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NONHOMOGENEOUS BIRTH-DEATH PROCESSES WITH CONSTANT RATIO OF RATES

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1. INTRODUCTION

This paper contains an investigation of nonhomogeneous birth-death processes in which the birth and death rates $\lambda(t)$ and $\mu(t)$ are positive and continous functions in the open interval $T = (0, \infty)$, and their ratio

$$\frac{\mu(t)}{\lambda(t)} = c$$

is constant everywhere in T. This is a special case of the process investigated by D. G. KENDALL [1].

Under the usual assumptions given at the beginning of Section 2, there are derived expressions for the probabilities $P_x(t)$, $t \geqq 0$, $x = 0, 1, 2, \ldots$, that the population size at time t will be exactly x, and also for the moments $\alpha_i(t)$, $i = 1, 2, 3$, of the distribution of the population size at time t; using these, a necessary condition that the process $\{\lambda(t), \mu(t)\}$ have the property $[\mu(t)/\lambda(t)] = c$ is obtained.

The present paper together with that already quoted [1], with the paper of M. S. BARTLETT [2] and with the previous paper of the author [3] completes the study of fundamental properties of nonhomogeneous birth-immigration-death processes, not only for the general case in which the ratios of rates are not mutually related, but also for the case where this ratio is constant everywhere in T or where one of the rates $\nu(t)$ or $\lambda(t)$ is identically zero everywhere in T.

The results of this paper have been used for the analysis of graphite nucleation in malleable cast iron.

2. FUNDAMENTAL PROPERTIES OF THE PROCESS

Consider the birth-death process with states E_x $(x = 0, 1, 2, \ldots)$.

Assumptions:

a) if at time t the system is in state E_x, then the probability of the transition $E_x \to$ $\to E_{x+1}$ in the interval $(t, t + \Delta t)$ is $x\,\lambda(t)\,\Delta t + o(\Delta t)$ for $x = 1, 2, \ldots$;

b) if at time t the system is in state E_x, then the probability of the transition $E_x \to \to E_{x-1}$ in the interval $(t, t + \Delta t)$ is $x\,\mu(t)\,\Delta t + o(\Delta t)$ for $x = 1, 2, \ldots$;

c) the probability of a transition to a state other than a neighbouring state is $o(\Delta t)$;

d) if at time t the system is in state E_x, then the probability of no change in the interval $(t, t + \Delta t)$ is $1 - x\{\lambda(t) + \mu(t)\}\,\Delta t + o(\Delta t)$;

e) at time $t = 0$ the system is in state E_1.

Let $\xi(t)$ be an integral-valued random variable which assumes the values x of the population size at time t, and let $P_x(t) = \mathscr{P}\{\xi(t) = x\}$, $x = 0, 1, 2, \ldots$

Now introduce the generating function

$$v = \varphi(z, t) = \sum_{x=-\infty}^{+\infty} P_x(t)\, z^x\,, \tag{1}$$

where we define $P_x(t) = 0$ for $x < 0$. From the fundamental differential-difference equations which the functions $P_x(t)$ must satisfy (see e.g. (3) and (4) of [1]) with the initial conditions

$$P_1(0) = 1 \quad \text{and} \quad P_x(0) = 0\,, \quad x \neq 1\,, \tag{2}$$

we find that the function v satisfies the linear partial differential equation

$$\frac{\partial v}{\partial t} - (z - 1)\left[z\,\lambda(t) - \mu(t)\right] \frac{\partial v}{\partial z} = 0 \tag{3}$$

with

$$v = z \quad \text{for} \quad t = 0 \quad \text{and} \quad v = 1 \quad \text{for} \quad z = 1\,. \tag{4}$$

Theorem 1. *Let $\lambda(t)$ and $\mu(t)$ be positive functions, continuous in the open interval $T = (0, \infty)$, and let the ratio*

$$\frac{\mu(t)}{\lambda(t)} = c \tag{5}$$

be constant everywhere in T. Then for the probabilities $P_x(t)$, $t > 0$, the relations

$$P_0(t) = \frac{1}{1 + \left(\int_0^t \lambda(\tau)\,\mathrm{d}\tau\right)^{-1}} \quad \textit{for} \quad c = 1\,,$$
$$= \frac{c(1 - e^{\varrho(t)})}{1 - c e^{\varrho(t)}} \quad \textit{for} \quad c \neq 1 \tag{6}$$

and

$$P_x(t) = \frac{e^{\varrho(t)}}{c^{x-1}} P_0^{x-1}(t)\left[1 - P_0(t)\right]^2 \quad \textit{for} \quad x = 1, 2, \ldots \tag{7}$$

hold, where

(8) $$\varrho(t) = (c - 1)\int_0^t \lambda(\tau)\,\mathrm{d}\tau\,, \quad c > 0\,.$$

Proof: In proving (6) and (7) we shall first look for the appropriate generating function.

For the case given by (5) we shall attempt to find an integral surface $v = \varphi(z, t)$, which fulfils the conditions (4).

From the canonical system of ordinary differential equations associated with the partial differential equation (3) there follow two equations,

(9) $$\frac{\mathrm{d}z}{\mathrm{d}t} = -(z - 1)(z - c)\,\lambda(t)$$

and

(10) $$\mathrm{d}v = 0\,.$$

The first integrals of the considered system are

(11) $$C_1 = \frac{1}{z - 1} - \int_0^t \lambda(\tau)\,\mathrm{d}\tau \qquad \text{for} \quad c = 1\,,$$

$$= \left(\frac{z - 1}{z - c}\right)^{1/(1-c)} e^{\int_0^t \lambda(\tau)\mathrm{d}\tau} \qquad \text{for} \quad c \neq 1$$

and

(12) $$C_2 = v\,.$$

Hence we obtain the general solution. Solving Cauchy's problem, for which we use the equations following from the first integrals (11) and (12) for $t = 0$, we obtain the required equation of the integral surface in the form

(13) $$v = \frac{z - (z - 1)\int_0^t \lambda(\tau)\,\mathrm{d}\tau}{1 - (z - 1)\int_0^t \lambda(\tau)\,\mathrm{d}\tau} \qquad \text{for} \quad c = 1\,,$$

$$= \frac{z(c - e^{\varrho(t)}) - c(1 - e^{\varrho(t)})}{z(1 - e^{\varrho(t)}) - 1 + c e^{\varrho(t)}} \qquad \text{for} \quad c \neq 1\,.$$

The expressions for $P_0(t)$ and $P_x(t)$, $x = 1, 2, \ldots$, in the form (6) and (7) respectively are obtained by expansion of $v = \varphi(z, t)$ given by (13) into a Maclaurin series in powers of z, and comparing the coefficients of the powers z^x of this series with those of the corresponding powers in the series (1). This completes the proof of Theorem 1.

Corollary 1. *For the process* $\{\lambda(t), \mu(t)\}$ *with the properties given in* (5), *the relations*

(14) $$\lim_{t\to\infty} P_0(t) = 1 \quad \textit{if} \quad \int_0^\infty \lambda(\tau)\,\mathrm{d}\tau = \infty\,, \quad < 1 \quad \textit{if} \quad \int_0^\infty \lambda(\tau)\,\mathrm{d}\tau < \infty$$

and for $c \geqq 1$ *hold; for* $c < 1$ *the relation*

(14′) $$\lim_{t\to\infty} P_0(t) < 1$$

always holds.

Proof: The equations (14) and (14′) follow by taking limits for $t \to \infty$ in (6).

Corollary 2. *Let* $\alpha_i(t)$, $i = 1, 2, \ldots$, *be the i-th moment of the distribution of the random variable* $\xi(t)$; *then for the process* $\{\lambda(t), \mu(t)\}$ *with properties* (5), *the relations*

(15) $$\alpha_1(t) = e^{-\varrho(t)} \quad \textit{for} \quad c > 0\,,$$

(16) $$\alpha_2(t) = 1 + 2\int_0^t \lambda(\tau)\,\mathrm{d}\tau \qquad \textit{for} \quad c = 1\,,$$

$$= \frac{2e^{-\varrho(t)} - c - 1}{(1 - c)\,e^{\varrho(t)}} \quad \textit{for} \quad c \neq 1\,,$$

(17) $$\alpha_3(t) = 1 + 6\int_0^t \lambda(\tau)\,\mathrm{d}\tau\left[\int_0^t \lambda(\tau)\,\mathrm{d}\tau + 1\right] \qquad \textit{for} \quad c = 1\,,$$

$$= \frac{e^{-3\varrho(t)}}{(1 - c)^2}\{6(1 - e^{\varrho(t)})(1 - ce^{\varrho(t)}) + (1 - c)^2\,e^{2\varrho(t)}\} \quad \textit{for} \quad c \neq 1$$

hold, where $\varrho(t)$ *is given by* (8).

Proof: The expressions (15), (16) and (17) follow from the known relations between the generating function of the probabilities $P_x(t)$, $x = 1, 2, \ldots$, and the moments $\alpha_i(t)$, $i = 1, 2, \ldots$, of this distribution.

Corollary 3. *For the process* $\{\lambda(t), \mu(t)\}$ *with the properties* (5) *the relations*

(18) $$\lim_{t\to\infty} \alpha_1(t) = 1 \qquad \textit{for} \quad c = 1\,,$$

$$= e^{(1-c)\int_0^\infty \lambda(\tau)\mathrm{d}\tau} \quad \textit{for} \quad c \neq 1\,, \quad \int_0^\infty \lambda(\tau)\,\mathrm{d}\tau < \infty\,,$$

$$= 0 \qquad \textit{for} \quad c > 1\,, \quad \int_0^\infty \lambda(\tau)\,\mathrm{d}\tau = \infty\,,$$

$$= \infty \qquad \textit{for} \quad c < 1\,, \quad \int_0^\infty \lambda(\tau)\,\mathrm{d}\tau = \infty$$

hold.

Proof: The equations (18) follow by taking limits for $t \to \infty$ in (15).

Corollary 4. *A necessary condition that the process* $\{\lambda(t), \mu(t)\}$ *has property* (5), *is that for every* $t \in T$ *the relations*

$$(19) \qquad \frac{\alpha_3(t) - 1}{\alpha_2^2(t) - 1} = \frac{3}{2} \quad \textit{for} \quad c = 1$$

and

$$(20) \qquad \mathsf{V}^2[\xi(t)]\,\{1 - \mathsf{E}^{-1}[\xi(t)]\}^{-1} = \frac{1 + c}{1 - c} = \text{konst.} \quad \textit{for} \quad c \neq 1$$

hold, where $\mathsf{E}[\xi(t)]$ *is the expected value and* $\mathsf{V}[\xi(t)]$ *is the coefficient of variation of the random variable* $\xi(t)$.

Proof: Condition (19) follows from (16) and (17), condition (20) from (15) and (16).

Corollary 5. *Let* $\mathsf{E}[\xi^*(t)]$ *be the expected value of the size of the population at time* t *due to the nonhomogeneous birth-immigration-death process* $\{\lambda(t), \nu(t), \mu(t)\}$ *with constant ratios of the rates*

$$(21) \qquad \frac{\nu(t)}{\lambda(t)} = b \quad \textit{and} \quad \frac{\mu(t)}{\lambda(t)} = c\,, \quad t \in T\,,$$

where all rates are positive and continuous in T. *Then between the expected values* $\mathsf{E}[\xi(t)]$ *and* $\mathsf{E}[\xi^*(t)]$, *the relations*

$$(22) \qquad \mathsf{E}[\xi(t)] = \mathsf{E}[\xi^*(t)] \left\{\int_0^t \nu(\tau)\,\mathrm{d}\tau\right\}^{-1} \quad \textit{for} \quad c = 1\,,$$

$$= 1 - \frac{c - 1}{b}\,\mathsf{E}[\xi^*(t)] \qquad \textit{for} \quad c \neq 1$$

hold for every $t \in T$.

Proof: (22) follow from (15) of the present paper and from (22) of [3].

Note. It is obvious that some of the expressions derived in this paper must correspond to those obtained by Kendall in [1]. For example, the expression for $P_0(t)$, $t \in T$, $c \leqq 1$, given by (6) can be obtained from equation (8) and (12) of [1] on applying the identity

$$e^{\varrho(t)} - 1 \equiv (c - 1) \int_0^t \lambda(\tau)\, e^{\varrho(\tau)}\,\mathrm{d}\tau\,, \quad c \neq 1\,.$$

The assertion of Corollary 1 is only an analogous transcript of (18) and (19) of [1]. The same also holds for (15) and (18).

On the other hand, the expression for $P_0(t)$ for $c = 1$ given by (6) cannot be derived without the knowledge of the corresponding generating function (13). A similar assertion holds also for probabilities $P_x(t)$, $x = 1, 2, \ldots$ The comparison for other assertions cannot be made.

It therefore appeared more reasonable to start out from the partial differential equation (3) modified in accordance with condition (5); to obtain the generating function directly by the simple solution (different from Kendall's) of this equation; to use this function for studying the fundamental properties of the process; and finally to verify the agreement of the results obtained with those of [1].

References

[1] *D. G. Kendall:* On the generalized birth and death process. Ann. Math. Statistics 19 (1948), 1–15.

[2] *M. S. Bartlett:* An Introduction to Stochastic Processes with Special Reference to Methods and Applications. Cambridge University Press 1955.

[3] *V. Horálek:* On some types of nonhomogeneous birth-immigration-death processes. Aplikace matematiky 9 (1964), 421–434.

Výtah

NEHOMOGENNÍ PROCES ROZENÍ-UMÍRÁNÍ S KONSTANTNÍM POMĚREM INTENSIT

VRATISLAV HORÁLEK

V práci je uvažován nehomogenní proces rození-umírání s intensitami rození $\lambda(t)$ a umírání $\mu(t)$. Za běžných předpokladů je studován speciální případ, kdy funkce $\lambda(t)$ a $\mu(t)$ jsou positivní a spojité v intervalu $T = (0, \infty)$ a poměr

$$\frac{\mu(t)}{\lambda(t)} = c$$

je všude v T konstantní.

Jsou odvozeny vzorce pro pravděpodobnosti $P_x(t)$, $t \geqq 0$, $x = 0, 1, 2, \ldots$, že soubor v čase t bude tvořen právě x částicemi, dále vzorce pro prvé tři obecné momenty $\alpha_i(t)$, $i = 1, 2, 3$, rozdělení počtu částic v čase t a je určena pravděpodobnost vymření souboru pro $t \to \infty$. Je ukázáno, že nutnou podmínkou pro to, aby proces $\{\lambda(t), \mu(t)\}$ měl poměr intensit všude v T konstantní, je aby pro každé $t \in T$

$$\frac{\alpha_3(t) - 1}{\alpha_2^2(t) - 1} = \frac{3}{2} \quad \text{pro} \quad c = 1$$

a

$$\mathsf{V}^2[\xi(t)] \{1 - \mathsf{E}^{-1} [\xi(t)]\}^{-1} = \frac{1 + c}{1 - c} = \text{konst}. \quad \text{pro} \quad c \neq 1,$$

kde $\mathsf{E}[\xi(t)]$ je matematická naděje a $\mathsf{V}[\xi(t)]$ je variační koeficient náhodné proměnné $\xi(t)$, nabývající hodnot počtu částic v souboru v čase t.

Резюме

НЕОДНОРОДНЫЙ ПРОЦЕСС РОЖДЕНИЯ – ГИБЕЛИ С ПОСТОЯННЫМ ОТНОШЕНИЕМ КОЭФФИЦИЕНТОВ РОЖДЕНИЯ И ГИБЕЛИ

ВРАТИСЛАВ ГОРАЛЕК (VRATISLAV HORÁLEK)

В работе рассматривается неоднородный процесс рождения — гибели с коэффициентами рождения $\lambda(t)$ и гибели $\mu(t)$. При обычных предположениях решается специальный случай, когда функции $\lambda(t)$ и $\mu(t)$ положительны и непрерывны в интервале $T = (0, \infty)$ и отношение

$$\frac{\mu(t)}{\lambda(t)} = c$$

всюду в T постоянно.

Выводятся формулы для вероятностей $P_x(t)$, $t \geqq 0$, $x = 0, 1, 2, \ldots$, что совокупность во времени t будет образована именно x частицами, далее формулы для первых трех моментов $\alpha_i(t)$, $i = 1, 2, 3$, распределения количества частиц во времени t, и установлена вероятность вымирания совокупности для $t \to \infty$. Еще показано, что еэли $\{\lambda(t), \mu(t)\}$ – неоднородный процесс рождения-гибели с постоянным отношением коэффициентов всюду в T, то для каждого $t \in T$

$$\frac{\alpha_3(t) - 1}{\alpha_2^2(t) - 1} = \frac{3}{2} \quad \text{для} \quad c = 1$$

и

$$\mathsf{V}^2[\xi|t)]\{1 - \mathsf{E}^{-1}[\xi|t)]\}^{-1} = \frac{1 + c}{1 - c} = \text{konst.} \quad \text{для} \quad c \neq 1\,,$$

где $\mathsf{E}[\xi(t)]$ – математическое ожидание и $\mathsf{V}[\xi(t)]$ – коэффициент вариации случайной переменной $\xi(t)$, принимающей значения, равные количеству частиц совокупности во времени t.

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