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THE L_2-NORM IN THE STUDY OF ERROR PROPAGATION IN INITIAL VALUE PROBLEMS

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(to topic b)

We consider[1]) computations in $r + 1$ dimensional rectangular grids where one coordinate — called time t — has been distinguished by the posing of the initial value problem. The grid points are $P^n_\nu := (nh; \nu_1 h_1, \ldots, \nu_r h_r)$ and the values of a vector-valued function u on the grid are correspondingly denoted by u^n_ν. The grid parameters h_ϱ in the spacial directions x_ϱ, $\varrho = 1(1)\, r$, are given functions of the time step h and tend to zero with h.

We assume that by the nature of the problem we may restrict our considerations in each grid level $t = nh$ to r-dimensional grid domains L_h with N_h grid points, N_h finite for $h > 0$. The values of a grid function u for $t = nh$ are measured by a norm $\|u\|^n_h$. The two ordinarily used norms are (see e.g. [1]):

1) The maximum norm: $\quad {}^\infty\|u\|^n_h := \max\limits_{\nu \in L_h} \|u^n_\nu\|$.

2) The (discretized) L_2-norm: $\quad {}^2\|u\|^n_h := \sqrt{\dfrac{1}{N_h} \sum\limits_{\nu \in L_h} \|u^n_\nu\|^2}$.

The right-hand norm $\|\ldots\|$ is some vector norm for the function vectors u^n_ν, its choice is of no influence on our considerations.

If an initial value problem for a partial differential equation is solved numerically by a $m + 1$ level discretization method in a rectangular grid, the error vectors E^n_ν obey a partial difference equation which is of m-th order with respect to t:

$$\sum_\tau A^0_\tau E^n_{\nu+\tau} = \sum_\tau A^1_\tau E^{n-1}_{\nu+\tau} + \ldots + \sum_\tau A^m_\tau E^{n-m}_{\nu+\tau} + \varepsilon^n_\nu\,, \tag{1}$$

$$\nu \in L_h\,, \quad n = m, m + 1, \ldots$$

The ε^n_ν are the local errors, both from discretization and round-off. The coefficient

[1]) Comp. [1] for more details of the problem and notation.

matrices A_τ^μ will in general depend on the grid parameters h and h_ϱ and on the independent variables t and x_ϱ. The summations over τ may be over fixed vicinities of 0 or over the whole grid domain L_h.

It is assumed that the initial value problem for the partial difference equation (1) is properly posed, i.e. that (1) may be solved for E^n at each point of the grid, $n \geqq m$. As an immediate consequence the stability properties of (1) with respect to the inhomogeneities ε_ν^n are equivalent to those with respect to initial values. Therefore the $m+1$ level algorithm is stable in a norm $\|\ldots\|_h^n$ if the solutions of the homogeneous equation (1) admit an estimate

$$\|E\|_h^n \leqq S \max_{\mu=1(1)m} \|E\|_h^{l-\mu}, \quad m \leqq l \leqq n, \tag{2}$$

$$\text{for} \quad m \leqq n \leqq T/h, \quad T > 0 \quad \text{fixed}, \quad 0 < h \leqq h_0 .$$

The important aspect of (2) is, of course, that the estimate must be uniform in h as h approaches zero and the number N_h of grid points in the domain L_h tends to infinity.

As a consequence of (2) the accumulated error E^n of the original algorithm may be estimated by

$$\|E\|_h^n \leqq K[\sum_{\mu=0}^{m-1} \|e\|_h^\mu + \sum_{l=m}^{n} (\|d\|_h^l + \|r\|_h^l)] \tag{3}$$

where e_ν^μ are the starting errors, d_ν^n the local discretization errors and r_ν^n the local round-off errors of the computation.

When $\|\ldots\|_h^n$ in (2) has been the L_2-norm the estimate (3) is also in this norm. This implies a bound on the individual errors E_ν^n for $t = nh$ which is $\sqrt{N_h}$ times as large as the one obtained from (2) and (3) in the max-norm. ($N_h \to \infty$ as $h \to 0$!) Nevertheless, the large majority of stability investigations for partial difference equations of type (1) have been based on the L_2-norm for two reasons:

a) The stability analysis in the L_2-norm is usually much easier than in the max-norm.

b) The error growth found in practical computations with L_2-stable algorithms never exceeded that which was to be expected for max-stability even if the particular algorithms were not max-stable at all (like the Lax-Wendroff scheme).

We will shortly analyze the reasons for this phenomenon b.

We will separate the treatment of discretization and round-off errors because they are of a different structure (although they both propagate according to (1)): The local discretization errors d_ν^n can ordinarily be regarded as discretizations of a smooth function $d(t, x)$ while the local round-off errors r_ν^n are ordinarily realizations of a random variable.

Let us first look at the global discretization error D_ν^n: It has been shown in [2] that for a p-th order method D_ν^n possesses an asymptotic expansion

$$\begin{aligned} D_\nu^n(h) &= h^p D_0(t_n, x_\nu) + h^{p+1} D_1(t_n, x_\nu) + \ldots + \\ &+ h^P D_{P-p}(t_n, x_\nu) + \hat{D}_\nu^n \quad \text{with} \quad {}^2\|\hat{D}\|_h^n = O(h^{P+1}) \end{aligned} \tag{4}$$

if the original problem as well as the algorithm are sufficiently differentiable[2]) and if the algorithm is L_2-stable. The functions $D_l(t, x)$ *do not depend upon h*, they are bounded in the regions considered.

Therefore, if $\sqrt{N_h} = O(h^{-q})$, $q > 0$, we have from (4)

$$\max_{n \leqq T/h, \nu \in L_h} \|D_\nu^n(h)\| = O(h^P) \quad \text{if} \quad P \geqq p + q - 1 \tag{5}$$

since ${}^\infty\|\hat{D}\|_h^n \leqq \sqrt{N_h} \cdot {}^2\|\hat{D}\|_h^n$ (see [1], Theorem 4.4). As P depends only on the differentiability properties of the problem[2]), for sufficiently smooth problems the growth of the discretization error in L_2-stable algorithms does not differ from that in max-stable ones[3]).

With respect to the local round-off errors r_ν^n we assume that they are independent random variables with mean zero. It is then reasonable to obtain a bound for the covariance matrix of the accumulated round-off error R_ν^n instead of a bound for R_ν^n itself since the first one will much better indicate the size of the error which is likely to occur (comp. e.g. [4]).

As a solution of (1), R_ν^n depends linearly on the local errors r_ν^n:

$$R_\nu^n = \sum_{l=m}^{n} \sum_{\lambda \in L_h} G_{\nu,\lambda}^{n,l} r_\lambda^l \,.$$

This implies (because of the independence of the various r_ν^n)

$$\operatorname{covar}(R_\nu^n) = \sum_{l=m}^{n} \sum_{\lambda \in L_h} G_{\nu,\lambda}^{n,l} \operatorname{covar}(r_\lambda^l) \; (G_{\nu,\lambda}^{n,l})^T$$

or

$$\|\operatorname{covar}(R_\nu^n)\| < n\sigma^2 \max_{l} \sum_{\lambda \in L_h} \|G_{\nu,\lambda}^{n,l}\|^2 \tag{6}$$

where σ^2 is a common bound for the covariance matrices of the r_ν^n. But the L_2-stability of (1) is equivalent to the uniform boundedness of $\sum_{\lambda} \|G_{\nu,\lambda}^{n,l}\|^2$ for arbitrary $l \leqq n$ and $\nu \in L_h$ as $h \to 0$. Hence (6) implies for an L_2-stable algorithm

$$\max_{n \leqq T/h, \nu \in L_h} \|\operatorname{covar}(R_\nu^n)\| \leqq M \frac{\sigma^2}{h} \,. \tag{7}$$

Thus the bound for a deviation which is not exceeded with given probability grows only like $1/\sqrt{h}$.

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[2]) For the concise differentiability assumptions see [2].

[3]) The above reasoning was employed — in a somewhat different and more special form — by STRANG ([3]).

As (5) and (7) are identical with the estimates which could have been obtained immediately for max-stable algorithms we have shown that *under the assumptions stated* L_2-stability guarantees the same restricted growth of the error as max-stability. Only in extreme situations a L_2-stable scheme which is not max-stable will behave worse than a max-stable one.

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