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THE MOTION OF A COLD FRONT[1])

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INTRODUCTION

The motion of a layer of cold air on the rotating earth, can be formulated as a symmetric hyperbolic first order system in terms of two horizontal velocity components and the depth. The following two-dimensional model incorporates the "average" effect that the upper warm air layer has on the cold air; neglects thermodynamics and incompressibility; and incorporates the hydrostatic pressure law (see [1]):

$$P_\tau + AP_\xi + BP_\eta = QP + K\,, \tag{1}$$

where

$$P \equiv \begin{pmatrix} u \\ v \\ \Phi \end{pmatrix}, \quad K \equiv \begin{pmatrix} 0 \\ G \\ 0 \end{pmatrix},$$

$$A \equiv \begin{pmatrix} u, & 0, & \frac{1}{2}\Phi \\ 0, & u, & 0 \\ \frac{1}{2}\Phi, & 0, & u \end{pmatrix}, \quad B \equiv \begin{pmatrix} v, & 0, & 0 \\ 0, & v, & \frac{1}{2}\Phi \\ 0, & \frac{1}{2}\Phi, & v \end{pmatrix}, \quad Q \equiv \begin{pmatrix} 0, & F, & 0 \\ -F, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix},$$

in dimensionless variables, with F and G constant, $(u, v) \equiv$ horizontal velocity components, $\Phi^2 \equiv 4 \cdot (\text{depth of cold air})$.

Equation (1) is to be solved with initial data given at $\tau = 0$, and appropriate boundary data in the changing domain $\Phi > 0$.

$$\Phi(\xi, \eta, \tau) = 0 \quad \text{is the "cold front."}$$

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The work herein reported on is a joint effort of A. Kasahara, J. J. Stoker and myself (see [3]). In the short space available to produce this paper, the author presents an abstract of that work and is ready to accept the sole blame for any shortcomings.

The motion of the "cold front" is determined by replacing (1) by finite differences and following the motion of "frontal points" $(\xi_i(\tau), \eta_i(\tau))$ satisfying

$$(2) \qquad \Phi(\xi_i(\tau),\ \eta_i(\tau), \tau) \equiv 0\ .$$

The meteorological "occlusion" process, which is characterized by the development of a cusp at the cold front, is found to begin in the numerical results.

The above example was offered at the Conference at Liblice, as a challenge to numerical analysts to develop efficient methods for the solution of "free boundary" problems.

METHOD OF SOLUTION

In the interior of the region of cold air, the finite difference scheme to approximate (1), of second order accuracy in the time step $\Delta\tau$, is found by setting at (ξ, η)

$$(2) \qquad \begin{aligned} P(\tau + \Delta\tau) &= P(\tau) - \langle A \rangle\, P_\xi(\tau + \tfrac{1}{2}\Delta\tau)\,.\,\Delta\tau - \\ &- \langle B \rangle\, P_\eta(\tau + \tfrac{1}{2}\Delta\tau)\,.\,\Delta\tau + (Q\langle P \rangle + K)\,.\,\Delta\tau\,, \end{aligned}$$

where

$$\langle g \rangle \equiv \tfrac{1}{2}[g(\tau + \Delta\tau) + g(\tau)]\,.$$

We then express

$$P_\xi(\tau + \tfrac{1}{2}\Delta\tau) = P_\xi(\tau) + \frac{\Delta\tau}{2}\frac{\partial}{\partial\xi}(P_\tau(\tau)) + O(\Delta\tau^2)\,,$$

where P_τ is replaced by using (1). (See [2] for a systematic discussion of difference schemes.)

Finally, since u, v and Φ appear linearly in A and B, (2) can be written in the explicit form

$$(3) \qquad P(\xi, \eta, \tau + \Delta\tau) = S\, P(\xi, \eta, \tau) + K^*\, \Delta\tau\,,$$

where S is a difference operator in (ξ, η) which involves the eight nearest neighbors, that is, for $i, j = -1, 0, 1$, $(\xi + i\,\Delta\xi, \eta + j\,\Delta\eta)$, while K^* is a vector. At interior points whose eight lattice neighbors are not interior to the cold air region at time τ, a slightly different procedure is used.

We solve the initial-boundary value problem for (1) in a domain bounded on the South by the front $\Phi = 0$; on the East and West by the periodicity condition $P(\xi, \eta, \tau) = P(\xi + D, \eta, \tau)$; and on the North by a mountain range with the condition $v(\xi, Y, \tau) \equiv 0$. The motion of the front is determined by following the motion of points that are originally selected to be about $0.8\ \Delta\xi$ apart.

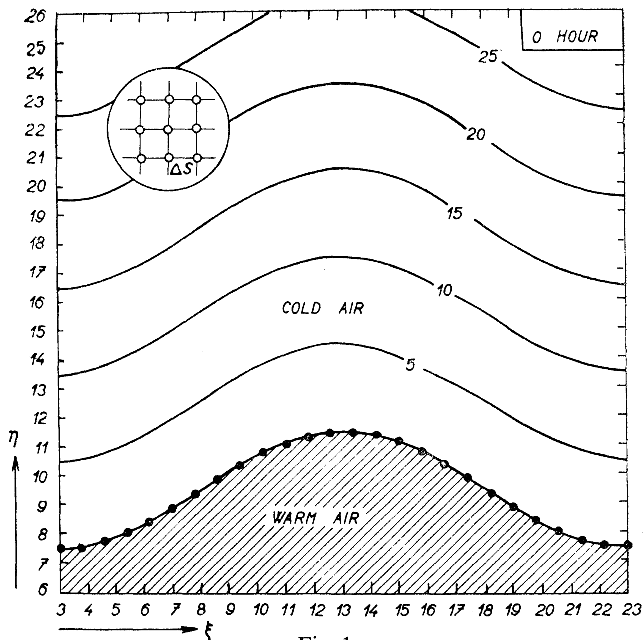
0 HOUR
25
20
15
10
5
COLD AIR
WARM AIR
ΔS
η
ξ

Fig. 1.

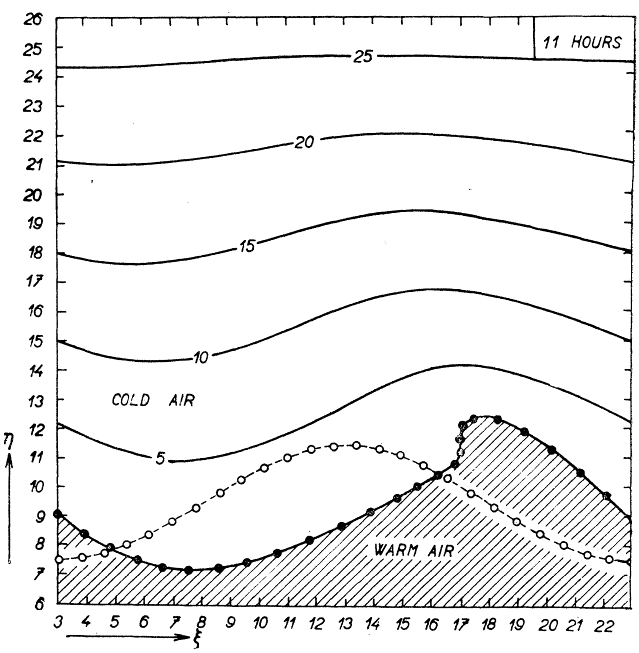
11 HOURS
25
20
15
10
5
COLD AIR
WARM AIR
η
ξ

Fig. 2.

In our calculations, we used $D \sim 1500$ km, $Y \sim 1500$ km, $\Delta\xi = \Delta\eta = \Delta s \sim 76$ km, $\Delta\tau \sim 600$ sec. We followed the motion which resulted from an initially "sinusoidal" front for approximately 11 hours. At the end of this time, the cold front had an asymmetric shape typical of the beginning of the occlusion process. In Figure 1 we plot the initial contour lines at 5,000 ft. intervals and in Figure 2 we plot the contour lines after 11 hours.

CONCLUSION

A large amount of human and machine effort was needed to formulate and partially solve the above mathematical problem. The only difficulty in solving (1) arose from the occurrence of the free boundary. Before such free surfaces can be incorporated into the solution of more complete meteorological problems, considerable simplification of the numerical method is needed.

Bibliography

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[3] *A. Kasahara, E. Isaacson, J. J. Stoker*, 1964: Numerical Studies of Frontal Motion in the Atmosphere — I. Report to be issued by Courant Institute of Mathematical Sciences, New York University.

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