamma > 0$ there does not exist for small ε any 2π-periodic solution of our problem.

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Souhrn

PERIODICKÁ ŘEŠENÍ PRVÉ OKRAJOVÉ ÚLOHY PRO LINEÁRNÍ A SLABĚ NELINEÁRNÍ ROVNICI PRO VEDENÍ TEPLA

VĚNCESLAVA ŠŤASTNOVÁ - OTTO VEJVODA

V § 1 se především vyšetřuje existence ω-periodického řešení rovnice (1.1) s okrajovými podmínkami (1.2) za předpokladu, že funkce g, h_0, h_1 jsou dostatečně hladké a ω-periodické. V případě, že $c \neq k^2$, k přirozené, takové řešení vždy existuje. Naproti tomu, je-li $c = l^2$, l přirozené, úloha má řešení, právě když je splněna podmínka (1.12). V § 3 se obdobně vyšetřuje slabě nelineární problém (3.1), (3.2). Též výsledky jsou obdobné. V druhém případě ($c = l^2$) lze existenci ω-periodického řešení pro dostatečně malé $\varepsilon > 0$ dokázat, požadujeme-li kromě nutných podmínek (1.12) a (3.11) splnění podmínky (3.12).

Резюме

ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ ПЕРВОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ЛИНЕЙНОГО И СЛАБО НЕЛИНЕЙНОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ

ВЕНЦЕСЛАВА ШТЬАСТНОВА, ОТТО ВЕЙВОДА (VĚNCESLAVA ŠŤASTNOVÁ, OTTO VEJVODA)

В § 1 исследуется существование ω-периодического решения уравнения (1.1) с краевыми условиями (1.2) в предположении, что функции g, h_0, h_1 достаточно гладкие и ω-периодические. В случае $c \neq k^2$, k натуральное, такое решение всегда существует. Наоборот, если $c = l^2$, l натуральное, задача имеет решение тогда и только тогда, когда исполнено условие (1.12). В § 3 изучается соответствующая слабо нелинейная проблемма (3.1), (3.2). Результаты аналогичны тем из § 1. Только в случае $c = l^2$ возможно существование ω-периодического решения для достаточно малого $\varepsilon > 0$ показать, если кроме необходимых условий (1.12) и (3.11) требовать исполнение условия 3.12.

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