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DERIVATION OF NON-CLASSICAL VARIATIONAL PRINCIPLES IN THE THEORY OF ELASTICITY

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1. INTRODUCTION

Methods of approximate solution of boundary-value problems in the mechanics of solids are often based on the variational principles, concerning the extreme (RITZ's and TREFFTZ's method) or merely the stationary value (BUBNOFF-GALERKIN's method) of certain functionals. Hence the effort follows to derive the variational principles also in the newer fields of the mechanics of solids, as e.g. in the geometrically or physically non-linear theory of elasticity, theory of viscoelasticity, theory of plasticity a.s.o.

It is the object of the present paper to suggest a certain scheme for deriving the complete group of variational principles, which is known already in the linear theory of elasticity. This group is composed partly of four fundamental principles: classical principles of the minimum of potential energy (LAGRANGE-DIRICHLET) and that of the minimum of complementary energy (CASTIGLIANO-MENABREA), generalized principles of HU HAI-CHANG-WASHIZU and of REISSNER-HELLINGER, partly of a series of the special variational theorems, following from the generalized principles. An analogous scheme would be possible to use for deriving similar variational principles in the newer branches of mechanics, too.

The group of variational principles, mentioned above, could, however, incite a research of the possibilities to define newly the weak (generalized) solutions of the boundary-value problems in the theory of partial differential equations. These definitions are based (for the elliptic equations and systems) solely on the principle of the minimum of potential energy. A question arises about the suitability of other definition following from some of the further three fundamental principles.

Finally the questions about the convergence of the approximate methods, based on the non-classical variational principles and theorems, stand out. The answer to one of them only – that of the "theorem for boundary conditions" – is presented here in the last section.

Let us consider the mixed boundary-value problem in the classical theory of elasticity for the body, occupying a bounded region Ω of the three-dimensional Euclidean space, having sufficiently smooth boundary Γ. Suppose that

$$\Gamma = \Gamma_u \cup \Gamma_P$$

where Γ_u and Γ_P are two mutually disjoint parts of the boundary. Let the displacements be given on the part Γ_u, i.e.

$$u_i = \bar{u}_i \quad \text{on} \quad \Gamma_u \tag{1}$$

and the surface charge be given on the part Γ_P, i.e.

$$\tau_{ik} n_k = P_i \quad \text{on} \quad \Gamma_P \,, \tag{2}$$

where τ_{ik} are the components of the stress tensor, n_k the components of the unit external normal-vector to Γ and the sums over repeated indices are implied; $i, k = 1, 2, 3$.

For this problem the well-known classical *principle of the minimum of potential energy* (Lagrange-Dirichlet) – see [3], § 26 – holds in the form

$$\mathscr{L} = \text{min.},$$

where

$$\mathscr{L}(u_i, \varepsilon_{ik}) = \int_\Omega \left(\tfrac{1}{2} c_{iklm} \varepsilon_{ik} \varepsilon_{lm} - K_i u_i\right) \mathrm{d}X - \int_{\Gamma_P} P_i u_i \, \mathrm{d}S \,. \tag{3}$$

In (3) the components of the strain tensor are given by

$$\varepsilon_{ik} = \tfrac{1}{2}(u_{i,k} + u_{k,i}) \tag{4}$$

on the class of sufficiently smooth vector-functions $\boldsymbol{u}(X)$ of the displacements, which satisfy the boundary condition (1), c_{iklm} are coefficients of the generalized Hooke's law in the relations

$$\tau_{ik} = c_{iklm} \varepsilon_{lm} \ ^{1)} \tag{5}$$

and $u_{i,k} = \partial u_i / \partial x_k$, K_i are the components of the vector of body forces.

The second classical variational principle is the *principle of the minimum of complementary energy* (Castigliano-Menabrea). It corresponds to the principle of the minimum of potential energy in the following sense: the components of stress-

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[1]) It holds $c_{iklm} = c_{kilm} = c_{lmik}$, $c_{iklm} \varepsilon_{ik} \varepsilon_{lm} \geqq \mu_0 \sum_{i,k=1}^{3} \varepsilon_{ik}^2$ ($\mu_0 = \text{const.} > 0$).

tensor correspond to the components of displacements, the equations of equilibrium to the equations of compatibility and the statical boundary conditions to the geometric boundary conditions. It is possible to derive this principle directly on the base of the positive-definiteness of the quadratic form, expressing the density of the strain energy by means of the stress components (see [5]), or by the method of orthogonal projections in the corresponding Hilbert space (see [3], § 54).[2]) With respect to the further procedure it is suitable, however, to show here the derivation of this principle from the principle of the minimum of potential energy using the Friedrichs' method (see e.g. [1] or [2]). The latter way of derivation is in the linear theory of elasticity more tedious than the method using positive definiteness of the energy, but only the Friedrichs' method is convenient for the non-linear cases of statical and dynamical problems of the theory of elasticity (see [9], [10]).

Let us sum up (4) and (1) as the side conditions with coefficients $\lambda_{ik}(X)$, $\mu_i(X)$ to the functional $\mathscr{L}(u_i, \varepsilon_{ik})$ according to the Lagrange's method of multipliers and let us express the components ε_{ik} everywhere by means of the components τ_{ik} according to the generalized Hooke's law in the form inverse to (5):

$$\varepsilon_{ik} = a_{iklm}\tau_{lm} \,. \tag{5'}$$

We obtain the functional

$$H(u_i, \tau_{ik}, \lambda_{ik}, \mu_i) \equiv \int_\Omega \{\tfrac{1}{2}a_{iklm}\tau_{ik}\tau_{lm} - K_i u_i + \lambda_{ik}[-a_{iklm}\tau_{lm} + \tfrac{1}{2}(u_{i,k} + u_{k,i})]\} \,\mathrm{d}X - \\ - \int_{\Gamma_P} P_i u_i \,\mathrm{d}S + \int_{\Gamma_u} \mu_i(u_i - \bar{u}_i) \,\mathrm{d}S \,,$$

where all variable functions are independent except the relation

$$\lambda_{ik} = \lambda_{ki} \,.$$

Let us form the corresponding variation and use the symmetry

$$a_{iklm} = a_{lmik}$$

and the integration by parts. Hence we obtain

$$\delta H = \int_\Omega \{a_{iklm}(\tau_{lm} - \lambda_{lm})\,\delta\tau_{ik} - (\lambda_{ik,k} + K_i)\,\delta u_i + [-a_{iklm}\tau_{lm} + \\ + \tfrac{1}{2}(u_{i,k} + u_{k,i})]\,\delta\lambda_{ik}\} \,\mathrm{d}X + \int_{\Gamma_u} \{(\mu_i + \lambda_{ik}n_k)\,\delta u_i + (u_i - \bar{u}_i)\,\delta\mu_i\} \,\mathrm{d}S - \\ - \int_{\Gamma_P} (P_i - \lambda_{ik}n_k)\,\delta u_i \,\mathrm{d}S \,.$$

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[2]) The method of orthogonal projections was used by the author to derive an analogous principle in the linear viscoelasticity (see [4]).

Now choose among the conditions which follow from $\delta H = 0$ only the following conditions:

(6) $$a_{iklm}(\tau_{lm} - \lambda_{lm}) = 0 \quad \text{on} \quad \Omega\,,$$

(7) $$\lambda_{ik,k} + K_i = 0 \quad \text{on} \quad \Omega\,,$$

(8) $$\mu_i + \lambda_{ik} n_k = 0 \quad \text{on} \quad \Gamma_u\,,$$

(9) $$-\lambda_{ik} n_k + P_i = 0 \quad \text{on} \quad \Gamma_P\,.$$

We have omitted just the conditions (1) and (4). The condition (6) implies, using that the matrix a_{iklm} (considered e.g. with double subscripts $[ik]$, $[lm]$) is regular,

(10) $$\lambda_{ik} = \tau_{ik}\,.$$

Insert (7), (8), (10) into $H(u_i, \tau_{ik}, \lambda_{ik}, \mu_i)$. We are led to a new functional

$$H_1(u_i, \lambda_{ik}) \equiv \int_\Omega \{\tfrac{1}{2} a_{iklm} \lambda_{ik} \lambda_{lm} + u_i \lambda_{ik,k} - a_{iklm} \lambda_{ik} \lambda_{lm} + $$
$$+ \tfrac{1}{2} \lambda_{ik}(u_{i,k} + u_{k,i})\}\, \mathrm{d}X + \int_{\Gamma_P} - P_i u_i \,\mathrm{d}S + \int_{\Gamma_u} - \lambda_{ik} n_k (u_i - \bar{u}_i)\,\mathrm{d}S\,.$$

Integrating by parts

$$\int_\Omega u_i \lambda_{ik,k}\,\mathrm{d}X = -\int_\Omega u_{i,k} \lambda_{ik}\,\mathrm{d}X + \int_\Gamma u_i \lambda_{ik} n_k\,\mathrm{d}S\,,$$

we obtain

$$H_1(u_i, \lambda_{ik}) \equiv -\int_\Omega \tfrac{1}{2} a_{iklm} \lambda_{ik} \lambda_{lm}\,\mathrm{d}X + \int_{\Gamma_P} (-P_i + \lambda_{ik} n_k)\, u_i\,\mathrm{d}S +$$
$$+ \int_{\Gamma_u} \lambda_{ik} n_k \bar{u}_i\,\mathrm{d}S\,.$$

Using (9) the functional becomes

(11) $$H_1(u_i, \lambda_{ik}) \equiv \mathscr{S}_1(\lambda_{ik}) \equiv -\int_\Omega \tfrac{1}{2} a_{iklm} \lambda_{ik} \lambda_{lm}\,\mathrm{d}X + \int_{\Gamma_u} \lambda_{ik} n_k \bar{u}_i\,\mathrm{d}S\,.$$

It holds the following

Lemma. *If the problem*

$$\mathscr{L}(u_i, \varepsilon_{ik}) = \min\,,$$

with the side conditions (1) *and* (4) *has the solution* $\mathring{u}_i$, $\mathring{\varepsilon}_{ik}$, *for which* $\mathscr{L}(\mathring{u}_i, \mathring{\varepsilon}_{ik}) = d$, *then the dual problem*

$$\mathscr{S}_1(\lambda_{ik}) = \max$$

with the side conditions (7), (9) *has a solution* $\mathring{\lambda}_{ik}$ *with the same extreme value*

$$\mathscr{S}_1(\mathring{\lambda}_{ik}) = d\,,$$

where

(12) $$\mathring{\lambda}_{ik} = \tfrac{1}{2}c_{iklm}(\mathring{u}_{i,k} + \mathring{u}_{k,i})\,.$$

Proof. The problem $\mathscr{L}(u_i, \varepsilon_{ik}) = \min.$, (1), (4) admits the representation

$$\mathscr{L}'(u_i, \tau_{ik}) \equiv \mathscr{L}(u_i, \varepsilon_{ik}(\tau_{ik})) = \min$$

with the side conditions (1) and

(4') $$a_{iklm}\tau_{lm} = \tfrac{1}{2}(u_{i,k} + u_{k,i})$$

expressing the components of strain by means of the components of stress according to (5').

It is well-known from the theory of the extrema with side conditions, that there exist functions $\mathring{\lambda}_{ik}$, $\mathring{\mu}_i$ such, that the solution $\mathring{u}_i$, $\mathring{\tau}_{ik}$ of this problem together with $\mathring{\lambda}_{ik}$, $\mathring{\mu}_i$ satisfy all Euler's equations and the natural boundary conditions (1), (4'), (6) to (9) of the problem $\delta H = 0$ without side conditions.[3]) For these functions all transformations are valid, on the base of equations (6) to (9), by means of which the functional H was transformed into $\mathscr{S}_1(\lambda_{ik})$ in (11). Thus we have

$$H(\mathring{u}_i, \mathring{\tau}_{ik}, \mathring{\lambda}_{ik}, \mathring{\mu}_i) = H_1(\mathring{u}_i, \mathring{\lambda}_{ik}) = \mathscr{S}_1(\mathring{\lambda}_{ik})\,.$$

On the other hand, since the conditions (1), (4') and the relations (5) are fulfilled, too, it holds

$$H(\mathring{u}_i, \mathring{\tau}_{ik}, \mathring{\lambda}_{ik}, \mathring{\mu}_i) = \mathscr{L}'(\mathring{u}_i, \mathring{\tau}_{ik}) = \mathscr{L}(\mathring{u}_i, \mathring{\varepsilon}_{ik}) = d\,.$$

Consequently

$$\mathscr{S}_1(\mathring{\lambda}_{ik}) = d\,.$$

Now choose fixed functions

$$\lambda_{ik}(X) = \bar{\lambda}_{ik}(X)\,, \quad \mu_i(X) = \bar{\mu}_i(X)\,.$$

Suppose, that for each $\bar{\lambda}_{ik}$, $\bar{\mu}_i$ from a certain neighbourhood of $\mathring{\lambda}_{ik}$, $\mathring{\mu}_i$ there exist functions $\bar{\mu}_i$, $\bar{\tau}_{ik}$, which minimize the functional $H(u_i, \tau_{ik}, \bar{\lambda}_{ik}, \bar{\mu}_i)$.[4])

Then it holds obviously

$$\min_{u_i,\tau_{ik}} H(u_i, \tau_{ik}, \bar{\lambda}_{ik}, \bar{\mu}_i) \leqq \min_{(1),(4')} H(u_i, \tau_{ik}, \bar{\lambda}_{ik}, \bar{\mu}_i) = \min_{(1),(4')} \mathscr{L}'(u_i, \tau_{ik}) =$$
$$= \min_{(1),(4)} \mathscr{L}(u_i, \varepsilon_{ik}) = d\,,$$

[3]) For the side conditions which involve partial derivatives, however, such an assertion probably was not yet proved, but it can be doubtless accepted — see [1], I., chapt. IV., § 7/3.

[4]) This assumption will be accepted without any further comment in the next theorem, too.

where (1), (4′) or (1), (4) means, that the minimum is bounded with the side conditions (1), (4′) or (1), (4) respectively.

At the same time we may write

$$\min_{u_i,\tau_{ik}} H(u_i, \tau_{ik}, \bar{\lambda}_{ik}, \bar{\mu}_i) = H(\bar{u}_i, \bar{\tau}_{ik}, \bar{\lambda}_{ik}, \bar{\mu}_i) = H_1(\bar{\mu}_i, \bar{\lambda}_{ik}) = \mathscr{S}_1(\bar{\lambda}_{ik}),$$

because for derivation of (11) only conditions (6) to (9) were used and the latter are satisfied by the minimizing functions. Consequently

$$\mathscr{S}_1(\bar{\lambda}_{ik}) \leqq d$$

for any $\bar{\lambda}_{ik}$ from the neighbourhood of $\mathring{\lambda}_{ik}$ mentioned above, which meet (7) and (9). The relations (10) for $\mathring{\lambda}_{ik}$, $\mathring{\tau}_{ik}$ and (5) imply (12) and the proof is completed.

If we substitute in (11) λ_{ik} according to (10) by the components of stress τ_{ik} and change the sign of the functional, we obtain the *principle of the minimum of complementary energy* (Castigliano-Menabrea) in the form of the variational problem

$$\mathscr{S}(\tau_{ik}) = -\mathscr{S}_1(\tau_{ik}) = \frac{1}{2}\int_\Omega a_{iklm}\tau_{ik}\tau_{lm}\,\mathrm{d}X - \int_{\Gamma_u} \tau_{ik}n_k\bar{u}_i\,\mathrm{d}S = \min \tag{13}$$

with side conditions

$$\tau_{ik,k} + K_i = 0 \quad \text{on} \quad \Omega\,, \tag{14}$$

$$\tau_{ik}n_k = P_i \quad \text{on} \quad \Gamma_P\,. \tag{15}$$

The lemma implies immediately a

Theorem. *Let the problem* $\mathscr{L}(u_i, \varepsilon_{ik}) = \min$ *with side conditions* (1), (4) *has the solution* $\mathring{u}_i$, $\mathring{\varepsilon}_{ik}$ *for which* $\mathscr{L}(\mathring{u}_i, \mathring{\varepsilon}_{ik}) = d$. *Then the dual Castigliano's variational problem* $\mathscr{S}(\tau_{ik}) = \min$ *with the side conditions* (14) *and* (15) *has a solution* $\mathring{\tau}_{ik}$, *corresponding to* $\mathring{u}_i$ *by the relation*

$$\tau_{ik} = \tfrac{1}{2}c_{iklm}(\mathring{u}_{l,m} + \mathring{u}_{m,l})\,.$$

Remark. We shall mention the Castigliano's principle once more later in connection which the principle of Reissner-Hellinger.

3. GENERALIZED PRINCIPLES

Recently, in the fifties, new variational principles, applicable to the theory of elasticity, generalizing the classical principles of minimal potential or complementary energy, were suggested. These are the principle of Hu Hai-Chang [6] – Washizu [7] and the principle of Hellinger-Reissner [8]. We shall show here, that both these principle may be derived from the classical principles using the analogous method as that used for derivation of Castigliano's principle from the principle of Lagrange-Dirichlet.

3.1 Principle of Hu Hai-Chang and Washizu

Let us add the conditions (1) and (4) to the functional $\mathscr{L}(u_i, \varepsilon_{ik})$ by means of coefficients λ_{ik}, μ_i in the same way as in the previous section, but keeping here the original expression by means of the strain components ε_{ik}. The new functional has the form

(16) $$\mathscr{J}_1(u_i, \varepsilon_{ik}, \lambda_{ik}, \mu_i) =$$

$$= \int_\Omega \{\tfrac{1}{2}c_{iklm}\varepsilon_{ik}\varepsilon_{lm} - K_i u_i + \lambda_{ik}[-\varepsilon_{ik} + \tfrac{1}{2}(u_{i,k} + u_{k,i})]\} \, \mathrm{d}X -$$

$$- \int_{\Gamma_P} P_i u_i \, \mathrm{d}S + \int_{\Gamma_u} \mu_i(u_i - \bar{u}_i) \, \mathrm{d}S \,,$$

where all variable functions are mutually independent, $\lambda_{ik} = \lambda_{ki}$. Integrating by parts, we obtain for the variation $\delta\mathscr{J}_1$

$$\delta\mathscr{J}_1 = \int_\Omega \{(c_{iklm}\varepsilon_{lm} - \lambda_{ik}) \, \delta\varepsilon_{ik} - (K_i + \lambda_{ik,k}) \, \delta u_i +$$

$$+ [\tfrac{1}{2}(u_{i,k} + u_{k,i}) - \varepsilon_{ik}] \, \delta\lambda_{ik}\} \, \mathrm{d}X + \int_{\Gamma_P} (-P_i + \lambda_{ik}n_k) \, \delta u_i \, \mathrm{d}S +$$

$$+ \int_{\Gamma_u} \{(\mu_i + \lambda_{ik}n_k) \, \delta u_i + (u_i - \bar{u}_i) \, \delta\mu_i\} \, \mathrm{d}S \,.$$

If we set $\delta\mathscr{J}_1 = 0$, then the following must hold:

(17) $$\lambda_{ik} = c_{iklm}\varepsilon_{lm}$$

and (7), (8), (9), (1) and (4). By virtue of (17), it is obvious, that λ_{ik} has the sense of the stress component τ_{ik}. Substituting $\lambda_{ik} = \tau_{ik}$ and according to (8) $\mu_i = -\tau_{ik}n_k$ into (16), we derive the functional

(18) $$\mathscr{J}(u_i, \varepsilon_{ik}, \tau_{ik}) = \int_\Omega \{\tfrac{1}{2}c_{iklm}\varepsilon_{ik}\varepsilon_{lm} - K_i u_i - \tau_{ik}\varepsilon_{ik} + \tfrac{1}{2}\tau_{ik}(u_{i,k} + u_{k,i})\} \, \mathrm{d}X -$$

$$- \int_{\Gamma_P} P_i u_i \, \mathrm{d}S + \int_{\Gamma_u} \tau_{ik}n_k(\bar{u}_i - u_i) \, \mathrm{d}S \,,$$

which was suggested by Hu Hai-Chang in [6] and by Washizu in [7]. Hu Hai-Chang calls the principle $\delta\mathscr{J} = 0$ the principle of the generalized potential energy.

Using the integration by parts for the term $\tfrac{1}{2}\tau_{ik}(u_{i,k} + u_{k,i})$, the functional admits the alternative representation

$$(18') \qquad \tilde{\mathscr{J}}(u_i, \varepsilon_{ik}, \tau_{ik}) = \int_\Omega \left(\tfrac{1}{2}c_{iklm}\varepsilon_{ik}\varepsilon_{lm} - K_i u_i - \tau_{ik}\varepsilon_{ik} - \tau_{ik,k}u_i\right) \mathrm{d}X + $$
$$+ \int_{\Gamma_P} (\tau_{ik}n_k - P_i)\, u_i \,\mathrm{d}S + \int_{\Gamma_u} \tau_{ik}n_k\bar{u}_i \,\mathrm{d}S .$$

Hu Hai-Chang calls the principle $\delta\tilde{\mathscr{J}} = 0$ the principle of the generalized complementary energy.

In the equation $\delta\mathscr{J} = 0$ or $\delta\tilde{\mathscr{J}} = 0$ respectively altogether 15 independent functions are varied: 3 components of displacements, 6 components of stress and 6 components of strain tensor. The resulting Euler's equations are the relation of the generalized Hooke's law, furthermore (4) and (14) on the region Ω. The natural boundary conditions are (1) on Γ_u and (15) on Γ_P.

3.2 Principle of Hellinger-Reissner

Proceed similarly to the derivation of Hu Hai-Chang's principle, but start with the dual Castigliano's principle. Add the side conditions (14) and (15) to the functional $\mathscr{S}(\tau_{ik})$ by means of multipliers λ_i, μ_i. We obtain the functional

$$(19) \qquad \mathscr{R}_1(\tau_{ik}, \lambda_i, \mu_i) \equiv \int_\Omega \left\{\tfrac{1}{2}a_{iklm}\tau_{ik}\tau_{lm} + \lambda_i(\tau_{ik,k} + K_i)\right\} \mathrm{d}X - $$
$$- \int_{\Gamma_u} \tau_{ik}n_k\bar{u}_i \,\mathrm{d}S + \int_{\Gamma_P} (\tau_{ik}n_k - P_i)\, \mu_i \,\mathrm{d}S ,$$

where all variable functions are mutually independent. Integration by parts leads in the variation $\delta\mathscr{R}_1$ to

$$\delta\mathscr{R}_1 = \int_\Omega \left\{\left[a_{iklm}\tau_{lm} - \tfrac{1}{2}(\lambda_{i,k} + \lambda_{k,i})\right] \delta\tau_{ik} + (\tau_{ik,k} + K_i)\, \delta\lambda_i\right\} \mathrm{d}X + $$
$$+ \int_{\Gamma_u} n_k(-\bar{u}_i + \lambda_i)\, \delta\tau_{ik} \,\mathrm{d}S + \int_{\Gamma_P} \left\{(\mu_i + \lambda_i)\, n_k\delta\tau_{ik} + (\tau_{ik}n_k - P_i)\, \delta\mu_i\right\} \mathrm{d}S .$$

If we set $\delta\mathscr{R}_1 = 0$, then besides (14) and (15)

$$(20) \qquad a_{iklm}\tau_{lm} = \tfrac{1}{2}(\lambda_{i,k} + \lambda_{k,i}) \quad \text{on} \quad \Omega ,$$

$$(20') \qquad \lambda_i = \bar{u}_i \quad \text{on} \quad \Gamma_u ,$$

$$(21) \qquad \mu_i = -\lambda_i \quad \text{on} \quad \Gamma_P$$

must hold. From the relations (20), (20') we conclude that λ_i have the sense of the components of displacements, provided the generalized Hooke's law holds. By substituting $\lambda_i = u_i$ and according to (21) $\mu_i = -u_i$ into (19) and changing the sign, we obtain the functional

$$\tilde{\mathscr{R}}(u_i, \tau_{ik}) = \int_\Omega \{-\tfrac{1}{2}a_{iklm}\tau_{ik}\tau_{lm} - \tau_{ik,k}u_i - K_i u_i\}\,\mathrm{d}X + \\ + \int_{\Gamma_u} \tau_{ik}n_k\bar{u}_i\,\mathrm{d}S + \int_{\Gamma_P} (\tau_{ik}n_k - P_i)\,u_i\,\mathrm{d}S\,. \tag{22'}$$

Finally, integrating the second term by parts, we derive the functional

$$\mathscr{R}(u_i, \tau_{ik}) = \int_\Omega \{-\tfrac{1}{2}a_{iklm}\tau_{ik}\tau_{lm} + \tfrac{1}{2}(u_{i,k} + u_{k,i})\,\tau_{ik} - K_i u_i\}\,\mathrm{d}X + \\ + \int_{\Gamma_u} \tau_{ik}n_k(\bar{u}_i - u_i)\,\mathrm{d}S - \int_{\Gamma_P} P_i u_i\,\mathrm{d}S\,, \tag{22}$$

which was suggested by HELLINGER and REISSNER (see e.g. [8]). In the equation $\delta\tilde{\mathscr{R}} = 0$ or $\delta\mathscr{R} = 0$ respectively, altogether 9 independent functions are varied: 3 components of displacements and 6 components of stress. In contradistinction to the Hu Hai-Chang's principle here either (A) the relation (4) between strain and displacements or (B) the Hooke's generalized law (5′) is supposed "a priori". The resulting Euler's conditions are

$$\tfrac{1}{2}(u_{i,k} + u_{k,i}) = a_{iklm}\tau_{lm}\,, \tag{23}$$

(which have to be comprehended in the case (A) as relations (5′) of the generalized Hooke's law or, in case (B), as the strain-displacement relations (4)), furthermore the equations of equilibrium (14) and the boundary conditions (1), (15).

Remark 1. At the same time we have derived, that the Eulers' conditions, corresponding to the Castigliano's principle, are the relations (20) and (20′). The equations (20) involve the generalized Hooke's law (5′) and the strain-displacement relations (4). They are equivalent to the assertion, that there exists a vector-function of displacements such, that from its gradient the strain components are formed according to (23).

Remark 2. The Reissner's functional $\mathscr{R}(u_i, \tau_{ik})$ follows directly from the Hu Hai-Chang's functional $\mathscr{J}(u_i, \varepsilon_{ik}, \tau_{ik})$ by the elimination of ε_{ik} according to the generalized Hooke's law (5), which may be, as we have just mentioned, supposed "a priori" for the Reissner's principle. It is impossible to derive $\mathscr{R}(u_i, \tau_{ik})$ from $\mathscr{J}(u_i, \varepsilon_{ik}, \tau_{ik})$ using the second starting assumption-(A), i.e. the strain-displacements relations (4).

4. SPECIAL VARIATIONAL THEOREMS

From the generalized principles, mentioned above, it is easy to derive not only the classical principles of the minimum of potential or complementary energy (see

[8]) but also a series of the variational theorems, which may be useful in some particular problems. These are

Theorem 4.1. *"For the boundary conditions", see* [8]. *From the relation*

$$\delta\mathscr{R}_2(u_i, \tau_{ik}) = 0\,,$$

where

$$\mathscr{R}_2(u_i, \tau_{ik}) = \int_{\Gamma_P} u_i(\tfrac{1}{2}\tau_{ik}n_k - P_i)\,\mathrm{d}S - \int_{\Gamma_u} \tau_{ik}n_k(\tfrac{1}{2}u_i - \bar{u}_i)\,\mathrm{d}S$$

and u_i, τ_{ik} satisfy on Ω the equations (4), (5), (14), *the boundary conditions* (1), (15) *follow as Euler's conditions.*

Further, it is possible from the Reissner's principle by substituting the equations of equilibrium to derive the

Theorem 4.2. *Suppose* (A) *the relations* (4) *or* (B) (5') *hold. Then from the condition*

$$\delta\mathscr{S}_2(u_i, \tau_{ik}) = 0\,,$$

where

$$\mathscr{S}_2(u_i, \tau_{ik}) = \mathscr{S}(\tau_{ik}) + \int_{\Gamma_P} u_i(P_i - \tau_{ik}n_k)\,\mathrm{d}S$$

and τ_{ik} satisfy the the equations of equilibrium (14) *on Ω,* (A) *equations* (5) *or* (B) *equations* (4) *on Ω and both the boundary conditions* (1) *and* (15) *follow as Euler's conditions.*

From the Hu Hai-Chang's principle by substitution of some side conditions we may derive the following theorems:

Theorem 4.3. *From the condition*

$$\delta\mathscr{S}_3(\varepsilon_{ik}, \tau_{ik}) = 0\,,$$

where

$$\mathscr{S}_3(\varepsilon_{ik}, \tau_{ik}) = \int_{\Omega} \{\tfrac{1}{2}c_{iklm}\varepsilon_{ik}\varepsilon_{lm} - \tau_{ik}\varepsilon_{ik}\}\,\mathrm{d}X + \int_{\Gamma_u} \tau_{ik}n_k\bar{u}_i\,\mathrm{d}S$$

and τ_{ik} satisfy the equations of equilibrium on Ω and the boundary conditions (15) *on Γ_P, the relations* (4), (5) *and the boundary conditions* (1) *follow as the Euler's conditions.*

Theorem 4.4. *From the condition*

$$\delta\mathscr{S}_4(\varepsilon_{ik}, \tau_{ik}) = 0\,,$$

where

$$\mathscr{S}_4(\varepsilon_{ik}, \tau_{ik}) = \mathscr{S}_3(\varepsilon_{ik}, \tau_{ik}) + \int_{\Gamma_P} (\tau_{ik} n_k - P_i)\, u_i \,\mathrm{d}S$$

and τ_{ik} *satisfy the equations of equilibrium* (14) *on* Ω, *the equations* (4), (5) *and the boundary conditions* (1), (15) *follow as Euler's conditions.*

If we substitute (4), (5) into the Hu Hai-Chang's functional (18), or insert the relation (4) or (5) respectively into the Reissner's functional (22), we obtain the functional

$$(24) \qquad \mathscr{L}_1(u_i, \varepsilon_{ik}) = \int_\Omega \{\tfrac{1}{2} c_{iklm} \varepsilon_{ik} \varepsilon_{lm} - K_i u_i\} \,\mathrm{d}X + \\ + \int_{\Gamma_u} c_{iklm} \varepsilon_{lm} n_k (\bar{u}_i - u_i) \,\mathrm{d}S - \int_{\Gamma_P} P_i u_i \,\mathrm{d}S = \mathscr{L}(u_i, \varepsilon_{ik}) + \\ + \int_{\Gamma_u} c_{iklm} \varepsilon_{lm} n_k (\bar{u}_i - u_i) \,\mathrm{d}S\,.$$

Hence it follows the

Theorem 4.5. *From the condition* $\delta \mathscr{L}_1(u_i, \varepsilon_{ik}) = 0$, *where* $\mathscr{L}_1(u_i, \varepsilon_{ik})$ *is defined through* (24) *and* u_i, ε_{ik} *satisfy* (4), *the equations of equilibrium* (14) *and both the boundary conditions* (1) *and* (15) *follow as Euler's conditions.*

5. PROOF OF CONVERGENCE OF THE APPROXIMATE SOLUTION ACCORDING TO THE THEOREM 4.1

Let us consider the boundary-value problem for prescribed tractions on the whole boundary and for zero body forces. Let the boundary Γ consists of a finite number of regular (smooth) surfaces. The equations of equilibrium may be written in the form

$$(25) \qquad (c_{iklm} u_{l,m})_{,k} = 0 \quad \text{on} \quad \Omega$$

and the boundary conditions (2) as

$$(26) \qquad c_{iklm} u_{l,m} n_k = P_i \quad \text{on} \quad \Gamma \\ (\Gamma_P = \Gamma, \Gamma_u = \emptyset, K_i \equiv 0)\,.$$

Let the conditions of statical equilibrium of the whole body, namely

$$\int_\Gamma P_i \,\mathrm{d}S = 0\,, \quad \int_\Gamma \boldsymbol{r} \times \boldsymbol{P} \,\mathrm{d}S = 0$$

be satisfied. We shall subject the displacements $\boldsymbol{u}$ to additional conditions, which exclude the possibility of small rigid body motions, (see [3], § 26), i.e. assume

$$\int_\Omega \boldsymbol{u} \, \mathrm{d}X = \int_\Omega \operatorname{rot} \boldsymbol{u} \, \mathrm{d}X = 0 . \tag{27}$$

Consider the linear manifold M of vector-functions $\boldsymbol{u}(X)$ with all the components $u_i \in C^{(1)}(\overline{\Omega})$ i.e. with continuous partial derivatives of the first order on Ω, extendible continuously on Γ, satisfying the equations (25) and (27). Let

$$(\boldsymbol{u}, \boldsymbol{v}) \equiv \int_\Gamma u_i v_i \, \mathrm{d}S \,, \quad \boldsymbol{u} \in M \,, \quad \boldsymbol{v} \in M$$

be the scalar product on M. This definition is justified, because the corresponding Dirichlet's problem ($u \in M$, $\boldsymbol{u} = 0$ on Γ) has only trivial solution.

Completing M in the associated norm, a Hilbert space arises, which may be interpreted as a subspace of $L_2(\Gamma)$, i.e. as a subspace of the space of vector-functions with each component square-integrable on Γ. Let us define the operator A through

$$(A\boldsymbol{u})_i = c_{iklm} u_{l,m} n_k \,,$$

i.e. mapping M into $L_2(\Gamma)$. Then, because of the integration by parts together with (25),

$$(A\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma (A\boldsymbol{u})_i \, v_i \, \mathrm{d}S = \int_\Gamma c_{iklm} n_k u_{l,m} v_i \, \mathrm{d}S = \int_\Omega c_{iklm} u_{l,m} v_{i,k} \, \mathrm{d}X = (\boldsymbol{u}, A\boldsymbol{v}) \,,$$

which implies that the operator A is symmetric. Moreover,

$$(A\boldsymbol{u}, \boldsymbol{u}) = \int_\Omega c_{iklm} u_{l,m} u_{i,k} \, \mathrm{d}X = \int_\Omega c_{iklm} \varepsilon_{ik} \varepsilon_{lm} \, \mathrm{d}X \geqq 0 \quad \text{on} \quad M \,, \tag{28}$$

because the density of the strain energy is a positive definite quadratic form of the strain components (see e.g. [11] § 39). Hence the operator is positive.

Assume there exists a (classical) solution $\boldsymbol{u}_0 \in M$ of the problem (25) to (27). Then it holds, that

$$(A\boldsymbol{u}_0, \boldsymbol{u}_0) - 2(\boldsymbol{P}, \boldsymbol{u}_0) = \min_{\boldsymbol{u} \in M} [(A\boldsymbol{u}, \boldsymbol{u}) - 2(\boldsymbol{P}, \boldsymbol{u})] \,,$$

i.e.

$$\mathscr{F}(\boldsymbol{u}) \equiv \int_\Gamma (\tfrac{1}{2} c_{iklm} n_k u_{l,m} u_i - P_i u_i) \, \mathrm{d}S = \min. \quad \text{for} \quad \boldsymbol{u} = \boldsymbol{u}_0 \,,$$

which corresponds exactly to the functional $\mathscr{R}_2(u_i, \tau_{ik})$ if $\Gamma_u = \emptyset$. In this case the condition $\delta \mathscr{R}_2 = 0$ of the theorem 4.1 expresses the condition of the minimal value and each minimizing sequence (constructed e.g. by means of Ritz's or some other method) converges to the solution $\boldsymbol{u}_0$ in the norm, derived by extension of the product $(A\boldsymbol{u}, \boldsymbol{u})$. By virtue of (28) and of the Korn's and Poincaré's inequalities, holding for the functions from M (see [11] § 42), it is easy to prove that the sequence converges even in the space $[W_2^{(1)}(\Omega)]^3$ [5]), i.e.

$$|\boldsymbol{u}_n - \boldsymbol{u}_0|^2_{[W_2^{(1)}(\Omega)]^3} = \sum_{i=1}^{3} \int_\Omega (u_{ni} - u_{0i})^2 \, \mathrm{d}X + \sum_{i,k=1}^{3} \int_\Omega (u_{ni,k} - u_{0i,k})^2 \, \mathrm{d}X \to 0 \, .$$

Remark. The method of proof used here is analogous to the "method of minimal surface integrals" as was presented by Michlin in [3], § 47.

6. APPENDIX- SURVEY OF VARIATIONAL PRINCIPLES AND THEOREMS

We shall try to sketch a systematic survey of all variational theorems and principles, mentioned above. In the following schedule we use the notation:

a ... for the relations, supposed "a priori",
E ... for the relations, following from the variational theorems as Euler's conditions.

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[5]) I.e. in the space of vector-functions each component of which has all first partial derivatives in the generalized sense, these components and all first derivatives being square-integrable in Ω.

<table>
<tr><th rowspan="2">Principle or theorem</th><th rowspan="2">Notation of the functional</th><th colspan="3">Relations, holding in Ω</th><th colspan="2">Boundary conditions</th></tr>
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<tr><td>Hu Hai-Chang, Washizu</td><td>$\mathcal{J}, \tilde{\mathcal{J}}$</td><td>E</td><td>E</td><td>E</td><td>E</td><td>E</td></tr>
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<tr><td>Lagrange, Dirichlet</td><td>$\mathcal{L}$</td><td>a</td><td>a</td><td>E</td><td>a</td><td>E</td></tr>
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<tr><td>Theorem 4·1</td><td>$\mathcal{R}_2$</td><td>a</td><td>a</td><td>a</td><td>E</td><td>E</td></tr>
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<tr><td>E</td><td>a</td></tr>
<tr><td>Theorem 4.3</td><td>$\mathcal{S}_3$</td><td>E</td><td>E</td><td>a</td><td>E</td><td>a</td></tr>
<tr><td>Theorem 4·4</td><td>$\mathcal{S}_4$</td><td>E</td><td>E</td><td>a</td><td>E</td><td>E</td></tr>
<tr><td>Theorem 4·5</td><td>$\mathcal{L}_1$</td><td>a</td><td>a</td><td>E</td><td>E</td><td>E</td></tr>
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Výtah

ODVOZENÍ NEKLASICKÝCH VARIAČNÍCH PRINCIPŮ V TEORII PRUŽNOSTI

IVAN HLAVÁČEK

Zobecněné principy, navržené Hu Hai-Changem a Washizu, resp. Hellingerem a Reissnerem, jsou v článku odvozeny z klasických principů minima potenciální resp. doplňkové energie. Dále je podán přehled speciálních variačních vět, které plynou z obecných principů a důkaz konvergence pro metodu založenou na jedné z nich.

Резюме

ВЫВОД НЕКЛАССИЧЕСКИХ ВАРИАЦИОННЫХ ПРИНЦИПОВ В ТЕОРИИ УПРУГОСТИ

ИВАН ГЛАВАЧЕК (IVAN HLAVÁČEK)

В статье выведены обобщенные вариационные принципы, предложенные Ху Хай-Чангом, Вашизу и Рейснером, из классических принципов минимума потенциальной или комплементарной энергии. Дальше приведен обзор специальных вариационных теорем, которые следуют из общих принципов, и доказательство сходимости для метода, обоснованного на одной из этих теорем.

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