

Aplikace matematiky

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Aplikace matematiky, Vol. 11 (1966), No. 6, 427–451

Persistent URL: <http://dml.cz/dmlcz/103052>

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ON FACTORIZATION METHOD

Jiří TAUFER

(Received September 20, 1965.)

1. INTRODUCTION

The problem of finding a solution of system of linear differential equations which satisfies given boundary conditions often occurs in many physical and technical fields.

The factorization method is one of very effective methods for the solution of this problem. The main advantages of factorization method are as follows:

1. The errors made in numerical realization of an algorithm can be represented as the errors of coefficients of the original problem.

2. The initial value problems are solved instead of boundary value problems.

An example of the factorization method:

Consider the following problem

$$(1.1) \quad \begin{aligned} y''(t) - p(t) y(t) &= q(t), \\ y'(a) &= \alpha_0 y(a) + \alpha_1, \\ y'(b) &= \beta_0 y(b) + \beta_1. \end{aligned}$$

By factorization method we get the solution in the following manner: first we solve the equations

$$(1.2) \quad \alpha'_0(t) + \alpha_0^2(t) = p(t),$$

$$(1.3) \quad \alpha'_1(t) + \alpha_0(t) \alpha_1(t) = q(t),$$

with initial conditions

$$(1.4) \quad \alpha_0(a) = \alpha_0, \quad \alpha_1(a) = \alpha_1.$$

Then we obtain the solution of problem (1.1) by solving equation (from the right to the left)

$$(1.5) \quad y'(t) = \alpha_0(t) y(t) + \alpha_1(t),$$

with initial condition

$$(1.6) \quad y(b) = \frac{\beta_1 - \alpha_1(b)}{\alpha_0(b) - \beta_0}.$$

As it will be shown, if Riccati equation (1.2) has a solution in the whole interval $\langle a, b \rangle$ and the problem (1.1) has a unique solution, then $\alpha_0(b) \neq \beta_0$. The cases in which the problem (1.1) has a unique solution and the equation (1.2) has not a solution in the whole interval $\langle a, b \rangle$ are not excluded.

Remark. The factorization method originated in this way: The differential operator of the n -th order is decomposed into differential operators of the first order — so called factors (hence the name). The technique mentioned above can be explained in the following manner: the solution of operator equation $Ly = q$ is to be found (it corresponds to the problem (1.1)); we find the operators L_1 and L_2 (in our case by means of the equation (1.2)) so that $L_1(L_2y) = Ly$ is valid. We solve equation $L_1\alpha_1 = q$ (it corresponds to the equation (1.3)) and then equation $L_2y = \alpha_1$ (it corresponds to the equation (1.5)). The term “factorization method” has been transferred to some difference methods as well.

The aim of this paper is:

1. To generalize this method for the boundary value problem of system of differential equations

$$(1.7) \quad \mathbf{x}'(t) + \mathbf{A}(t) \mathbf{x}(t) = \mathbf{f}(t)$$

(where \mathbf{x} and \mathbf{f} are vectors and \mathbf{A} is a matrix).

2. To achieve that the factorization method should lead to the result providing that the problem itself has a unique solution (applied to the problem (1.1) it means to remove the difficulty caused by the fact that Riccati equation (1.2) need not have a solution in the whole interval $\langle a, b \rangle$).

3. To develop this method so that it might be applied to the problems in which the elements of the matrix \mathbf{A} and the components of the vector \mathbf{f} are generalized functions of the first order.

2. FORMULATION OF THE PROBLEM

In this paragraph, the boundary value problem for the system of linear differential equations with the linear interface conditions is formulated.

Let there be given:

1. a sequence of numbers $M = \{\gamma_i\}_{i=0}^{r+1}$ such that $a = \gamma_0 < \gamma_1 < \dots < \gamma_r < \gamma_{r+1} = b$;
2. a constant matrix \mathbf{U}_1 of type (n_1, N) and of rank n_1 (where $1 \leq n_1 < N$);
3. a constant matrix \mathbf{U}_2 of type (n_2, N) and of rank n_2 (where $n_2 = N - n_1$);
4. a constant vector \mathbf{u}_1 with n_1 components;
5. a constant vector \mathbf{u}_2 with n_2 components;

6. constant non-singular matrices \mathbf{W}_i of type (N, N) for $i = 1, 2, \dots, r$;
7. constant vectors \mathbf{w}_i with N components for $i = 1, 2, \dots, r$;
8. a matrix $\mathbf{A}(t) = \{A_{ij}(t)\}$ of type (N, N) such that $A_{ij}(t) \in \mathcal{L}\langle a, b \rangle$ for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, N$;
($\mathcal{L}\langle a, b \rangle$ denotes the class of functions integrable in Lebesgue sense in the interval $\langle a, b \rangle$);
9. a vector $\mathbf{f}(t) = (f_1(t), f_2(t), \dots, f_N(t))^T$ such that $f_i(t) \in \mathcal{L}\langle a, b \rangle$ for $i = 1, 2, \dots, N$.

Further, we shall deal with the problem which is introduced in the following definition.

Definition 2.1. *The following problem is called "problem ψ ":*

an $\mathbf{x}(t)$ is to be found for which it holds:

- a) $\mathbf{x}'(t) + \mathbf{A}(t)\mathbf{x}(t) = \mathbf{f}(t)$ almost everywhere on $\langle a, b \rangle$;
- b) $\mathbf{U}_1 \mathbf{x}^+(a) = \mathbf{u}_1$, $\mathbf{U}_2 \mathbf{x}^-(b) = \mathbf{u}_2$;
- c) $\mathbf{x}(t)$ has discontinuities at most of the first type;
- d) $\mathbf{x}^-(\gamma_i) = \mathbf{W}_i \mathbf{x}^+(\gamma_i) + \mathbf{w}_i$ for $i = 1, 2, \dots, r$;
- e) $\mathbf{x}^i(t)$ are absolutely continuous in the interval $\langle \gamma_i, \gamma_{i+1} \rangle$; for $i = 0, 1, 2, \dots, r$.

$\mathbf{x}^-(\tau)$ (or $\mathbf{x}^+(\tau)$) means the limit of vector $\mathbf{x}(t)$ when approaches τ from the left (or from the right) and $\mathbf{x}^i(t)$ are defined as follows:

$$\mathbf{x}^i(t) = \begin{cases} \mathbf{x}(t) & \text{for } t \in (\gamma_i, \gamma_{i+1}). \\ \mathbf{x}^-(t) & \text{for } t = \gamma_i, \\ \mathbf{x}^+(t) & \text{for } t = \gamma_{i+1}. \end{cases}$$

3. TRANSFORMATION OF THE BOUNDARY VALUE PROBLEM TO THE INITIAL VALUE PROBLEMS

In this paragraph we shall construct (by solving the initial value problems) such matrices $\Phi_i(t)$ and such vectors $\varphi_i(t)$ that the following statement is valid for every solution $\mathbf{x}(t)$ of the problem ψ :

$$\Phi_i(t) \mathbf{x}^i(t) = \varphi_i(t)$$

for $t \in \langle \gamma_i, \gamma_{i+1} \rangle$ and for $i = 0, 1, \dots, r$.

We shall prove for these systems that they have a solution (or a unique one) if and only if the problem ψ has a solution (or resp. a unique one).

Lemma 3.1. Let $\mathbf{c}(t)$ be an absolutely continuous solution of the equation

$$(3.1) \quad \mathbf{c}'(t) + \mathbf{A}(t) \mathbf{c}(t) = \mathbf{f}(t)$$

almost everywhere on $\langle \xi_1, \xi_2 \rangle$ and let

$$(3.2) \quad \mathbf{V}_0 \mathbf{c}(\xi_0) = \mathbf{v}_0$$

where $\xi_0 \in \langle \xi_1, \xi_2 \rangle$ holds where \mathbf{V}_0 is a constant matrix of type (n, N) and \mathbf{v}_0 is a constant vector with n components. Let a matrix $\mathbf{V}(t)$ and a vector $\mathbf{v}(t)$ be an absolutely continuous solution of the equations

$$(3.3) \quad \begin{aligned} \mathbf{V}'(t) &= \mathbf{V}(t) \mathbf{A}(t) \quad \text{almost everywhere on } \langle \xi_1, \xi_2 \rangle, \\ \mathbf{v}'(t) &= \mathbf{V}(t) \mathbf{f}(t) \quad \text{almost everywhere on } \langle \xi_1, \xi_2 \rangle, \end{aligned}$$

with the initial conditions

$$(3.4) \quad \mathbf{V}(\xi_0) = \mathbf{V}_0 \quad \text{and} \quad \mathbf{v}(\xi_0) = \mathbf{v}_0.$$

Then

$$(3.5) \quad \mathbf{V}(t) \mathbf{c}(t) = \mathbf{v}(t) \quad \text{for } t \in \langle \xi_1, \xi_2 \rangle$$

holds.

Proof. We premultiply the equations (3.1) by the matrix \mathbf{V} : $\mathbf{V}\mathbf{c}' + \mathbf{V}\mathbf{A}\mathbf{c} = \mathbf{V}\mathbf{f}$ almost everywhere on $\langle \xi_1, \xi_2 \rangle$. We substitute $\mathbf{V}\mathbf{A}$ and $\mathbf{V}\mathbf{f}$ by \mathbf{V}' and \mathbf{v}' according to the equation (3.3), and hence we get $(\mathbf{V}\mathbf{c})' = \mathbf{v}'$ almost everywhere on $\langle \xi_1, \xi_2 \rangle$. The components of the vectors $\mathbf{V}\mathbf{c}$ and \mathbf{v} are absolutely continuous in the interval $\langle \xi_1, \xi_2 \rangle$ and thus $\mathbf{V}\mathbf{c} = \mathbf{v} + \mathbf{k}$ (where \mathbf{k} is a constant vector) follows from the last equations. As follows from equation (3.2) \mathbf{k} is a zero-vector.

This assertion states in which way it is possible to displace the condition of the form (3.2) from one to the other point if both the points lie in an interval in which the vector $\mathbf{c}(t)$ is continuous.

Definition 3.1. Let $\{\mathbf{D}_i(t)\}_{i=0}^r$ be a sequence of matrices of type (n_1, N) and $\{\mathbf{d}_i(t)\}_{i=0}^r$ be a sequence of vectors with n_1 components. In the interval $\langle \gamma_i, \gamma_{i+1} \rangle$ let the matrices $\mathbf{D}_i(t)$ and the vectors $\mathbf{d}_i(t)$ be absolutely continuous solutions of the equations

$$(3.6) \quad \begin{aligned} \mathbf{D}'_i &= \mathbf{D}_i \mathbf{A}, \\ \mathbf{d}'_i &= \mathbf{D}_i \mathbf{f}, \end{aligned}$$

with the initial conditions

$$(3.7) \quad \begin{aligned} \mathbf{D}_0(a) &= \mathbf{U}_1, \\ \mathbf{d}_0(a) &= \mathbf{u}_1, \\ \mathbf{D}_i(\gamma_i) &= \mathbf{D}_{i-1}(\gamma_i) \mathbf{W}_i \quad \text{for } i = 1, 2, \dots, r, \\ \mathbf{d}_i(\gamma_i) &= \mathbf{d}_{i-1}(\gamma_i) \mathbf{w}_i \quad \text{for } i = 1, 2, \dots, r. \end{aligned}$$

Definition 3.2. Let $\{\hat{\mathbf{D}}_i(t)\}_{i=0}^r$ be a sequence of matrices of type (n_2, N) and $\{\hat{\mathbf{d}}_i(t)\}_{i=0}^r$ be a sequence of vectors with n_2 components. In the interval $\langle \gamma_i, \gamma_{i+1} \rangle$ let the matrices $\hat{\mathbf{D}}_i(t)$ and the vectors $\hat{\mathbf{d}}_i(t)$ be absolutely continuous solutions of the equations

$$(3.8) \quad \begin{aligned} \hat{\mathbf{D}}_i' &= \hat{\mathbf{D}}_i \mathbf{A}, \\ \hat{\mathbf{d}}_i' &= \hat{\mathbf{D}}_i \mathbf{f}, \end{aligned}$$

with the initial conditions

$$(3.9) \quad \begin{aligned} \hat{\mathbf{D}}_r(b) &= \mathbf{U}_2, \\ \hat{\mathbf{d}}_r(b) &= \mathbf{u}_2, \\ \hat{\mathbf{D}}_i(\gamma_{i+1}) &= \hat{\mathbf{D}}_{i+1}(\gamma_{i+1}) \mathbf{W}_{i+1}^{-1} \\ &\text{for } i = r-1, r-2, \dots, 0. \\ \hat{\mathbf{d}}_i(\gamma_{i+1}) &= \hat{\mathbf{d}}_{i+1}(\gamma_{i+1}) + \hat{\mathbf{D}}_{i+1}(\gamma_{i+1}) \mathbf{W}_{i+1}^{-1} \mathbf{w}_{i+1} \\ &\text{for } i = r-1, r-2, \dots, 0. \end{aligned}$$

The matrices $\mathbf{D}_i(t)$ and the vectors $\mathbf{d}_i(t)$ are defined recursively, successively for $i = 0, 1, 2, \dots, r$ and the matrices $\hat{\mathbf{D}}_i(t)$ and the vectors $\hat{\mathbf{d}}_i(t)$ are defined recursively too, but successively for $i = r, r-1, \dots, 1, 0$.

Definition 3.3. Let $\Phi_i(t)$ be a matrix of the type (N, N) and let $\varphi_i(t)$ be vector with N components such that

$$(3.10) \quad \begin{aligned} \Phi_i(t) &= \begin{pmatrix} \mathbf{D}_i(t) \\ \hat{\mathbf{D}}_i(t) \end{pmatrix}, \\ \varphi_i(t) &= \begin{pmatrix} \mathbf{d}_i(t) \\ \hat{\mathbf{d}}_i(t) \end{pmatrix}, \end{aligned}$$

for $i = 0, 1, \dots, r$ and for $t \in \langle \gamma_i, \gamma_{i+1} \rangle$.

Let us formulate some properties of the matrices $\Phi_i(t)$ and the vectors $\varphi_i(t)$ mentioned above:

Theorem 3.1. For every solution $\mathbf{x}(t)$ of the problem ψ the following assertion is valid:

$$(3.11) \quad \Phi_i(t) \mathbf{x}^i(t) = \varphi_i(t) \quad \text{for } t \in \langle \gamma_i, \gamma_{i+1} \rangle.$$

Theorem 3.2. If i exists so that the system (3.11) has a solution (or a unique one) in at least one of the points of the interval $\langle \gamma_i, \gamma_{i+1} \rangle$, then all the systems (3.11) (for all i) have a solution (or a unique one) for all the t for which they are defined.

Remark. Now the sentences that the systems (3.11) have the solution, or have no solution, or have the unique solution, are evident.

Theorem 3.3. *The systems (3.11) have a solution (or a unique solution) if and only if the problem ψ has a solution (or a unique one).*

Remark. From the theorem 3.3 and from the fact that $\Phi_i(t)$ are square matrices the following statement follows: $\Phi_i(t)$ are non-singular if and only if the problem has a unique solution.

Proof of theorem 3.1. Let $\mathbf{x}(t)$ be a solution of the problem ψ . At the point a

$$(3.12) \quad \mathbf{U}_1 \mathbf{x}(a) = \mathbf{u}_1$$

is valid.

In the interval $\langle a, \gamma_1 \rangle$ $\mathbf{D}_0(t) \mathbf{x}_0(t) = \mathbf{d}_0(t)$ holds (it follows from the equation (3.12) and the lemma (3.1)).

At the point γ_1

$$\begin{aligned} \mathbf{D}_0(\gamma_1) \mathbf{x}^-(\gamma_1) &= \mathbf{d}_0(\gamma_1), \\ \mathbf{x}^-(\gamma_1) &= \mathbf{W}_1 \mathbf{x}^+(\gamma_1) + \mathbf{w}_1 \end{aligned}$$

is true.

We readily get

$$\mathbf{D}_0(\gamma_1) \mathbf{W}_1 \mathbf{x}^+(\gamma_1) = \mathbf{d}_0(\gamma_1) - \mathbf{D}_0(\gamma_1) \mathbf{w}_1.$$

The last equation can be rewritten

$$(3.13) \quad \mathbf{D}_1(\gamma_1) \mathbf{x}^+(\gamma_1) = \mathbf{d}_1(\gamma_1).$$

From the equality (3.13) and the lemma (3.1) the validity of the bellow given equation follows

$$\mathbf{D}_1(t) \mathbf{x}^1(t) = \mathbf{d}_1(t) \quad \text{for } t \in \langle \gamma_1, \gamma_2 \rangle.$$

We could prove by induction that for $i = 0, 1, \dots, r$

$$(3.14) \quad \mathbf{D}_i \mathbf{x}^i(t) = \mathbf{d}_i(t) \quad \text{for } t \in \langle \gamma_i, \gamma_{i+1} \rangle$$

is valid.

Analogously it is possible to prove that for $i = 0, 1, \dots, r$

$$(3.15) \quad \hat{\mathbf{D}}_i(t) \mathbf{x}^i(t) = \hat{\mathbf{d}}_i(t) \quad \text{for } t \in \langle \gamma_i, \gamma_{i+1} \rangle.$$

holds.

The validity of the equations (3.11) follows from the equations (3.14) and (3.15).

We need the following lemma for the proof of the theorem 3.2:

Lemma 3.2. *In the interval $\langle \xi_1, \xi_2 \rangle$ let the matrix $\mathbf{V}(t)$ (generally rectangular) be an absolutely continuous solution of the equation $\mathbf{V}' = \mathbf{V}\mathbf{H}$ almost everywhere on $\langle \xi_1, \xi_2 \rangle$, where \mathbf{H} is a square matrix and $H_{ij} \in \mathcal{L}\langle \xi_1, \xi_2 \rangle$. Thus if the rank of the matrix $\mathbf{V}(t)$ equals h in at least one point of the interval $\langle \xi_1, \xi_2 \rangle$ then the rank of the matrix $\mathbf{V}(t)$ is equal to h in the whole interval $\langle \xi_1, \xi_2 \rangle$.*

Proof of the lemma 3.2. Let \mathbf{C} be a square absolutely continuous matrix such that $\mathbf{C}' = \mathbf{C}\mathbf{H}$ almost everywhere on $\langle \xi_1, \xi_2 \rangle$; $\mathbf{C}(\xi_1) = \mathbf{E}$ (where \mathbf{E} is an identity matrix). As known, \mathbf{C} is non-singular in the whole interval $\langle \xi_1, \xi_2 \rangle$ (see [13]).

The matrix $\mathbf{V}(t)$ can be represented as a product

$$\mathbf{V}(t) = \mathbf{V}(\xi_1) \mathbf{C}(t).$$

In view of $\mathbf{C}(t)$ being a non-singular matrix the ranks of the matrices $\mathbf{V}(t)$ and $\mathbf{V}(\xi_1)$ are equal.

Proof of the theorem 3.2. The matrix Φ_i satisfies the equation $\Phi_i'(t) = \Phi_i(t) \mathbf{A}(t)$ almost everywhere on $\langle \gamma_i, \gamma_{i+1} \rangle$ and thus according to the lemma 3.2 it has a constant rank in the interval $\langle \gamma_i, \gamma_{i+1} \rangle$. The augmented matrix $(\Phi_i(t), \varphi_i(t))$ of the system (3.11) satisfies the equation

$$(\Phi_i(t), \varphi_i(t))' = (\Phi_i(t), \varphi_i(t)) \begin{pmatrix} \mathbf{A}(t) & \mathbf{f}(t) \\ 0 & 0 \end{pmatrix}$$

almost everywhere on $\langle \gamma_i, \gamma_{i+1} \rangle$ and so it has also constant rank in the interval $\langle \gamma_i, \gamma_{i+1} \rangle$.

From the conditions (3.7) and (3.9) it follows:

$$\begin{aligned} \Phi_i(\gamma_i) &= \Phi_{i-1}(\gamma_i) \mathbf{W}_i, \\ (\Phi_i(\gamma_i), \varphi_i(\gamma_i)) &= (\Phi_{i-1}(\gamma_i), \varphi_{i-1}(\gamma_i)) \begin{pmatrix} \mathbf{W}_i & -\mathbf{w}_i \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The ranks of all the matrices $\Phi_i(t)$ are equal and the ranks of all the augmented matrices (Φ_i, φ_i) are equal too, as follows from the equations mentioned above. The theorem 3.2 follows from these statements and Frobenius theorem.

Proof of the theorem 3.3. Let the systems (3.11) have solutions. Let \mathbf{v} be a constant vector that is a solution of the system (3.11) at the point b , i.e.

$$\Phi_r(b) \mathbf{v} = \varphi_r(b).$$

Let $\mathbf{x}^i(t)$ ($i = r, r-1, \dots, 0$) be absolutely continuous solutions of the equations

$$(\mathbf{x}^i(t))' + \mathbf{A}(t) \mathbf{x}^i(t) = \mathbf{f}(t)$$

almost everywhere on $\langle \gamma_i, \gamma_{i+1} \rangle$, with the initial conditions

$$\begin{aligned} \mathbf{x}^r(b) &= \mathbf{v}, \\ \mathbf{x}^i(\gamma_{i+1}) &= \mathbf{W}_{i+1} \mathbf{x}^{i+1}(\gamma_{i+1}) + \mathbf{w}_i \quad \text{for } i = r-1, r-2, \dots, 1, 0. \end{aligned}$$

Define

$$\mathbf{x}(t) = \mathbf{x}^i(t) \quad \text{for } t \in (\gamma_i, \gamma_{i+1}).$$

We shall show that thus defined vector $\mathbf{x}(t)$ is a solution of the problem ψ . It is sufficient to show that $\mathbf{x}(t)$ satisfies the condition

$$\mathbf{U}_1 \mathbf{x}^+(a) = \mathbf{u}_1.$$

From the lemma 3.1 and the definition of the matrices Φ_i and the vectors φ_i it follows

$$\Phi_i(t) \mathbf{x}^i(t) = \varphi_i(t) \quad \text{for } t \in \langle \gamma_i, \gamma_{i+1} \rangle.$$

Then particularly $\mathbf{U}_1 \mathbf{x}^+(a) = \mathbf{u}_1$ is valid.

Hence the following statement has been proved: For each solution \mathbf{v} of the system (3.11) at the point b there exists the solution $\mathbf{x}(t)$ of the problem such that $\mathbf{x}(b) = \mathbf{v}$.

Theorem 3.3 follows readily from this statement and theorems 3.1 and 3.2.

The above explained procedure by which the matrices Φ_i have been obtained, transforms the boundary value problem ψ to initial value problems. If this procedure were used as an algorithm to solve the problem ψ we should have to solve the equations (3.6) with the conditions (3.7) for $i = 0, 1, \dots, r$ from the left to the right, then the equations (3.8) with the conditions (3.9) for $i = r, r - 1, \dots, 0$, from the right to the left and eventually to solve the systems (3.11). It is possible to show that this algorithm possesses several properties which are not adequate for numerical computations. It is in fact the method of combination of solutions, the disadvantage of which is described in [14].

To define this procedure has only an auxiliary character.

4. FACTORIZATION EQUATIONS

In this paragraph we shall construct systems of equations which will be equivalent to the systems (3.11) and which will be constructed by solving initial value problems. It will be shown as well that the new procedure, contrary to that described in sec. 3, has a certain property suitable for numerical realization.

From here on let us suppose that the problem ψ has exactly one solution. From this assumption it follows, as already discussed above, that the matrices $\Phi_i(t)$ are non-singular and the matrices $\mathbf{D}_i(t)$ and $\hat{\mathbf{D}}_i(t)$ have the rank equal to the number of rows.

Further on we shall deal with selected submatrices of the matrices $\mathbf{D}_i(t)$, $\hat{\mathbf{D}}_i(t)$ and $\mathbf{A}(t)$. To make the writing easier we shall introduce certain notations.

Definition 4.1. Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$ be some order of numbers $1, 2, \dots, N$. Let $\mathbf{P}^\sigma = \{p_{ij}^\sigma\}$ be a square permutation matrix of order N defined as follows:

$$p_{\sigma_i j}^\sigma = \delta_i^j, \quad i, j = 1, 2, \dots, N,$$

where δ_i^j is Kronecker delta, i.e.

$$\delta_i^j = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases}$$

Definition 4.2. Let $n < N$, let $\mathbf{L}_n = \{l_{ij}^n\}$ be a rectangular matrix of type (N, n) defined as

$$l_{ij}^{(n)} = \delta_i^j, \quad i = 1, 2, \dots, N; j = 1, 2, \dots, n.$$

Definition 4.3. Let $n < N$, let $\mathbf{R}_n = \{r_{ij}^n\}$ be a rectangular matrix of type $(N, N - n)$ defined in this way:

$$r_{ij}^{(n)} = \delta_{i-n}^j, \quad i = 1, 2, \dots, N; j = 1, 2, \dots, N - n.$$

If we postmultiply some matrix by the matrix \mathbf{P}^σ (or premultiply by the transpose matrix) we re-arrange the columns (or the rows) of the original matrix according to the ordering σ .

If we postmultiply a matrix by the matrix \mathbf{L}_n (or premultiply by \mathbf{R}_n) we obtain a matrix selected from the original matrix such that it consists of the n first left (or $N - n$ last right) columns of the original matrix.

Lemma 4.1. Let \mathbf{E}_n be an identity matrix of type (n, n) . Then the following statements are valid:

1. $\mathbf{L}_n^T \mathbf{L}_n = \mathbf{E}_n$,
2. $\mathbf{R}_n^T \mathbf{R}_n = \mathbf{E}_{N-n}$,
3. $(\mathbf{P}^\sigma)^T \mathbf{P}^\sigma = \mathbf{P}^\sigma (\mathbf{P}^\sigma)^T = \mathbf{E}_N$,
4. $(\mathbf{L}_n, \mathbf{R}_n) = \mathbf{E}_N$.
5. $\mathbf{L}_n \cdot \mathbf{L}_n^T + \mathbf{R}_n \cdot \mathbf{R}_n^T = \mathbf{E}_N$

(the last relation will often be used in the proofs).

This statement can readily be verified by direct computation.

Theorem 4.1. There exists a partition $a = \vartheta_0 < \vartheta_1 < \dots < \vartheta_{v+1} = b$ of the interval $\langle a, b \rangle$ such that the following assertions are valid:

1. For each interval $\langle \vartheta_i, \vartheta_{i+1} \rangle$ there exists $j(i)$ so that

$$\langle \vartheta_i, \vartheta_{i+1} \rangle \subset \langle \gamma_{j(i)}, \gamma_{j(i)+1} \rangle. \quad (i = 0, 1, \dots, v).$$

2. For each interval $\langle \vartheta_i, \vartheta_{i+1} \rangle$ there exists an ordering σ_i such that the matrix $\mathbf{D}_{j(i)} \cdot \mathbf{P}^{\sigma_i} \cdot \mathbf{L}_{n_1}$ is non-singular in the whole interval $\langle \vartheta_i, \vartheta_{i+1} \rangle$ ($i = 0, 1, \dots, v$).

Proof. According to the assumptions of the theorem the rank of the matrix $\mathbf{D}_{j(i)}(t)$ is equal to n_1 and so at each point t there exists a selected non-singular matrix of order n_1 and the existence of such ordering follows from the continuity of the matrix $\mathbf{D}_{j(i)}(t)$ and from Borel covering theorem.

Analogously, this theorem holds even for matrices $\hat{\mathbf{D}}_j(t)$.

Theorem 4.2. *There exists a partition $a = \lambda_0 < \lambda_1 < \dots < \lambda_{\varrho+1} = b$ of the interval $\langle a, b \rangle$ such that:*

1. *For each interval $\langle \lambda_i, \lambda_{i+1} \rangle$ there exists $j(i)$ such that*

$$\langle \lambda_i, \lambda_{i+1} \rangle \subset \langle \gamma_{j(i)}, \gamma_{j(i)+1} \rangle.$$

2. *For each interval $\langle \lambda_i, \lambda_{i+1} \rangle$ there exists an ordering \varkappa_i such that the matrix $\hat{\mathbf{D}}_{j(i)} \mathbf{P}^{\varkappa_i} \mathbf{L}_{n_2}$ is non-singular in the whole interval $\langle \lambda_i, \lambda_{i+1} \rangle$ ($i = 0, 1, \dots, \varrho$).*

Now let us define the operators G , G^* and g :

Definition 4.4. *Let \mathbf{D} be a matrix of type (n, N) and \mathbf{d} be a vector with n components. Let σ be a certain ordering of numbers $1, 2, \dots, N$, let the matrix $\mathbf{D}\mathbf{P}^\sigma \mathbf{L}_n$ be non-singular. Then let us introduce operators:*

1. $G(\mathbf{D}, \sigma, n) = (\mathbf{D}\mathbf{P}^\sigma \mathbf{L}_n)^{-1} \mathbf{D}\mathbf{P}^\sigma \mathbf{R}_n,$
2. $G^*(\mathbf{D}, \sigma, n) = (\mathbf{D}\mathbf{P}^\sigma \mathbf{L}_n)^{-1} \mathbf{D},$
3. $g(\mathbf{D}, \mathbf{d}, \sigma, n) = (\mathbf{D}\mathbf{P}^\sigma \mathbf{L}_n)^{-1} \mathbf{d}.$

The matrix $\mathbf{D}\mathbf{P}^\sigma \mathbf{L}_n$ is a selected non-singular matrix of the matrix \mathbf{D} , the matrix $\mathbf{D}\mathbf{P}^\sigma \mathbf{R}_n$ consists of those columns of the matrix \mathbf{D} which do not occur in the matrix $\mathbf{D}\mathbf{P}^\sigma \mathbf{L}_n$. The matrix $G(\mathbf{D}, \sigma, n)$ is of type (n, n) , the matrix $G^*(\mathbf{D}, \sigma, n)$ is of the same type as the matrix \mathbf{D} , i.e. of type (n, N) . The vector $g(\mathbf{D}, \mathbf{d}, \sigma, n)$ has n components. The matrix G arises evidently from the matrix G^* if we delete the columns which lie at the places $\sigma_1, \sigma_2, \dots, \sigma_n$ from the matrix G^* (these columns are evidently identity vectors) and rearrange the remaining columns in a certain manner. Conversely, the matrix G^* arises from the matrix G according to the simple prescription $G^* = [\mathbf{E}_n, G](\mathbf{P}^\sigma)^T = [\mathbf{E}_n, G](\mathbf{L}_n \mathbf{L}_n^T + \mathbf{R}_n \mathbf{R}_n^T)(\mathbf{P}^\sigma)^T = \mathbf{L}_n^T (\mathbf{P}^\sigma)^T + G \mathbf{R}_n^T (\mathbf{P}^\sigma)^T.$

Lemma 4.2. *Let \mathbf{T} be a non-singular matrix of order n . Let σ be a certain ordering of numbers $1, 2, \dots, n$. Let \mathbf{D} be a matrix of type (n, N) such that the symbol $G(\mathbf{D}, \sigma, n)$ is meaningful and let \mathbf{d} be a vector with n components. Then the following relations are valid:*

1. $G(\mathbf{T}\mathbf{D}, \sigma, n) = G(\mathbf{D}, \sigma, n),$
2. $G^*(\mathbf{T}\mathbf{D}, \sigma, n) = G^*(\mathbf{D}, \sigma, n),$
3. $g(\mathbf{T}\mathbf{D}, \mathbf{T}\mathbf{d}, \sigma, n) = g(\mathbf{D}, \mathbf{d}, \sigma, n).$

Definition 4.5. *Let $a = \vartheta_0 < \vartheta_1 < \dots < \vartheta_{v+1} = b$ be such a partition of the interval $\langle a, b \rangle$ and let σ_i be such orderings that the assertions of the theorem 4.1. are true. For each interval $\langle \vartheta_i, \vartheta_{i+1} \rangle$ ($i = 0, 1, \dots, v$) let us define:*

1. *matrices*

$$\mathbf{G}_i(t) = G(\mathbf{D}_{j(i)}, \sigma_i, n_1)$$

and

$$\mathbf{G}_i^*(t) = G^*(\mathbf{D}_{j(i)}, \sigma_i, n_1) \text{ for } t \in \langle \vartheta_i, \vartheta_{i+1} \rangle;$$

2. vectors

$$\mathbf{g}_i(t) = g(\mathbf{D}_{j(i)}, \mathbf{d}_{j(i)}, \sigma_i, n_1) \text{ for } t \in \langle \vartheta_i, \vartheta_{i+1} \rangle;$$

3. submatrices of the matrix \mathbf{A} for $t \in \langle \vartheta_i, \vartheta_{i+1} \rangle$ thus:

$$\mathbf{A}_1^i(t) = \mathbf{L}_{n_1}^T(\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1},$$

$$\mathbf{A}_2^i(t) = \mathbf{L}_{n_1}^T(\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1},$$

$$\mathbf{A}_3^i(t) = \mathbf{R}_{n_1}^T(\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1},$$

$$\mathbf{A}_4^i(t) = \mathbf{R}_{n_1}^T(\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1};$$

4. vectors

$$\mathbf{F}_2^i(t) = \mathbf{L}_{n_1}^T(\mathbf{P}^{\sigma_i})^T \mathbf{f},$$

$$\mathbf{F}_1^i(t) = \mathbf{R}_{n_1}^T(\mathbf{P}^{\sigma_i})^T \mathbf{f} \text{ for } t \in \langle \vartheta_i, \vartheta_{i+1} \rangle;$$

5. vectors

$$\mathbf{y}^i(t) = \mathbf{L}_{n_1}^T(\mathbf{P}^{\sigma_i})^T \mathbf{x}^{j(i)},$$

and

$$\mathbf{z}^i(t) = \mathbf{R}_{n_1}^T(\mathbf{P}^{\sigma_i})^T \mathbf{x}^{j(i)} \text{ for } t \in \langle \vartheta_i, \vartheta_{i+1} \rangle,$$

where $j(i)$ is such an index that

$$\langle \vartheta_i, \vartheta_{i+1} \rangle \subset \langle \gamma_{j(i)}, \gamma_{j(i)+1} \rangle.$$

Definition 4.6. Let $a = \lambda_0 < \lambda_1 < \dots < \lambda_{\varrho+1} = b$ be such a partition of the interval $\langle a, b \rangle$ and let \varkappa_i be such orderings that the assertion of the theorem 4.2 are true. For each interval $\langle \lambda_i, \lambda_{i+1} \rangle$ ($i = 0, 1, \dots, \varrho$) let us define:

1. matrices

$$\hat{\mathbf{G}}_i(t) = G(\hat{\mathbf{D}}_{j(i)}, \varkappa_i, n_2)$$

and

$$\mathbf{G}_i^*(t) = G^*(\mathbf{D}_{j(i)}, \varkappa_i, n_2) \text{ for } t \in \langle \lambda_i, \lambda_{i+1} \rangle;$$

2. vectors

$$\hat{\mathbf{g}}_i(t) = g(\hat{\mathbf{D}}_{j(i)}, \hat{\mathbf{d}}_{j(i)}, \varkappa_i, n_2) \text{ for } t \in \langle \lambda_i, \lambda_{i+1} \rangle;$$

3. submatrices of the matrix \mathbf{A} for $t \in \langle \lambda_i, \lambda_{i+1} \rangle$ in this way:

$$\hat{\mathbf{A}}_1^i = \mathbf{L}_{n_2}^T(\mathbf{P}^{\varkappa_i})^T \mathbf{A} \mathbf{P}^{\varkappa_i} \mathbf{L}_{n_2},$$

$$\hat{\mathbf{A}}_2^i = \mathbf{L}_{n_2}^T(\mathbf{P}^{\varkappa_i})^T \mathbf{A} \mathbf{P}^{\varkappa_i} \mathbf{R}_{n_2},$$

$$\hat{\mathbf{A}}_3^i = \mathbf{R}_{n_2}^T(\mathbf{P}^{\varkappa_i})^T \mathbf{A} \mathbf{P}^{\varkappa_i} \mathbf{L}_{n_2},$$

$$\hat{\mathbf{A}}_4^i = \mathbf{R}_{n_2}^T(\mathbf{P}^{\varkappa_i})^T \mathbf{A} \mathbf{P}^{\varkappa_i} \mathbf{R}_{n_2};$$

4. vectors

$$\begin{aligned}\hat{\mathbf{F}}_2^i(t) &= \mathbf{L}_{n_2}^T(\mathbf{P}^{\sigma_i})^T \mathbf{f}, \\ \hat{\mathbf{F}}_1^i(t) &= \mathbf{R}_{n_2}^T(\mathbf{P}^{\sigma_i})^T \mathbf{f} \quad \text{for } t \in \langle \lambda_i, \lambda_{i+1} \rangle ;\end{aligned}$$

5. vectors

$$\hat{\mathbf{y}}^i(t) = \mathbf{L}_{n_2}^T(\mathbf{P}^{\sigma_i})^T \mathbf{x}^{j(i)},$$

and

$$\hat{\mathbf{z}}^i(t) = \mathbf{R}_{n_2}^T(\mathbf{P}^{\sigma_i})^T \mathbf{x}^{j(i)} \quad \text{for } t \in \langle \lambda_i, \lambda_{i+1} \rangle ,$$

where $j(i)$ is such an index, that

$$\langle \lambda_i, \lambda_{i+1} \rangle \subset \langle \gamma_{j(i)}, \gamma_{j(i)+1} \rangle .$$

Theorem 4.3. For $i = 0, 1, \dots, v$ the following relations are valid:

$$(4.1) \quad \mathbf{y}^i(t) + \mathbf{G}_i(t) \mathbf{z}^i(t) = \mathbf{g}_i(t) \quad \text{for } t \in \langle \vartheta_i, \vartheta_{i+1} \rangle ,$$

$$(4.2) \quad \mathbf{G}_i' = \mathbf{G}_i \mathbf{A}_4^i - \mathbf{A}_1^i \mathbf{G}_i - \mathbf{G}_i \mathbf{A}_3^i \mathbf{G}_i + \mathbf{A}_2^i$$

almost everywhere on $\langle \vartheta_i, \vartheta_{i+1} \rangle$

$$(4.3) \quad \mathbf{g}_i' = -(\mathbf{A}_1^i + \mathbf{G}_i \mathbf{A}_3^i) \mathbf{g}_i + \mathbf{F}_1^i + \mathbf{G}_i \mathbf{F}_2^i$$

almost everywhere on $\langle \vartheta_i, \vartheta_{i+1} \rangle$

$$(4.4) \quad \mathbf{z}_i' = -(\mathbf{A}_4^i - \mathbf{A}_3^i \mathbf{G}_i) \mathbf{z}_i + \mathbf{F}_2^i - \mathbf{A}_3^i \mathbf{g}_i$$

almost everywhere on $\langle \vartheta_i, \vartheta_{i+1} \rangle$.

Proof. For $t \in \langle \vartheta_i, \vartheta_{i+1} \rangle$

$$(4.5) \quad \mathbf{D}_{j(i)} \mathbf{x}^{j(i)} = \mathbf{d}_{j(i)}$$

is valid, where $j(i)$ is such an index that

$$\langle \vartheta_i, \vartheta_{i+1} \rangle \subset \langle \gamma_{j(i)}, \gamma_{j(i)+1} \rangle$$

is true.

Let us premultiply (4.5) by the matrix

$$(\mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1})^{-1} .$$

Thus we obtain

$$\mathbf{G}_i^* \mathbf{x}^{j(i)} = \mathbf{g}_i$$

and hence

$$(\mathbf{L}_{n_1}^T(\mathbf{P}^{\sigma_i})^T + \mathbf{G}_i \mathbf{R}_{n_1}^T(\mathbf{P}^{\sigma_i})^T) \mathbf{x}^{j(i)} = \mathbf{g}_i .$$

The last equation will be arranged according to the definition

$$\mathbf{y}^i(t) + \mathbf{G}_i(t) \mathbf{z}^i(t) = \mathbf{g}_i(t) .$$

The validity of equations (4.2) and (4.3) will be verified by differentiating the equations

$$(4.6) \quad \mathbf{G}_i(t) = (\mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1})^{-1} \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1},$$

$$(4.7) \quad \mathbf{g}_i(t) = (\mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1})^{-1} \mathbf{d}_{j(i)}.$$

The following identities are valid:

$$(4.8) \quad \begin{aligned} \mathbf{D}_{j(i)} \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} &= \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} (\mathbf{L}_{n_1} \mathbf{L}_{n_1}^T + \mathbf{R}_{n_1} \mathbf{R}_{n_1}^T) (\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} = \\ &= \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} \mathbf{L}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} + \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} \mathbf{R}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} = \\ &= \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} \mathbf{A}_1^i + \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} \mathbf{A}_3^i; \end{aligned}$$

$$(4.9) \quad \begin{aligned} \mathbf{D}_{j(i)} \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} &= \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} (\mathbf{L}_n \mathbf{L}_n^T + \mathbf{R}_n \mathbf{R}_n^T) (\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} = \\ &= \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} \mathbf{L}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} + \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} \mathbf{R}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} = \\ &= \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} \mathbf{A}_2^i + \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} \mathbf{A}_4^i. \end{aligned}$$

Let us differentiate the equation (4.6):

$$(4.10) \quad \begin{aligned} \mathbf{G}'_i(t) &= -(\mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1})^{-1} \mathbf{D}_{j(i)} \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} (\mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1})^{-1} \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} + \\ &+ (\mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1})^{-1} \mathbf{D}_{j(i)} \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1}. \end{aligned}$$

Let us substitute into the equation (4.10) according to the identities (4.8) and (4.9):

$$\mathbf{G}'_i(t) = -\mathbf{A}_1^i \mathbf{G}_i - \mathbf{G}_i \mathbf{A}_3^i \mathbf{G}_i + \mathbf{A}_2^i + \mathbf{G}_i \mathbf{A}_4^i.$$

Let us differentiate (4.7):

$$(4.11) \quad \begin{aligned} \mathbf{g}'_i(t) &= -(\mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1})^{-1} \mathbf{D}_{j(i)} \mathbf{A} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} (\mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1})^{-1} \mathbf{d}_{j(i)} + \\ &+ (\mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1})^{-1} \mathbf{D}_{j(i)} \mathbf{f}. \end{aligned}$$

The following identity is valid:

$$(4.12) \quad \begin{aligned} \mathbf{D}_{j(i)} \mathbf{f} &= \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} (\mathbf{L}_{n_1} \mathbf{L}_{n_1}^T + \mathbf{R}_{n_1} \mathbf{R}_{n_1}^T) (\mathbf{P}^{\sigma_i})^T \mathbf{f} = \\ &= \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} \mathbf{L}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{f} + \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} \mathbf{R}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{f} = \\ &= \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{L}_{n_1} \mathbf{F}_1 + \mathbf{D}_{j(i)} \mathbf{P}^{\sigma_i} \mathbf{R}_{n_1} \mathbf{F}_2. \end{aligned}$$

We put into the equations (4.11) according to the identities (4.8) and (4.12) and hence we obtain

$$\mathbf{g}'_i(t) = -(\mathbf{A}_1^i + \mathbf{G}_i \mathbf{A}_3^i) \mathbf{g}_i + \mathbf{F}_1^i + \mathbf{G}_i \mathbf{F}_2^i.$$

Let us differentiate the equation

$$\mathbf{z}^i = \mathbf{R}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{x}^{j(i)}.$$

Hence we obtain

$$(\mathbf{z}^i)' = \mathbf{R}_{n_1}^T (\mathbf{P}^{\sigma_i})^T (\mathbf{x}^{j(i)})'.$$

We substitute $\mathbf{f} - \mathbf{A}\mathbf{x}^{j(i)}$ for $(\mathbf{x}^{j(i)})'$ and hence we obtain

$$\begin{aligned} (\mathbf{z}^i)' &= \mathbf{R}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{f} - \mathbf{R}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{A}\mathbf{x}^{j(i)} = \\ &= \mathbf{F}_2^i - \mathbf{R}_{n_1}^T (\mathbf{P}^{\sigma_i})^T \mathbf{A}\mathbf{P}^{\sigma_i} (\mathbf{L}_{n_1} \mathbf{L}_{n_1}^T + \mathbf{R}_{n_1} \mathbf{R}_{n_1}^T) (\mathbf{P}^{\sigma_i})^T \mathbf{x}^{j(i)} = \\ &= -\mathbf{A}_3 \mathbf{y}^i - \mathbf{A}_4 \mathbf{z}^i + \mathbf{F}_2^i. \end{aligned}$$

We substitute for \mathbf{y} from equation (4.1) and hence we get the equation (4.4).

Similarly the following theorem is valid for the matrices $\hat{\mathbf{G}}_i$ and vectors $\hat{\mathbf{g}}_i$:

Theorem 4.4. For $i = 0, 1, \dots, \varrho$ the following relations are valid:

$$(4.13) \quad \hat{\mathbf{y}}_i + \hat{\mathbf{G}}_i \hat{\mathbf{z}}_i = \hat{\mathbf{g}}_i \quad \text{for } t \in \langle \lambda_i, \lambda_{i+1} \rangle,$$

$$(4.14) \quad \hat{\mathbf{G}}_i' = \hat{\mathbf{G}}_i \hat{\mathbf{A}}_4^i - \hat{\mathbf{A}}_1^i \hat{\mathbf{G}}_i - \hat{\mathbf{G}}_i \hat{\mathbf{A}}_3^i \hat{\mathbf{G}}_i + \hat{\mathbf{A}}_2^i \quad \text{almost everywhere on } \langle \lambda_i, \lambda_{i+1} \rangle,$$

$$(4.15) \quad \hat{\mathbf{g}}_i' = -(\hat{\mathbf{A}}_3^i + \hat{\mathbf{G}}_i \hat{\mathbf{A}}_3^i) \hat{\mathbf{g}}_i + \hat{\mathbf{F}}_1^i + \hat{\mathbf{G}}_i \hat{\mathbf{F}}_2^i \quad \text{almost everywhere on } \langle \lambda_i, \lambda_{i+1} \rangle,$$

$$(4.16) \quad (\hat{\mathbf{z}}^i)' = -(\hat{\mathbf{A}}_4^i - \hat{\mathbf{A}}_3^i \hat{\mathbf{G}}_i) \hat{\mathbf{z}}^i + \hat{\mathbf{F}}_2^i - \hat{\mathbf{A}}_3^i \hat{\mathbf{g}}_i \quad \text{almost everywhere on } \langle \lambda_i, \lambda_{i+1} \rangle.$$

The equations (4.1), (4.2), (4.3), (4.4), (4.13), (4.14), (4.15) and (4.16) will be used for numerical solution of the problem ψ and will be called factorization equations. Now we introduce the initial conditions for factorization equations.

Theorem 4.5. (The initial conditions for the equations (4.2) and (4.3).)

$$(4.17) \quad 1. \mathbf{G}_0(a) = G(\mathbf{U}_1, \sigma_0, n_1),$$

$$\mathbf{g}_0(a) = g(\mathbf{U}_1, \mathbf{u}_1, \sigma_0, n_1);$$

2. for $i = 1, 2, \dots, \nu$ the following statements are valid:

if $\vartheta_i \notin M$ then

$$\mathbf{G}_i(\vartheta_i) = G(\mathbf{G}_{i-1}^*(\vartheta_i), \sigma_i, n_1),$$

$$\mathbf{g}_i(\vartheta_i) = g(\mathbf{G}_{i-1}^*(\vartheta_i), \mathbf{g}_{i-1}(\vartheta_i), \sigma_i, n_1),$$

if $\vartheta_i = \gamma_l (\in M)$ then

$$\mathbf{G}_i(\vartheta_i) = G(\mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{W}_l, \sigma_i, n_1),$$

$$\mathbf{g}_i(\vartheta_i) = g(\mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{W}_l, \mathbf{g}_{i-1}(\vartheta_i) - \mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{w}_l, \sigma_i, n_1).$$

Theorem 4.6. (The initial conditions for the equations (4.14) and (4.15).)

$$(4.18) \quad 1. \hat{\mathbf{G}}_0(b) = G(\mathbf{U}_2, \alpha_0, n_2),$$

$$\hat{\mathbf{g}}_0(b) = g(\mathbf{U}_2, \mathbf{u}_2, \alpha_0, n_2);$$

2. for $i = 0, 1, \dots, \varrho - 1$ the following statements are valid:

if $\lambda_{i+1} \notin M$ then

$$\hat{\mathbf{G}}_i(\lambda_{i+1}) = G(\hat{\mathbf{G}}_{i+1}^*(\lambda_{i+1}), \varkappa_i, n_2),$$

$$\hat{\mathbf{g}}_i(\lambda_{i+1}) = g(\hat{\mathbf{G}}_{i+1}^*(\lambda_{i+1}), \hat{\mathbf{g}}_{i+1}(\lambda_{i+1}), \varkappa_i, n_2),$$

if $\lambda_{i+1} = \gamma_l (\in M)$ then

$$\hat{\mathbf{G}}_i(\lambda_{i+1}) = G(\hat{\mathbf{G}}_{i+1}^*(\lambda_{i+1}) \mathbf{W}_l^{-1}, \varkappa_i, n_2),$$

$$\hat{\mathbf{g}}_i(\lambda_{i+1}) = g(\hat{\mathbf{G}}_{i+1}^*(\lambda_{i+1}) \mathbf{W}_l^{-1}, \hat{\mathbf{g}}_{i+1}(\lambda_{i+1}) + \hat{\mathbf{G}}_{i+1}^*(\lambda_{i+1}) \mathbf{W}_l^{-1} \mathbf{w}_l, n_2).$$

Now we can find all the matrices \mathbf{G}_i and the vectors \mathbf{g}_i by solving the equations (4.2) and (4.3) with the initial conditions (4.17) successively for $i = 0, 1, \dots, \nu$ from the left to the right and the matrices $\hat{\mathbf{G}}_i$ and vectors $\hat{\mathbf{g}}_i$ by solving the equations (4.14) and (4.15) with initial conditions (4.18) successively for $i = \varrho, \varrho - 1, \dots, 0$ from the right to the left. Let us derive the initial conditions for the equations (4.4) and (4.16) furthermore.

Theorem 4.7. (The initial conditions for the equation (4.4).)

$$(4.19) \quad 1. \quad \mathbf{U}_2 \mathbf{x}^-(b) = \mathbf{u}_2,$$

$$\mathbf{G}_\nu^*(b) \mathbf{x}^-(b) = \mathbf{g}_\nu(b);$$

2. for $i = \nu - 1, \nu - 2, \dots, 0$ the following statements are valid:

if $\vartheta_{i+1} \notin M$ then

$$\mathbf{x}^-(\vartheta_{i+1}) = \mathbf{x}^+(\vartheta_{i+1}),$$

if $\vartheta_{i+1} = \gamma_l (\in M)$ then

$$\mathbf{x}^-(\vartheta_{i+1}) = \mathbf{W}_l \mathbf{x}^+(\vartheta_{i+1}) + \mathbf{w}_l.$$

Theorem 4.8. (The initial conditions for the equation (4.16).)

$$(4.20) \quad 1. \quad \hat{\mathbf{G}}_0^*(a) \mathbf{x}^+(a) = \hat{\mathbf{g}}_0(a),$$

$$\mathbf{U}_1 \mathbf{x}^+(a) = \mathbf{u}_1;$$

2. for $i = 1, 2, \dots, \varrho$ the following statements are valid:

if $\lambda_i \notin M$ then

$$\mathbf{x}^+(\lambda_i) = \mathbf{x}^-(\lambda_i)$$

and if $\lambda_i = \gamma_l (\in M)$ then

$$\mathbf{x}^+(\lambda_i) = \mathbf{W}_l^{-1} \mathbf{x}^-(\lambda_i) - \mathbf{W}_l^{-1} \mathbf{w}_l.$$

Remark. The systems of equations which occur in the theorems 4.7 and 4.8 have under our assumptions a unique solution, i.e. we can calculate the components of the vector $\mathbf{x}^-(\vartheta_{i+1})$ or $\mathbf{x}^+(\lambda_i)$ from them.

Now let us describe the method of simple direct factorization, the method of simple inverse factorization, and the method of composite factorization.

The method of the direct factorization.

We solve the equations (4.2) and (4.3) with the initial conditions (4.17) successively for $i = 0, 1, \dots, v$ from the left to the right. Thus we obtain all the matrices \mathbf{G}_i and vectors \mathbf{g}_i . Then we solve the equations (4.4) with the initial conditions (4.19) successively for $i = v, v - 1, \dots, 0$ from the right to the left. (Solving the equations (4.4) we obtain the vector \mathbf{z}^i and the remaining components of the vector \mathbf{x} are calculated from the equation (4.1)). The partition $\vartheta_0, \vartheta_1, \dots, \vartheta_{v+1}$ of the interval $\langle a, b \rangle$ and the corresponding orderings σ_i ($i = 0, 1, \dots, v$) are determined in the course of solving equations (4.2). This will be described more accurately further on.

The method of the simple inverse factorization.

We solve the equations (4.14) and (4.15) with the initial conditions (4.18) successively for $i = \varrho, \varrho - 1, \dots, 0$ from the right to the left. We find, therefore, all the matrices $\hat{\mathbf{G}}_i$ and the vectors $\hat{\mathbf{g}}_i$. Then we solve the equations (4.16) with the initial conditions (4.20) successively for $i = 0, 1, \dots, \varrho$ from the left to the right. The partition $\lambda_0, \lambda_1, \dots, \lambda_{\varrho+1}$ of the interval $\langle a, b \rangle$ and the corresponding orderings \varkappa_i ($i = 0, 1, \dots, \varrho$) are similarly determined solving the equations (4.14).

The method of the composite factorization.

Let $\{\tau_{ij}\}_{i=0}^{k+1} = \{\vartheta_{ij}\}_{i=0}^{v+1} \cup \{\lambda_{ij}\}_{i=0}^{\varrho+1}$ and let τ_i be ordered so that $\tau_0 < \tau_1 < \dots < \tau_{k+1}$. Let us define:

$$\Phi_i^*(t) = \begin{pmatrix} \mathbf{G}_{i(i)}^*(t) \\ \mathbf{G}_{j(i)}^*(t) \end{pmatrix}, \quad \varphi_i^*(t) = \begin{pmatrix} \mathbf{g}_{i(i)}(t) \\ \hat{\mathbf{g}}_{j(i)}(t) \end{pmatrix} \quad \text{for } t \in \langle \tau_i, \tau_{i+1} \rangle,$$

where i_l and i_j are such indices that

$$\langle \tau_i, \tau_{i+1} \rangle \subset \langle \vartheta_{i(i)}, \vartheta_{i(i)+1} \rangle \cap \langle \lambda_{j(i)}, \lambda_{j(i)+1} \rangle.$$

Evidently the systems of the equations $\Phi_i(t) x^i(t) = \varphi_i(t)$ and the systems of the equations $\Phi_i^*(t) \mathbf{x}^i(t) = \varphi_i^*(t)$ are equivalent for $i \in \langle \tau_i, \tau_{i+1} \rangle$ and for i such that $\langle \tau_i, \tau_{i+1} \rangle \subset \langle \gamma_i, \gamma_{i+1} \rangle$. The method of composed factorization consists in the following procedure: we solve equations (4.2) and (4.3) with initial conditions (4.17) successively for $i = 0, 1, \dots, v$ from the left to the right, then equations (4.14) and (4.18) with initial conditions (4.18) successively for $i = \varrho, \varrho - 1, \dots, 0$ from the right to the left. Thus we find the matrices Φ_i^* and the vectors $\varphi_i^*(t)$. We get the solution of the problem ψ by solving the equations

$$(4.21) \quad \begin{aligned} \Phi_i^*(t) \mathbf{x}(t) &= \varphi_i^*(t) \quad \text{for } t \in (\tau_i, \tau_{i+1}), \\ \Phi_i^*(\tau_i) \mathbf{x}^+(\tau_i) &= \varphi_i^*(\tau_i), \\ \Phi_i^*(\tau) \mathbf{x}^-(\tau_{i+1}) &= \varphi_i^*(\tau_{i+1}). \end{aligned}$$

There remains now to show how to construct the partition $\mathfrak{D}_0, \mathfrak{D}_1, \dots, \mathfrak{D}_{\nu+1}$ and the orderings σ_i ($i = 0, 1, \dots, \nu$) or the partition $\lambda_0, \lambda_1, \dots, \lambda_{\varrho+1}$ and the orderings \varkappa_i ($i = 0, 1, \dots, \varrho$).

Let $\mathbf{D}_j(t)$ be a matrix defined by the equation (3.6) with conditions (3.7). Let σ_i be such an ordering that $\mathbf{D}_{j(i)}(t) \mathbf{P}^{\sigma_i} \mathbf{L}_{n_i}$ is non-singular for $t \in \langle \xi_1, \xi_2 \rangle$, where $j(i)$ is such that $\langle \vartheta_i, \vartheta_{i+1} \rangle \subset \langle \gamma_{j(i)}, \gamma_{j(i)+1} \rangle$. Let $\det(\mathbf{D}_{j(i)}(t) \mathbf{P}^{\sigma_i} \mathbf{L}_{n_i}) \rightarrow 0$ when $t \rightarrow \xi_2^-$. Then there exist l and k so that $|G_{lk}^i(t)| \rightarrow \infty$ when $t \rightarrow \xi_2^-$. From this fact it follows that the solution of equation (4.2) for fixed i has a pole at the point ξ_2 if and only if the corresponding matrix $\mathbf{D}_{j(i)}(\xi_2) \mathbf{P}^{\sigma_i} \mathbf{L}_{n_i}$ is singular. Thus we must be careful integrating the equations (4.2) not to pass the point, where the solution of the equation (4.2) has a pole. The following statement will be useful for this purpose:

Lemma 4.3. *Let $|A_{ij}(t)| \leq K$ for all the i and j . Let h and μ satisfy the inequality*

$$h < \frac{4}{NK(N\mu + 2)}.$$

Let

$$(4.22) \quad |G_{lk}^i(t)| \leq \mu$$

be valid for all l and k . Then the matrix $(\mathbf{D}_{j(i)}(\tau) \mathbf{P}^{\sigma_i} \mathbf{L}_{n_i})$ is non-singular for all $\tau \in \langle t - h, t + h \rangle$.

Proof. Let q be a solution of the equation

$$q' = K \left(Nq + \frac{N^2}{4} q^2 + 1 \right) = K \left(\frac{Nq}{2} + 1 \right)^2,$$

with the initial condition $q(0) = \mu$; $q(\tau)$ is defined for

$$\tau < \frac{4}{NK(N\mu + 2)}.$$

Evidently

$$|G_{lk}^i(t + \tau)| \leq q(\tau)$$

is valid for

$$|\tau| < \frac{4}{NK(N\mu + 2)}$$

and for every l and k .

If $\mu \geq 1$ then, by a suitable choice of σ_i , we can achieve for each fixed t the following inequality $|G_{lk}^i(t)| \leq \mu$ for all l and k , as the following is valid:

Lemma 4.4. *Let \mathbf{D} be a matrix of type (n, N) and rank n . Then there exists an ordering $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$ of the numbers $1, 2, \dots, N$ such that the matrix*

$\mathbf{DP}^\sigma \mathbf{L}_n$ is non-singular and the absolute values of the elements of the matrix $(\mathbf{DP}^\sigma \mathbf{L}_n)^{-1} \mathbf{D}$ are not greater than 1.

Proof. Let σ be such an ordering that

$$|\det(\mathbf{DP}^\sigma \mathbf{L}_n)| \geq |\det(\mathbf{DP}^\kappa \mathbf{L}_n)|$$

is valid for all the orderings κ . We shall prove that σ is the sought ordering. This will be proved indirectly. Thus let there exists a column β of the matrix \mathbf{D} such that the absolute value of at least one component of the vector $(\mathbf{DP}^\sigma \mathbf{L}_n)^{-1} \beta$ is greater than 1. Let $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be the columns of the matrix $\mathbf{DP}^\sigma \mathbf{L}_n$. In view of the fact that the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ are independent, the vector β may be represented as a linear combination of the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$:

$$\beta = \sum_{k=1}^n \xi_k \alpha_k.$$

Then

$$(\mathbf{DP}^\sigma \mathbf{L}_n)^{-1} \beta = (\xi_1, \xi_2, \dots, \xi_n)^T$$

is valid.

Let l be such an index that $|\xi_l| > 1$ is valid (we suppose that such an index exists). Let us consider the absolute value of the determinant of the matrix

$$\begin{aligned} & (\alpha_1, \alpha_2, \dots, \alpha_{l-1}, \beta, \alpha_{l+1}, \dots, \alpha_n) |\det(\alpha_1, \alpha_2, \dots, \alpha_{l-1}, \beta, \alpha_{l+1}, \dots, \alpha_n)| = \\ & = |\det(\alpha_1, \alpha_2, \dots, \alpha_{l-1}, \sum_{k=1}^n \xi_k \alpha_k, \alpha_{l+1}, \dots, \alpha_n)| = \\ & = |\det(\alpha_1, \alpha_2, \dots, \alpha_{l-1}, \xi_l \alpha_l, \alpha_{l+1}, \dots, \alpha_n)| = \\ & = |\xi_l| |\det(\alpha_1, \alpha_2, \dots, \alpha_{l-1}, \alpha_l, \alpha_{l+1}, \dots, \alpha_n)| = |\xi_l| |\det(\mathbf{DP}^\sigma \mathbf{L}_n)| > |\det(\mathbf{DP}^\sigma \mathbf{L}_n)| \end{aligned}$$

which is a contradiction.

Remark. The ordering σ as well as the resulting matrix $(\mathbf{DP}^\sigma \mathbf{L}_n)^{-1} \mathbf{D}$ simultaneously may be found by a modified Gauss algorithm.

Suppose that we know the numbers $\vartheta_0, \vartheta_1, \dots, \vartheta_i$, the orderings $\sigma_0, \sigma_1, \dots, \sigma_i$, the systems $(\mathbf{G}_0^*, \mathbf{g}_0), (\mathbf{G}_1^*, \mathbf{g}_1), \dots, (\mathbf{G}_{i-1}^*, \mathbf{g}_{i-1})$. Then the matrix $\mathbf{G}_i(\vartheta_i)$ and the vector $\mathbf{g}_i(\vartheta_i) \vartheta_{i+1}$ will be determined as follows: we solve the equation (4.2) with the integration step h so far till either $|G_{lk}^i(\vartheta_i + \tilde{n}h)| > \mu$ at least for one pair l, k or $\langle \vartheta_i, \vartheta_i + (\tilde{n} + 1)h \rangle \cap M \neq \emptyset$ (where \tilde{n} means the number of integration steps performed). In the former case we put $\vartheta_{i+1} = \vartheta_i + \tilde{n}h$ in the latter case ϑ_{i+1} will be the least number of the set $\langle \vartheta_i, \vartheta_i + (\tilde{n} + 1)h \rangle \cap M$ and in the case that $\vartheta_{i+1} \neq \vartheta_i + \tilde{n}h$ we finish solving the equation (4.2) up to the point ϑ_{i+1} . We construct the matrix (according to the initial conditions (4.17)):

$$\mathbf{U}_1, \text{ or } \mathbf{G}_i^*(\vartheta_{i+1}) \text{ or } \mathbf{G}_i^*(\gamma_l) \mathbf{W}_l,$$

and we find the σ_{i+1} for this matrix so that:

$$|G_{l,k}^{i+1}(\vartheta_{i+1})| \leq \mu$$

is valid for all l and k . We choose the number μ and the integration step h so that $\mu \geq 1$ and

$$h < \frac{4}{NK(N\mu + 2)}.$$

Similarly we proceed solving the equations (4.14).

Memory requirements and number of the operations of factorization method.

For the sake of simplicity let us compare the number of operations of the simple and the composed factorization in the case $2n = N$ and $n_1 = n_2 = n$. Let us choose, as a unit of effort, the effort which must be expended on solving one equation in the interval $\langle a, b \rangle$.

1. The simple direct factorization.

1a. The solution is sought at one point t_1 only.

We must solve the equations (4.2) and (4.3) in the interval $\langle a, b \rangle$ which is equivalent to the effort of solving $n(n+1)$ equations in the interval $\langle a, b \rangle$. Then we solve the equations (4.4) in the interval $\langle t_1, l \rangle$ backwards which is equivalent to the effort of solving h equation in the interval $\langle t_1, l \rangle$. Thus the total effort is equivalent to the effort of solving $n(l-t_1)/(l-a) + n(n+1)$ equations in interval $\langle a, b \rangle$.

1b. The solution is sought at every point.

From the point 1a. it follows that the total effort is equivalent to the effort of solving $n + (n+1)n$ equations in the interval $\langle a, b \rangle$. In both cases we must store the matrix \mathbf{G}_i and the vector \mathbf{g}_i , i.e. we must store $n(n+1)$ functions in the interval $\langle a, b \rangle$. In addition, we must store the partition $(\vartheta_0, \vartheta_1, \dots, \vartheta_{v+1})$ and the corresponding ordering σ_i ($i = 0, 1, \dots, v$).

2. The composite factorization.

2a. The solution is sought at one point t_1 only. We must solve the equations (4.2) and (4.3) in the interval $\langle a, t_1 \rangle$ and then the equations (4.14) and (4.15) in the interval $\langle t_1, l \rangle$. This is equivalent to the effort of solving $n(n+1)$ equations in the interval $\langle a, b \rangle$.

2b. The solution is sought at all the points. We must solve the equations (4.2) and (4.3) in the interval $\langle a, b \rangle$ and the equations (4.14) and (4.15) in the interval $\langle a, b \rangle$. This is equivalent to the effort of solving $2n(n+1)$ equations in the interval $\langle a, b \rangle$.

In both cases we need not store any function, but only the system $(\mathbf{G}_i^*, \mathbf{g}_i)$ at the points at which the solution is sought.

Remark. If the problem ψ follows from the problem

$$\sum_{i=0}^n (-1)^i (P_i y^{(i)})^{(i)} = f$$

with n conditions at the point a and with n conditions at the point b which guarantee self-adjointness of the corresponding operator, then it is possible, in some cases, to reduce the number of the factorization equations (4.2) and (4.3) or (4.14) and (4.15) by $(n^2 - n)/2$. If, in addition, the conditions are such that the corresponding operator is positive definite then it is guaranteed that the factorization equations (4.2) and (4.3) have the solutions in the whole interval $\langle a, b \rangle$ and so the effort connected with testing condition (4.22) or seeking new ordering σ_i and new matrix \mathbf{G}_i disappears.

Remark. The matrix and the vectors defined by the definition (4.5) can be interpreted as follows:

$$\begin{aligned} \mathbf{A}_1^i(t) &= \{A_{\sigma_j^i \sigma_k^i}\} \quad \text{where } 1 \leq j \leq n, 1 \leq k \leq n_1, \\ \mathbf{A}_2^i(t) &= \{A_{\sigma_j^i \sigma_k^i}\} \quad \text{where } 1 \leq j \leq n_1, n_1 < k \leq N, \\ \mathbf{A}_3^i(t) &= \{A_{\sigma_j^i \sigma_k^i}\} \quad \text{where } 1 < j \leq N, 1 \leq k \leq n_1, \\ \mathbf{A}_4^i(t) &= \{A_{\sigma_j^i \sigma_k^i}\} \quad \text{where } n_1 < j \leq N, n_1 < k \leq N, \\ \mathbf{F}_1^i(t) &= (f_{\sigma_1^i}, f_{\sigma_2^i}, \dots, f_{\sigma_{n_1}^i})^T, \\ \mathbf{F}_2^i(t) &= (f_{\sigma_{n_1+1}^i}, f_{\sigma_{n_1+2}^i}, \dots, f_{\sigma_N^i})^T, \\ \mathbf{Y}^i(t) &= \{x_{\sigma_1^i}^{j(i)}, x_{\sigma_1^i}^{k(i)}, \dots, x_{\sigma_{n_1}^i}^{j(i)}\}^T, \\ \mathbf{Z}^i(t) &= \{x_{\sigma_{n_1+1}^i}^{j(i)}, x_{\sigma_{n_1+2}^i}^{j(i)}, \dots, x_{\sigma_N^i}^{j(i)}\}^T, \end{aligned}$$

where $j(i)$ is such an index that $\langle \mathcal{D}_i, \mathcal{D}_{i+1} \rangle \subset \langle \gamma_{j(i)}, \gamma_{j(i)+1} \rangle$. We introduce notations:

1. $A_{j, N+1} = f_j$,
2. $(\mathbf{G}_i, \mathbf{g}_i) = \mathbf{B}_i = \{B_{kl}^{(i)}\}$ where $k = 1, 2, \dots, n_1$ and $l = n_1 + 1, \dots, N + 1$,
3. $(\sigma_i, N + 1) = (\sigma_1^i, \sigma_2^i, \dots, \sigma_{N+1}^i)$,
4. $z^i = \{z_k^{(i)}\}$ where $k = n_1 + 1, \dots, N + 1$ and $z_{N+1}^{(i)} = -1$.

The equations (4.2) and (4.3) can be written as a single equation

$$(\mathbf{B}_{kj}^{(i)})' = \sum_{s=n_1+1}^N B_{ks}^{(i)} A_{\sigma_s^i \sigma_j^i} - \sum_{r=n_1+1}^{n_1} (A_{\sigma_k^i \sigma_r^i} + \sum_{s=n_1+1}^{N+1} B_{ks}^{(i)} A_{\sigma_s^i \sigma_r^i}) B_{rj}^{(i)} + A_{\sigma_k^i \sigma_j^i}.$$

The equation (4.4) can be written

$$(z_k^{(i)})' = - \sum_{s=n_1+1}^{N+1} (A_{\sigma_k^i \sigma_s^i} - \sum_{r=1}^{n_1} A_{\sigma_k^i \sigma_r^i} B_{rs}^{(i)}) z_s^{(i)}.$$

5. ON THE STABILITY OF FACTORIZATION METHOD

When numerically solving the factorization equations we make certain errors. Let us investigate how the errors affect the solution obtained by the algorithm described above. The fact that the factorization equations are solved inaccurately can be represented so that the following equations

$$(5.1) \quad \mathbf{G}'_i = \mathbf{G}_i \mathbf{A}_4^i - \mathbf{A}_1^i \mathbf{G}_i - \mathbf{G}_i \mathbf{A}_3^i \mathbf{G}_i + \mathbf{A}_2^i + \Delta_1^i,$$

$$\mathbf{g}'_i = -(\mathbf{A}_1^i + \mathbf{G}_i \mathbf{A}_3^i) \mathbf{g}_i + \mathbf{F}_1^i + \mathbf{G}_i \mathbf{F}_2^i + \Delta_2^i,$$

$$(5.2) \quad \hat{\mathbf{G}}'_i = \hat{\mathbf{G}}_i \hat{\mathbf{A}}_4^i - \hat{\mathbf{A}}_1^i \hat{\mathbf{G}}_i - \hat{\mathbf{G}}_i \hat{\mathbf{A}}_3^i \hat{\mathbf{G}}_i + \hat{\mathbf{A}}_2^i + \Delta_3^i,$$

$$\hat{\mathbf{g}}'_i = -(\hat{\mathbf{A}}_1^i + \hat{\mathbf{G}}_i \hat{\mathbf{A}}_3^i) \hat{\mathbf{g}}_i + \hat{\mathbf{F}}_1^i + \hat{\mathbf{G}}_i \hat{\mathbf{F}}_2^i + \Delta_4^i,$$

$$(5.3) \quad (\mathbf{z}^i)' = -(\mathbf{A}_4^i - \mathbf{A}_3^i \mathbf{G}_i) \mathbf{z}^i + \mathbf{F}_2^i - \mathbf{A}_3^i \mathbf{g}_i + \Delta_5^i,$$

$$(5.4) \quad (\hat{\mathbf{z}}^i)' = -(\hat{\mathbf{A}}_4^i - \hat{\mathbf{A}}_3^i \hat{\mathbf{G}}_i) \hat{\mathbf{z}}^i + \hat{\mathbf{F}}_2^i - \hat{\mathbf{A}}_3^i \hat{\mathbf{g}}_i + \Delta_6^i,$$

are solved accurately instead of solving the equations (4.2), (4.3), (4.4), (4.14), (4.15), (4.16) inaccurately. In the equations written above Δ_j^i ($j = 1, 2, \dots, 6$) there are certain disturbances. The disturbance Δ_1^i can be interpreted as the disturbance of the matrix \mathbf{A}_2^i , the disturbance Δ_2^i as the disturbance of the vector \mathbf{F}_1^i , the disturbance Δ_3^i as the disturbance of the vector \mathbf{F}_2^i etc. Thus the disturbances of the equations (5.1), (5.2), (5.3), (5.4) can be represented as the disturbances of the matrix \mathbf{A} and the vector \mathbf{f} .

This conception indicates how the obtained solution satisfies the equation (1.7).

It now remains to resolve how the solution of the problem ψ is affected by errors which appear in calculating the initial conditions for the factorization equations or in solving the equations (4.21). In all these cases it is a matter of solving systems of equations. Suppose that the calculation of the initial conditions for the factorization equations and solving the equations (4.21) are performed by such a method that the numerical realization of this method leads to the same result as the exact solution of systems disturbed in a certain manner. This assumption is satisfied e.g. in Gauss algorithm (in [12] an estimation of the disturbance of the system is shown which resulted from an inaccurate solution of the system by Gauss algorithm).

From these assumptions it follows that an inaccurate calculation of the initial condition for equation (4.2) and (4.3) at the point a can be interpreted so that instead of the condition $\mathbf{U}_1 \mathbf{x}(a) = \mathbf{u}_1$ there is the condition $(\mathbf{U}_1 + \bar{\Delta}_1) \mathbf{x}(a) = \mathbf{u}_1 + \bar{\delta}_1$.

Let us suppose that all the ϑ_i for $i = 1, 2, \dots, v$ are at M . This assumption can be easily reacted: we add the conditions $\mathbf{x}^-(\vartheta_i) = \mathbf{E} \mathbf{x}^+(\vartheta_i)$ at the points ϑ_i which do not lie at M .

The calculation of the initial condition for the equations (4.2) and (4.3) at the point ϑ_i which is performed inaccurately can be similarly represented so that we perform the calculation accurately but we start with the disturbed system

$$(\mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{W}_i + \bar{\Delta}_{i-1}, \mathbf{g}_{i-1}(\vartheta_i) - \mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{w}_i + \bar{\delta}_{i-1}),$$

where $\bar{A}_{i-1} = \{\bar{A}_{lk}^{i-1}\}$ and $\bar{\delta}_{i-1} = (\delta_1^{i-1}, \delta_2^{i-1}, \dots, \delta_n^{i-1})$ are certain disturbances. Let us define the matrix $\Omega_{i-1} = \{\omega_{kl}^{i-1}\}$ and the vector $\varepsilon_{i-1} = (\varepsilon_1^{i-1}, \varepsilon_2^{i-1}, \dots, \varepsilon_n^{i-1})^T$ in the subsequent way:

$$\omega_{\sigma_i i^{-1}k} = \begin{cases} \bar{A}_{lk}^{i+1} & \text{for } 1 \leq l \leq n_1, 1 \leq k \leq N, \\ 0 & \text{for } n_1 < l \leq N, 1 \leq k \leq N, \end{cases}$$

$$\varepsilon_{\sigma_i i^{-1}}^{i-1} = \begin{cases} \bar{\delta}_l^{i+1} & \text{for } 1 \leq l \leq n_1, \\ 0 & \text{for } n_1 < l < N. \end{cases}$$

We can verify calculating directly that

$$\mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{W}_i + \bar{A}_{i-1} = \mathbf{G}_{i-1}^*(\vartheta_i) (\mathbf{W}_i + \Omega_{i-1}),$$

$$\mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{w}_i - \bar{\delta}_{i-1} = \mathbf{G}_{i-1}^*(\vartheta_i) (\mathbf{w}_i - \varepsilon_{i-1})$$

is valid.

Thus the disturbances \bar{A}_{i-1} and $\bar{\delta}_{i-1}$ can be imagined as disturbances of the matrix \mathbf{W}_i and the vector \mathbf{w}_i .

Now let us investigate the influence of errors arising from numerical calculation of the conditions (4.19) and (4.20). In part 2 of conditions (4.19) or (4.20) the errors made in multiplication and addition or in inversion of the matrix \mathbf{W}_i , can be represented as the disturbances of the matrix \mathbf{W}_i and the vector \mathbf{w}_i (see [12]). Thus let us investigate part 1 of the conditions (4.19). Calculating $\mathbf{x}(b)$ we solve the equations

$$(5.5) \quad \mathbf{y}^v(b) + \mathbf{G}_v(b) \mathbf{z}^v(b) = \mathbf{g}_v(b),$$

$$\mathbf{U}_2 \mathbf{x}^-(b) = \mathbf{u}_2.$$

Let us define the matrices \mathbf{V}_1 and \mathbf{V}_2 as

$$\mathbf{V}_1 = \{U_{i\sigma_j^2}^2\} \quad \text{where } 1 \leq i \leq n_2, n_1 < j < N.$$

$$\mathbf{V}_2 = \{U_{i\sigma_j^2}^2\} \quad \text{where } 1 \leq i \leq n_2, n_1 < j \leq N.$$

The solution of the equation (5.5) will be obtained in the following way: we solve the equation

$$(5.6) \quad (\mathbf{V}_2 - \mathbf{V}_1 \mathbf{G}_v(b)) \mathbf{z}^v(b) = \mathbf{u}_2 - \mathbf{V}_1 \mathbf{g}_v(b)$$

and the vector $\mathbf{y}^v(b)$ will be determined from the equation

$$\mathbf{y}^v(b) = \mathbf{g}_v(b) - \mathbf{G}_v(b) \mathbf{z}^v(b).$$

An inaccurate solution of the equation (5.6) will be again imagined so that the disturbed system (5.6) is accurately solved. It is evident that these disturbances may be represented as the disturbances of the matrix \mathbf{V}_2 and the vector \mathbf{u}_2 .

In composite factorization we must solve the systems (4.21) at the points at which the solution is sought. Let us suppose that the point where the system is solved lies at M and that the solution at this point is sought from the right. For instance at the point ϑ_i the following equations are solved

$$(5.7) \quad \begin{aligned} \mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{W}_i \mathbf{x}^+(\vartheta_i) &= \mathbf{g}_{i-1}(\vartheta_i) - \mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{w}_i, \\ \hat{\mathbf{y}}^i(\vartheta_i) + \hat{\mathbf{G}}_i(\vartheta_i) \mathbf{z}^i(\vartheta_i) &= \hat{\mathbf{g}}_i(\vartheta_i). \end{aligned}$$

Let us define the matrices $\bar{\mathbf{V}}_1$ and $\bar{\mathbf{V}}_2$ in this way:

$$\begin{aligned} \bar{\mathbf{V}}_1 &= \{(\mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{W}_i)_{l\sigma_k^{i-1}}\} \quad \text{where } 1 \leq l \leq n_1, 1 \leq k \leq n_2, \\ \bar{\mathbf{V}}_2 &= \{(\mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{W}_i)_{l\sigma_k^{i-1}}\} \quad \text{where } 1 \leq l \leq n_1, n_2 < k \leq N. \end{aligned}$$

The solution of the equations (5.7) will be obtained as follows:

We solve the equations

$$(5.8) \quad (\bar{\mathbf{V}}_2 - \bar{\mathbf{V}}_1 \hat{\mathbf{G}}_i(\vartheta_i)) \hat{\mathbf{z}}^i(\vartheta_i) = \mathbf{g}_{v-1}(\vartheta_i) - \mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{w}_v - \bar{\mathbf{V}}_1 \hat{\mathbf{g}}_i(\vartheta_i)$$

and the vector $\mathbf{y}^i(\vartheta_i)$ will be obtained from the equation

$$\hat{\mathbf{y}}^i(\vartheta_i) = \hat{\mathbf{g}}_i(\vartheta_i) - \hat{\mathbf{G}}_i(\vartheta_i) \mathbf{z}^i(\vartheta_i).$$

The inaccurate solution of the equations (5.8) can be again imagined so that the disturbed system (5.8) is accurately solved. These disturbances can be represented as the disturbances of the matrix $\bar{\mathbf{V}}_2$ and the vector $\mathbf{g}_{i-1}(\vartheta_i) \mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{w}_i$. The disturbances of the matrix $\mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{W}_i$ and the disturbances of the vector $\mathbf{g}_{i-1}(\vartheta_i) - \mathbf{G}_{i-1}^*(\vartheta_i) \mathbf{w}_i$ may be represented, as it was shown, as the disturbances of the matrix \mathbf{W}_i and the vector \mathbf{w}_i .

Thus the errors made in numerical realization of the method may be represented as the disturbances of the original problem (i.e. the disturbances of the matrix $\mathbf{A}(t)$, the vector $\mathbf{f}(t)$, of the boundary conditions b and eventually of the interface conditions d). These disturbances are of such a character that they may be estimated a priori in dependence on the methods used for solving the factorization equations and for calculation of initial conditions.

The question what influence these disturbances have upon a solution in the original problem will not be treated here. In order to answer this question special knowledge of our differential operator is required, e.g. the knowledge of the magnitude of the norm of corresponding Green function or that of the eigenvalues of the operator.

If the elements of the matrix \mathbf{A} and the components of the vector \mathbf{f} are given inaccurately (or cannot be realized with the required accuracy) and if the problem is solved by the algorithm described above then it is of no value to solve the factorization equations and to calculate the initial conditions much more precisely than the elements of the matrix \mathbf{A} and the components of the vector \mathbf{f} are given.

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Výtah

O METODĚ FAKTORISACE

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V tomto článku je zobrazena metoda faktorizace pro okrajový problém soustavy diferenciálních rovnic s přechodovými podmínkami. Popsaná metoda vede k cíli za předpokladu, že úloha má jediné řešení.

Nejprve se okrajová úloha převede na úlohy počáteční. Toto převedení se uskutečňuje pomocí matic $\mathbf{D}_i(t)$ a $\hat{\mathbf{D}}_i(t)$ a vektorů $\mathbf{d}_i(t)$ a $\hat{\mathbf{d}}_i(t)$ zavedených definicemi 3.1 a 3.2. Pomocí matic $\mathbf{D}_i(t)$ a $\hat{\mathbf{D}}_i(t)$ a vektorů $\mathbf{d}_i(t)$ a $\hat{\mathbf{d}}_i(t)$ jsou pak definovány matice $\mathbf{G}_i(t)$ a $\hat{\mathbf{G}}_i(t)$ a vektory $\mathbf{g}_i(t)$ a $\hat{\mathbf{g}}_i(t)$. Pro tyto matice a vektory jsou uvedeny diferenciální rovnice (tzv. faktorizační) a příslušné počáteční podmínky, takže je možno tyto matice a vektory získávat řešením Cauchyových úloh. Známe-li již matice $\mathbf{G}_i(t)$ a $\hat{\mathbf{G}}_i(t)$ a vektory $\mathbf{g}_i(t)$ a $\hat{\mathbf{g}}_i(t)$ (příp. jen matice $\mathbf{G}_i(t)$ a vektory $\mathbf{g}_i(t)$), můžeme již snadno nalézt řešení původní okrajové úlohy.

Dále je v článku řešena otázka vlivu nepřesnosti na řešení, vyplývajících z numerické realizace metody. Ukáže se, že nepřesnosti, kterých se dopouštíme během numerické realizace metody si můžeme představit jako poruchy v koeficientech původní úlohy.

Резюме

О МЕТОДЕ ФАКТОРИЗАЦИИ

ИРЖИ ТАУФЕР (JIŘÍ TAUFER)

В этой статье обобщен метод факторизации для краевой задачи системы дифференциальных уравнений с переходными условиями. Описанный метод дает результат в том случае, если задача имеет единственное решение.

Сначала краевая задача сводится на задачи начальные. Это делается при помощи матриц $\mathbf{D}_i(t)$ и $\hat{\mathbf{D}}_i(t)$ и векторов $\mathbf{d}_i(t)$ и $\hat{\mathbf{d}}_i(t)$ определенных в 3.1 и 3.3. Потом при помощи матриц $\mathbf{D}_i(t)$ и $\hat{\mathbf{D}}_i(t)$ и векторов $\mathbf{d}_i(t)$ и $\hat{\mathbf{d}}_i(t)$ определяются матрицы $\mathbf{G}_i(t)$ и $\hat{\mathbf{G}}_i(t)$ и векторы $\mathbf{g}_i(t)$ и $\hat{\mathbf{g}}_i(t)$. Для матриц и этих векторов получены дифференциальные уравнения (т. н. уравнения факторизации) с соответствующими начальными условиями, так что эти матрицы и векторы можно получить решением задач Коши. Если мы знаем матрицы $\mathbf{G}_i(t)$ и $\hat{\mathbf{G}}_i(t)$ и векторы $\mathbf{g}_i(t)$ и $\hat{\mathbf{g}}_i(t)$ (или только матрицы $\mathbf{G}_i(t)$ и векторы $\mathbf{g}_i(t)$), можем уже легко найти решение исходной задачи.

Дальше в статье решен вопрос влияния на решение неточностей, которые вытекают из вычислительной реализации метода. Выходит, что неточности, которые делаем в течении вычислительной работы можно представить как возмущения в коэффициентах исходной задачи.

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