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QUASI-STATIC THERMAL DEFLECTION IN A SOLID CIRCULAR PLATE IN THE AXISYMMETRIC CASE

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1. INTRODUCTION

The problem of the determination of thermal stresses and deflections in plates of different contours has got wide considerations as it has practical applications in air-craft structures. Forray and Newmann (1960), Forray (1958) and Forray and Zaid (1958) have discussed the problem of thermal stresses and vibrations in various papers. In the first mentioned paper [2], the authors have obtained thermal stresses in a circular plate with various edge conditions for a particular temperature distribution in the axisymmetric case by solving the differential equation directly. In the second [3] and third paper [4], they have introduced a stress function and obtained the stresses.

In this paper, Forray and Newmann's method will be followed. It is assumed that the temperature varies through the thickness and the deflections are small in comparison with the thickness. The thermal deflections of the plate for various edge conditions in a circular plate is obtained in the axisymmetric case. In this case, the temperature is found to be a function of r and t only and t is considered as a parameter.

2. NOMENCLATURE

r, ϑ, z	– Cylindrical polar co-ordinates
ω_{st}	– Quasi-static deflection in the Z-direction
∇^2	– Laplacian operator
t	– time parameter
E	– Young's Modulus
h	– thickness of the plate
ν	– Poisson's ratio (assumed 0·3 in numerical calculations)
D	– Flexural rigidity
α_t	– Coefficient of thermal expansion

K	–	thermal conductivity (constant)
k	–	Thermal diffusivity (constant)
a	–	radias of the plate
τ	–	Non-dimensional time parameter $= kt/a^2$
T	–	Temperature
T_D	–	Temperature difference between the upper and lower surfaces of the plate
F_0	–	Flux into the plate
n	–	integers
α_r	–	roots of $J_1(\alpha_n) = 0$
C_1, C_2, C_3, C_4	–	Constants

3. METHOD OF SOLUTION

We take a solid circular plate of thickness h with zero initial temperature and constant flux F_0 into the plate, we take the centre of the plate in the middle surface to be the origin and Z-axis downwards. Since T is a function of r and t only, the equation of heat conduction is,

$$(1) \qquad \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} - \frac{1}{k}\frac{\partial T}{\partial t} = 0\,, \quad 0 < r < a;\ t > 0$$

with the initial conditions,

$$(2) \qquad T = 0 \quad \text{when} \quad t = 0\,, \quad K\frac{\partial T}{\partial r} = F_0 \quad \text{on} \quad r = a$$

The solution of equation (1) with the given initial conditions is

$$(3) \qquad T = \frac{2F_0 kt}{Ka} + \frac{F_0 a}{K}\left[\frac{r^2}{2a^2} - \frac{1}{4} - 2\sum_{n=1}^{\infty} \frac{\exp\left(-\frac{k\alpha_n^2}{a^2}t\right)}{\alpha_n^2 J_0(\alpha_n)} J_0\left(r\frac{\alpha_n}{a}\right)\right]$$

where α_n are the positive roots of the transcendental equation

$$(4) \qquad J_1(\alpha_n) = 0$$

The equilibrium equation for the deflection of the plate is given by

$$(5) \qquad \nabla^4 \omega_{st} = \frac{(1 + \nu)\,\alpha_t}{h} \nabla^2 T_D$$

where ∇^4 in the axisymmetric case is given by

$$\nabla^4 \equiv \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)$$

and we take for T_D the expression given by equation (3).

The solution of $\nabla^4 \omega_{st} = 0$ is

$$\omega_{st} = C_1 + C_2 \frac{r^2}{a^2} + C_3 \ln \frac{r}{a} + C_4 r^2 \ln \frac{r}{a} \tag{6}$$

For a solid plate, the displacement ω_{st} must be finite at $r = 0$ so that C_3 and C_4 comes out to be zero. Hence, the complete solution becomes,

$$\omega_{st} = C_1 + C_2 \frac{r^2}{a^2} \tag{7}$$

In order to find the general solution of the non-homogeneous equation (5), we must add to the complete solution, the particular integral of equation (5). Now, we note that a particular integral of

$$\nabla^2 \omega_{st} = \frac{(1 + v)\,\alpha_t}{h} T_D \tag{8}$$

is necessarily a particular solution of equation (5). This is obtained by integration, so that,

(9)

$$\omega_{st} = \frac{(1 + v)\,\alpha_t}{h}\left[\frac{F_0 k t}{2Ka} r^2 + \frac{F_0 a}{K}\left\{\frac{r^4}{32a^2} - \frac{r^2}{16} + a^2 \sum_{n=1}^{\infty} \frac{\exp\left(-\frac{k\alpha_n^2 t}{a^2}\right)}{\alpha_n^4 J_0(\alpha_n)} J_0\left(\frac{r\alpha_n}{a}\right)\right\}\right]$$

Hence, the general solution of equation (5) is,

$$\omega_{st} = C_1 + C_2 \frac{r^2}{a^2} + \frac{(1 + v)\,\alpha_t}{h} \cdot \tag{10}$$

$$\cdot \left[\frac{F_0 k t}{2Ka} r^2 + \frac{F_0 a}{K}\left\{\frac{r^4}{32a^2} - \frac{r^2}{16} + a^2 \sum_{n=1}^{\infty} \frac{\exp\left(-\frac{k\alpha_n^2 t}{a^2}\right)}{\alpha_n^4 J_0(\alpha_n)} J_0\left(\frac{r\alpha_n}{a}\right)\right\}\right]$$

4. CLAMPED PLATE

The boundary condition for a clamped plate is

$$\omega_{st} = \frac{d\omega_{st}}{dr} = 0 \quad \text{at} \quad r = a. \tag{11}$$

Imposing these conditions, we get from equation (10), the values of the constants,

$$(12) \qquad C_1 = \frac{(1+\nu)\,\alpha_t F_0 a^3}{hK}\left[\frac{1}{32} - 2\sum_{n=1}^{\infty}\frac{\exp\left(-\dfrac{k\alpha_n^2 t}{a^2}\right)}{\alpha_n^4}\right]$$

$$C_2 = -\frac{(1+\nu)\,\alpha_t F_0 a}{2hK}\,kt$$

Therefore, the deflection w_{st} for a clamped plate is given by

(13)

$$\omega_{st} = \frac{(1+\nu)\,\alpha_t F_0 a}{hK}\left[\frac{a^2}{32}\left(1-\frac{r^4}{a^4}\right) - \frac{r^2}{16} + 2a^2\sum_{n=1}^{\infty}\frac{\exp\left(-\dfrac{k\alpha_n^2 t}{a^2}\right)}{\alpha_n^4}\left\{\frac{J_0(r\alpha_n/a)}{J_0(\alpha_n)} - 1\right\}\right]$$

Introducing the non-dimensional time-parameter τ, we can write equation (13) in the form,

(14)

$$\omega_{st} = \frac{(1+\nu)\,\alpha_t F_0 a}{hK}\left[\frac{a^2}{32}\left(1-\frac{r^4}{a^4}\right) - \frac{r^2}{16} + 2a^2\sum_{n=1}^{\infty}\frac{\exp(-\alpha_n^2\tau)}{\alpha_n^4}\left\{\frac{J_0(r\alpha_n/a)}{J_0(\alpha_n)} - 1\right\}\right]$$

The maximum deflection is at the centre of the plate and is obtained by substituting $r = 0$ in equation (14), so that,

$$(15) \qquad \omega_{st_{\max}} = \frac{(1+\nu)\,\alpha_t F_0 a^3}{hK}\left[\frac{1}{32} + 2\sum_{n=1}^{\infty}\frac{\exp(-\alpha_n^2\tau)}{\alpha_n^4}\left\{\frac{1}{J_0(\alpha_n)} - 1\right\}\right]$$

For numerical calculations, we write equation (15) in the form,

$$(16) \qquad \frac{hK\omega_{st_{\max}}}{(1+\nu)\,\alpha_t F_0 a^3} = \frac{1}{32} + 2\sum_{n=1}^{\infty}\frac{\exp(-\alpha_n^2\tau)}{\alpha_n^4}\left[\frac{1}{J_0(\alpha_n)} - 1\right]$$

The values of $[hK\omega_{st_{\max}}]/[(1+\nu)\,\alpha_t F_0 a^3]$ for different values of $\tau = 0{\cdot}02$, 0·04, 0·06, 0·08, 0·10 are presented in Table I (Reference 6).

Table I

τ	0·02	0·04	0·06	0·08	0·10
$\dfrac{hK\omega_{st_{\max}}}{(1+\nu)\,\alpha_t F_0 a^3}$	0·007862	0·013557	0·017950	0·021271	0·023818

Curve for the variation of $[hK\omega_{st_{\max}}]/[(1+\nu)\,\alpha_t F_0 a^3]$ with respect to τ is presented in Figure 1.

5. SIMPLY — SUPPORTED PLATE

The condition to be satisfied at the edge for a simply-supported plate is that the deflection and the moment should be zero at the edge, so that,

$$\omega_{st} = \left[\frac{d^2\omega_{st}}{dr^2} + \frac{\nu}{r}\frac{d\omega_{st}}{dr} + \frac{(1+\nu)\,\alpha_t}{h}T_D\right] = 0 \quad \text{at} \quad r = a\,. \tag{17}$$

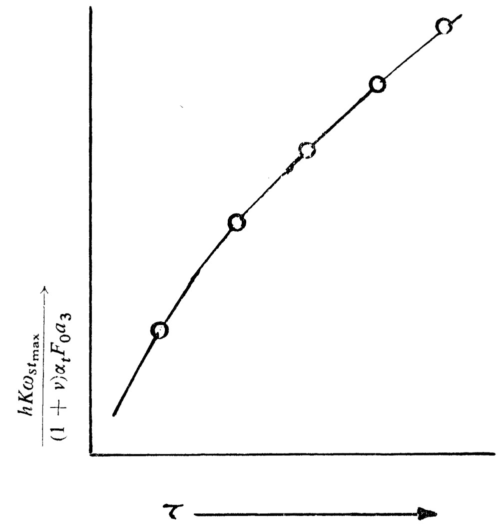

Fig. 1.

These conditions determine the constants of equation (10), such that,

$$C_1 = \frac{(1+\nu)\,\alpha_t F_0 a^3}{hK}\left[\frac{1}{32} - \frac{kt}{a^2(1+\nu)} - 2\sum_{n=1}^{\infty}\frac{\exp\left(-\frac{k\alpha_n^2 t}{a^2}\right)}{\alpha_n^4}\right] \tag{18}$$

$$C_2 = \frac{(1-\nu)\,\alpha_t F_0 a^3}{2hK}\,\frac{kt}{a^2}$$

Hence, the deflection ω_{st} for a simply supported plate is given by, (introducing the non-dimensional time parameter)

$$\omega_{st} = \frac{(1+\nu)\,\alpha_t F_0 a^3}{hK}\,. \tag{19}$$

$$\cdot\left[\frac{1}{32} - \frac{\tau}{1+\nu} + \frac{3+\nu}{2(1+\nu)}\frac{\tau r^2}{a^2} + \frac{r^4}{32a^4} - \frac{r^2}{16a^2} + 2\sum_{n=1}^{\infty}\frac{\exp(-\alpha_n^2\tau)}{\alpha_n^4}\left\{\frac{J_0(r\alpha_n/a)}{J_0(\alpha_n)} - 1\right\}\right]$$

The maximum deflection is at the centre of the plate and is obtained by putting $r = 0$ in equation (19), so that,

$$(20) \quad \omega_{st_{max}} = \frac{(1+\nu)\,\alpha_t F_0 a^3}{hK}\left[\frac{1}{32} - \frac{\tau}{1+\nu} + 2\sum_{n=1}^{\infty}\frac{\exp(-\alpha_n^2\tau)}{\alpha_n^4}\left\{\frac{1}{J_0(\alpha_n)} - 1\right\}\right]$$

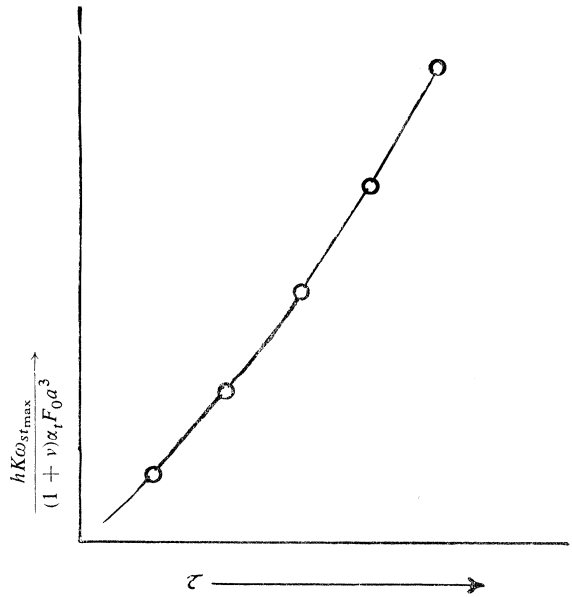

Fig. 2.

For numerical calculations, we write this in the form,

$$(21) \quad \frac{hK\omega_{st_{max}}}{(1+\nu)\,\alpha_t F_0 a^3} = \frac{1}{32} - \frac{\tau}{1+\nu} + 2\sum_{n=1}^{\infty}\frac{\exp(-\alpha_n^2\tau)}{\alpha_n^4}\left(\frac{1}{J_0(\alpha_n)} - 1\right)$$

The values of $[hK\omega_{st_{max}}]/[(1+\nu)\,\alpha_t F_0 a^3]$ for values of $\tau = 0{\cdot}02$, $0{\cdot}04$, $0{\cdot}06$, $0{\cdot}08$, $0{\cdot}10$, are presented in Table II and the corresponding curve in Figure 2.

Table II

0·02	0·04	0·06	0·08	0·10
—·007522	—·017212	—·028203	—·040226	—·053105

6. CONCLUSION

It is to be noted that for a clamped plate with the given temperature distribution, the maximum quasi—static thermal deflection continually increases, with the time parameter. In the case of simply supported plates, the maximum quasi-static thermal deflection continually diminishes as the time parameter increases.

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Výtah

QUASISTATICKÉ TERMICKÉ OHÝBÁNÍ KRUHOVÉ DESKY V OSOVĚ SYMETRICKÉM PŘÍPADĚ

S. K. Sarkar

Na základě Forray-Newmannovy metody je v článku řešena úloha termických ohýbání kruhové desky v osově symetrickém případě. Teplota je uvažována jako funkce souřadnice r a času t.

Резюме

ПОЧТИ-СТАТИЧЕСКОЕ ТЕРМИЧЕСКОЕ ИЗГИБАНИЕ КРУГОВОЙ ДЕСКИ В СЛУЧАЕ ОСЕВОЙ СИММЕТРИИ

С. К. САРКАР (S. K. SARKAR)

На основе метода Форраи и Ньюманна в статье решена проблема термических изгибаний круговой доски в случае осевой симметрии. Температура является функцией координаты r и временного параметра t.

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