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PŘEDBĚŽNÁ SDĚLENÍ

CONSTITUTIVE RELATIONS FOR THE PLANE-STRAIN PROBLEM OF THE DEVIATION THEORY OF PLASTICITY

[ADVANCED NOTE]

ZDENĚK SOBOTKA

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The paper contains the constitutive statical and kinematical equations for the plane-strain problem of the deviation theory of plasticity.

The deviation theory of plasticity is characterized by the non-coincidence of the directions of principal stresses and of principal strain rates. The directions of principal stresses are given by the axes of angles formed by the slip lines, and the directions of principal strain rates may be found from stream lines. In the general case, these directions are inclined at an angle which has been called by the author the angle of deviation.

The deviation is influenced by the internal friction, compressibility and dilatancy of the material. It is in some relations with the angle of internal friction and with the effect of the normal mean stress on the yield condition.

The stress field is defined by two differential equations of equilibrium and by the yield condition; this, in the general case of a plane-strain state, may be expressed by

$$\Psi(\sigma_x, \sigma_y, \tau_{xy}; x, y) = 0\,. \tag{1}$$

Differentiating this equation with respect to x and y, and introducing for the partial derivatives of the shear stress the expressions from the equations of equilibrium

$$\frac{\partial \tau_{yx}}{y\varrho} = X - \frac{\partial \sigma_x}{\partial x}\,, \tag{2}$$

$$\frac{\partial \tau_{xy}}{\partial x} = Y - \frac{\partial \sigma_y}{\partial y}\,, \tag{3}$$

the author has obtained the set of two partial differential equations of limit equilibrium, containing the partial derivatives of normal stresses only

$$\lambda \frac{\partial \sigma_x}{\partial x} + \eta \frac{\partial \sigma_y}{\partial x} - \chi \frac{\partial \sigma_y}{\partial y} + = A\,0\,, \tag{4}$$

$$\lambda \frac{\partial \sigma_x}{\partial y} + \eta \frac{\partial \sigma_y}{\partial y} - \chi \frac{\partial \sigma_x}{\partial x} + B = 0\,, \tag{5}$$

where

$$\lambda = \frac{\partial \Psi}{\partial \sigma_x}\,, \quad \eta = \frac{\partial \Psi}{\partial \sigma_y}\,, \quad \chi = \frac{\partial \Psi}{\partial \tau_{xy}}\,, \tag{6}$$

$$A = \frac{\partial \Psi}{\partial x} + Y \frac{\partial \Psi}{\partial \tau_{xy}}\,, \qquad B = \frac{\partial \Psi}{\partial y} + X \frac{\partial \Psi}{\partial \tau_{xy}}\,. \tag{7}$$

If (1) permits an explicit expression of the shear stress

$$\tau_{xy} = \psi(\sigma_x, \sigma_y; x, y)\,, \tag{8}$$

it is easily eliminated in terms of λ, η, χ, A and B. The direct substitution of (8) into the equations of equilibrium leads to the following values

$$\lambda = \frac{\partial \psi}{\partial \sigma_x}\,, \quad \eta = \frac{\partial \psi}{\partial \sigma_y}\,, \quad \chi = -1\,, \tag{9}$$

$$A = \frac{\partial \psi}{\partial x} - Y\,, \quad B = \frac{\partial \psi}{\partial y} - X\,. \tag{10}$$

The system (4) and (5) has two families of characteristics defined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\eta}\left(-\chi \pm \sqrt{\chi^2 - 4\lambda\eta}\right), \tag{11}$$

$$\xi_{1,2}^2 \frac{\mathrm{d}\sigma_x}{\mathrm{d}x} - \eta^2 \frac{\mathrm{d}\sigma_y}{\mathrm{d}x} = A\eta + B\xi_{1,2}\,, \tag{12}$$

where

$$\xi_{1,2} = \frac{1}{2}\left(\chi \pm \sqrt{\chi^2 - 4\lambda\eta}\right). \tag{13}$$

The condition of reality of characteristics representing the slip lines is given by

$$\chi^2 \geqq 4\lambda\eta\,. \tag{14}$$

The velocity field may be defined by the condition of continuity and by the condition of deviation. The condition of continuity of the non-homogeneous, compressible and dilatant material

$$\frac{\partial(\varrho \dot{u})}{\partial x} + \frac{\partial(\varrho \dot{v})}{\partial y} = 0\,, \tag{15}$$

where $\dot{u}$ and $\dot{v}$ are the displacement rates and $\varrho(\sigma_x, \sigma_y, \tau_{xy}; x, y)$ the density, may be rewritten as follows

$$(16)\qquad \frac{\partial \dot{u}}{\partial x} + \frac{\partial \dot{v}}{\partial y} + \frac{1}{\varrho}\left[\dot{u}\left(\frac{\partial \varrho}{\partial x} + \frac{\partial \varrho}{\partial \sigma_x}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \varrho}{\partial \sigma_y}\frac{\partial \sigma_y}{\partial x} + \frac{\partial \varrho}{\partial \tau_{xy}}\frac{\partial \tau_{xy}}{\partial x}\right) + \right.$$
$$\left. + \dot{v}\left(\frac{\partial \varrho}{\partial y} + \frac{\partial \varrho}{\partial \sigma_x}\frac{\partial \sigma_x}{\partial y} + \frac{\partial \varrho}{\partial \sigma_y}\frac{\partial \sigma_y}{\partial y} + \frac{\partial \varrho}{\partial \tau_{xy}}\frac{\partial \tau_{xy}}{\partial y}\right)\right] = 0\,.$$

If there is no dilatancy, the density is a function of the normal mean stress $\sigma = \frac{1}{2}(\sigma_x + \sigma_y)$ only and we have

$$(17)\qquad \frac{\partial \varrho}{\partial \sigma_x} = \frac{\partial \varrho}{\partial \sigma_y}\,, \quad \frac{\partial \varrho}{\partial \tau_{xy}} = 0\,.$$

The condition of deviation expresses the relation between the principal stresses and strain rates.

Denoting by α the angle formed by the major principal stress and by β that formed by the major principal strain rate with the positive direction of the x-axis, we may write

$$(18)\qquad \beta = \alpha + \vartheta\,,$$

where ϑ is the angle of deviation; this may be constant or a function of $\sigma_x, \sigma_y, \tau_{xy}, x, y$.

The angles α and β are given by

$$(19)\qquad \operatorname{tg} 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\dot{\gamma}_{xy} - (\dot{\varepsilon}_x - \dot{\varepsilon}_y)\operatorname{tg} 2\vartheta}{\dot{\varepsilon}_x - \dot{\varepsilon}_y + \dot{\gamma}_{xy}\operatorname{tg} 2\vartheta}\,,$$

$$(20)\qquad \operatorname{tg} 2\beta = \frac{\dot{\gamma}_{xy}}{\dot{\varepsilon}_x - \dot{\varepsilon}_y} = \frac{2\tau_{xy} + (\sigma_x - \sigma_y)\operatorname{tg} 2\vartheta}{\sigma_x - \sigma_y - 2\tau_{xy}\operatorname{tg} 2\vartheta}\,.$$

The preceding equations give the well known Lévy-Mises relations as a particular case for $\vartheta = 0$.

Introducing into (19) or (20), respectively, the relations between the strain and the displacement rates, we obtain the second differential equation for the velocity field

$$(21)\qquad \frac{\partial \dot{u}}{\partial x}\left[(\sigma_x - \sigma_y)\operatorname{tg} 2\vartheta + 2\tau_{xy}\right] - \frac{\partial \dot{u}}{\partial y}(\sigma_x - \sigma_y - 2\tau_{xy}\operatorname{tg} 2\vartheta) -$$
$$- \frac{\partial \dot{v}}{\partial x}(\sigma_x - \sigma_y - 2\tau_{xy}\operatorname{tg} 2\vartheta) - \frac{\partial \dot{v}}{\partial y}\left[(\sigma_x - \sigma_y)\operatorname{tg} 2\vartheta + 2\tau_{xy}\right] = 0\,.$$

The system of (16) and (21) has two families of real characteristics defined by

$$(22)\quad \kappa_{1,2} \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\sigma_x - \sigma_y)\,\mathrm{tg}\,2\vartheta + 2\tau_{xy} \pm \sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]\,(1 + \mathrm{tg}^2\,2\vartheta)}}{\sigma_x - \sigma_y - 2\tau_{xy}\,\mathrm{tg}\,2\vartheta},$$

$$(23)\quad \kappa_{1,2}\frac{\mathrm{d}\dot{u}}{\mathrm{d}x} - \frac{\mathrm{d}\dot{v}}{\mathrm{d}x} = \mp C\,\frac{\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]\,(1 + \mathrm{tg}^2\,2\vartheta)}}{\sigma_x - \sigma_y - 2\tau_{xy}\,\mathrm{tg}\,2\vartheta},$$

where

$$(24)\quad C = \frac{1}{\varrho}\left[\dot{u}\left(\frac{\partial\varrho}{\partial x} + \frac{\partial\varrho}{\partial\sigma_x}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\varrho}{\partial\sigma_y}\frac{\partial\sigma_y}{\partial x} + \frac{\partial\varrho}{\partial\tau_{xy}}\frac{\partial\tau_{xy}}{\partial x}\right) + \right.$$
$$\left. + \dot{v}\left(\frac{\partial\varrho}{\partial y} + \frac{\partial\varrho}{\partial\sigma_x}\frac{\partial\sigma_x}{\partial y} + \frac{\partial\varrho}{\partial\sigma_y}\frac{\partial\sigma_y}{\partial y} + \frac{\partial\varrho}{\partial\tau_{xy}}\frac{\partial\tau_{xy}}{\partial y}\right)\right].$$

Considering the complete system of four equations (4), (5), (16) and (21) for the stress and velocity field simultaneously, we obtain for the four families of characteristics the relations (11), (12), (22), whereas instead of (23), we get after introducing (8):

$$(25)\quad \{(\sigma_x - \sigma_y)\,\mathrm{tg}\,2\vartheta + 2\psi \pm \sqrt{[(\sigma_x - \sigma_y)^2 + 4\psi^2]\,(1 + \mathrm{tg}^2\,2\vartheta)}\}\,\frac{\mathrm{d}\dot{u}}{\mathrm{d}x} -$$
$$- (\sigma_x - \sigma_y - 2\psi\,\mathrm{tg}\,2\vartheta)\,\frac{\mathrm{d}\dot{v}}{\mathrm{d}x} \pm$$
$$\pm \frac{\dot{u}}{\varrho}\left[\left(\frac{\partial\varrho}{\partial\sigma_x} + \frac{\partial\varrho}{\partial\psi}\frac{\partial\psi}{\partial\sigma_x}\right)\frac{\mathrm{d}\sigma_x}{\mathrm{d}x} + \left(\frac{\partial\varrho}{\partial\sigma_y} + \frac{\partial\varrho}{\partial\psi}\frac{\partial\psi}{\partial\sigma_y}\right)\frac{\mathrm{d}\sigma_y}{\mathrm{d}x}\right].$$
$$.\sqrt{[(\sigma_x - \sigma_y)^2 + 4\psi^2]\,(1 + \mathrm{tg}^2\,2\vartheta)} =$$
$$= \mp\frac{1}{\varrho}\left(\dot{u}\,\frac{\partial\varrho}{\partial x} + \dot{v}\,\frac{\partial\varrho}{\partial y}\right)\sqrt{[(\sigma_x - \sigma_y)^2 + 4\psi^2]\,(1 + \mathrm{tg}^2\,2\vartheta)}.$$

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