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ON SOME TYPES OF NONHOMOGENEOUS BIRTH – IMMIGRATION – DEATH PROCESSES

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This paper deals with some special types of nonhomogeneous birth-immigration-death processes. The formulae obtained permit the determination of the probabilities that the population will be of a certain size at time t, further of the moments of the distribution of the population size at time t and of the probabilities of extinction of the population. The necessary conditions holding for the different types of processes considered are given.

1. INTRODUCTION

In recent years, birth and death processes (cf. e.g. BARTLETT [1], [2]) have been finding more and more use in physics, biology and the technical sciences. Due to new fields of applications, attention has shifted from the basic types of processes introduced e.g. in FELLER's book [3] to nonhomogeneous birth and death processes in which the birth and death rates are specified functions of the time t. This nonhomogeneous process was studied first by KENDALL [4].

This paper deals with a nonhomogeneous process which includes in addition to the above mentioned birth and death rates $\lambda(t)$ and $\mu(t)$ an immigration rate $\nu(t)$. When solving the equations characterizing the process $\{\lambda(t), \nu(t), \mu(t)\}$ we investigate only these special cases:

a) the functions $\lambda(t)$, $\nu(t)$ and $\mu(t)$ are positive and continuous in the open interval $T = (0, \infty)$ and the ratios $[\nu(t)/\lambda(t)] = b$ and $[\mu(t)/\lambda(t)] = c$ are constant everywhere in T;

b) $\lambda(t) \equiv 0$ everywhere in T, the functions $\mu(t)$ and $\nu(t)$ are positive and continous in T and the ratio $[\mu(t)/\nu(t)] = a$ is constant everywhere in T;

c) $\lambda(t) \equiv 0$ everywhere in T and the functions $\mu(t)$ and $\nu(t)$ are positive and continuous in T and satisfy the equality

$$\frac{\nu(t)}{\mu(t)} = \int_0^t \nu(\tau)\, d\tau$$

identically everywhere in T.

The expressions for the probabilities $P_x(t)$, $t \geqq 0$, $x = 0, 1, 2, \ldots$, that the population size at time t will be exactly x, further the expressions for the first two general moments $\alpha_i(t)$, $i = 1, 2$, of the distribution of the population size at time t are given and the probabilities of extinction of the population for $t \to \infty$ are determined. Assumptions of solution are introduced at the beginning of Section 2.

The influence of immigration on the nonhomogeneous birth and death process has been studied in [2]. The results obtained there do not permit the direct derivation of the required expressions for the probabilities $P_x(t)$, $t \geqq 0$, $x = 0, 1, 2, \ldots$ nor for the general moments $\alpha_i(t)$, $t \geqq 0$, $i = 1, 2$, for the cases solved in this paper, but they are used at the conclusion of this paper in an analysis of some derived relations between the moments $\alpha_i(t)$ and the probabilities $P_x(t)$.

This paper arose from a study of the kinetics of metal phase transformation. Here one of the basic questions is to derive the phase transformation rate which is determined by two actions: the formation of nuclei of a new phase and by their growth. It appears from an analysis of the first of these actions that for certain types of phase transformations the process of nucleus formation may be very well described by the types of nonhomogeneous procesess $\{\lambda(t), \nu(t), \mu(t)\}$ investigated in this paper.

2. SOME SPECIAL TYPES OF PROCESSES $\{\lambda(t), \nu(t), \mu(t)\}$

2.1 Consider the birth-immigration-death process with states E_x $(x = 0, 1, 2, \ldots)$.

Assumptions:

a) if at time t the system is in the state E_x, then the probability of the transition $E_x \to E_{x+1}$ in the interval $(t, t + \Delta t)$ is $\{\nu(t) + x\,\lambda(t)\}\,\Delta t + o(\Delta t)$ for $x = 0, 1, 2, \ldots$;

b) if at time t the system is in the state E_x, then the probability of the transition $E_x \to E_{x-1}$ in the interval $(t, t + \Delta t)$ is $x\,\mu(t)\,\Delta t + o(\Delta t)$ for $x = 1, 2, \ldots$;

c) the probability of a transition to a state other than a neighbouring state is $o(\Delta t)$;

d) if at time t the system is in the state E_x, then the probability of no change in the interval $(t, t + \Delta t)$ is $1 - \{\nu(t) + x[\lambda(t) + \mu(t)]\}\,\Delta t + o(\Delta t)$;

e) at time $t = 0$ the system is in the state E_0.

Let $\xi(t)$ be a integer-valued random variable, which assumes the values x of the population size at time t, and let $P_x(t) = \mathscr{P}\{\xi(t) = x\}$, $x = 0, 1, 2, \ldots$.

From the given assumptions it then follows that the $P_x(t)$ must satisfy the differential-difference equations:

$$(1) \qquad \frac{\partial}{\partial t} P_x(t) = (x + 1)\,\mu(t)\,P_{x+1}(t) - \\ - \{[\lambda(t) + \mu(t)]\,x + \nu(t)\}\,P_x(t) + \{(x - 1)\,\lambda(t) + \nu(t)\}\,P_{x-1}(t), \\ x = 1, 2, \ldots$$

and

$$\frac{\partial}{\partial t} P_0(t) = \mu(t)\, P_1(t) - \nu(t)\, P_0(t) \tag{2}$$

with the initial conditions

$$P_0(0) = 1 \quad \text{and} \quad P_x(0) = 0 \quad \text{for} \quad x = 1, 2, \ldots . \tag{3}$$

Let us now introduce the generating function

$$v = \varphi(z, t) = \sum_{x=-\infty}^{\infty} P_x(t)\, z^x , \tag{4}$$

if we define $P_x(t) = 0$ for $x < 0$. With the aid of equations (1) and (2) we find that the function v satisfies the linear partial differential equation

$$\frac{\partial v}{\partial t} - (z - 1)\left[z\, \lambda(t) - \mu(t)\right] \frac{\partial v}{\partial z} = \nu(t)\,(z - 1)\, v , \tag{5}$$

where

$$v = 1 \quad \text{for} \quad t = 0 \quad \text{and} \quad v = 1 \quad \text{for} \quad z = 1 . \tag{6}$$

Theorem 1. *Let $\lambda(t)$, $\nu(t)$ and $\mu(t)$ be positive functions, continuous in the open interval $T = (0, \infty)$, and let the ratios*

$$\frac{\nu(t)}{\lambda(t)} = b \quad \textit{and} \quad \frac{\mu(t)}{\lambda(t)} = c \tag{7}$$

be constant everywhere in T. Then for the probabilities $P_x(t)$, $t \geqq 0$, the relations

$$P_0(t) = [1 - f(t)]^b \tag{8}$$

and

$$P_x(t) = \frac{\Gamma(b + x)}{\Gamma(b)\, \Gamma(x + 1)} P_0(t)\left[1 - P_0(t)\right]^x \quad \textit{for} \quad x = 1, 2, \ldots \tag{9}$$

hold, where

$$f(t) = \frac{1}{1 + \left(\int_0^t \mu(\tau)\, d\tau\right)^{-1}} \quad \textit{for} \quad c = 1 , \tag{10}$$

$$= \frac{e^{\varrho(t)} - 1}{c e^{\varrho(t)} - 1} \quad \textit{for} \quad c \neq 1$$

and

$$\varrho(t) = 0 \quad \textit{for} \quad c = 1 ; \tag{11}$$

$$= (c - 1) \int_0^t \lambda(\tau)\, d\tau \quad \textit{for} \quad c \neq 1$$

and $\Gamma(n)$ denotes the gamma-function.

Proof: Equations (8) and (9) are easily proved, using the generating function $\varphi(z, t)$.

Therefore for the case determined by the equations (7) we shall attempt to find an integral surface $v = \varphi(z, t)$, which passes through the straight line $t = 0$, $v = 1$.

From the canonical system of ordinary differential equations associated with the partial differential equation (5) there follow two equations

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -b \frac{v}{z - c}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\lambda(t) z^2 + [\lambda(t) + \mu(t)] z - \mu(t). \tag{12}$$

The first integrals of the considered system are

$$C_1 = (z - c)^b . v \tag{13}$$

and

$$C_2 = \frac{1}{z - 1} - \int_0^t \lambda(\tau) \, \mathrm{d}\tau \quad \text{for} \quad c = 1, \tag{14}$$

$$= \frac{e^{\varrho(t)}}{z - 1} - \frac{e^{\varrho(t)} - 1}{c - 1} \quad \text{for} \quad c \neq 1,$$

where $\varrho(t)$ is given by equation (11). The first integral (13) is evident. Taking into account that one particular solution of equation (12), which is of Riccati type, is $z_{(1)} = 1$, we may determine the first integral (14). The knowledge of this particular solution permits the transformation of Riccati's equation into a linear differential equation (cf. e. g. [5]).

With the aid of (13) and (14) we obtain for the case determined by (7) the general solution of (5) in the form

$$\Phi \left\{ (z - 1)^b . v ; \frac{1}{z - 1} - \int_0^t \lambda(\tau) \, \mathrm{d}\tau \right\} = 0 \quad \text{for} \quad c = 1$$

and

$$\Phi^* \left\{ (z - c)^b . v ; \frac{e^{\varrho(t)}}{z - 1} - \frac{e^{\varrho(t)} - 1}{c - 1} \right\} = 0 \quad \text{for} \quad c \neq 1,$$

where Φ and Φ^* are arbitrary differentiable functions. Solving Cauchy's problem, to which purpose we use the equations following from the first integrals (13) and (14) for $t = 0$, we obtain the required solution for the integral surface satisfying the initial condition $v = 1$ for $t = 0$:

$$v = \left[1 - (z - 1)\int_0^t \lambda(\tau)\,\mathrm{d}\tau\right]^{-b} \quad \text{for} \quad c = 1 \tag{15}$$

and

$$v = \left[\frac{e^{\varrho(t)}(c - 1)}{z(1 - e^{\varrho(t)}) - 1 + ce^{\varrho(t)}}\right]^b \quad \text{for} \quad c \neq 1 \tag{15'}$$

In view of the notation introduced in (10) we may express the required generating function as

$$\varphi(z, t) = \left\{\frac{1 - f(t)}{1 - z\,f(t)}\right\}^b. \tag{16}$$

Expanding the function $\varphi(z, t)$ given in the form (16) in a Maclaurin series in powers of z and comparing the coefficients of the powers z^x of this series with those of corresponding powers in the series (4), we obtain the expression for $P_0(t)$ and $P_x(t)$, $x = 1, 2, \ldots$, in the form (8) and (9) respectively. This completes the proof of theorem 1.

Corollary 1.1. *In the process $\{\lambda(t), \nu(t), \mu(t)\}$ with the properties given in Theorem 1, the relation*

$$\mathsf{V}^2\left[\xi(t)\right] - \frac{1}{\mathsf{E}\left[\xi(t)\right]} = \frac{1}{b} = \text{const}. \tag{17}$$

holds for every $t \in T$, where $\mathsf{V}[\xi(t)]$ is the coefficient of variation and and $\mathsf{E}[\xi(t)]$ the expected value of the random variable $\xi(t)$.

Proof: Let

$$\alpha_i(t) = \mathsf{E}[\xi^i(t)], \quad i = 1, 2, \ldots$$

be the i-th moment of the distribution of the random variable $\xi(t)$.

From the known relations between the generating function of the probabilities $P_x(t)$, $x = 0, 1, 2, \ldots$, and the moments $\alpha_i(t)$, $i = 1, 2$ (cf. e.g. [3]), there follow according to (16) the expressions for the first two moments,

$$\alpha_1(t) = \frac{b\,f(t)}{1 - f(t)}, \tag{18}$$

$$\alpha_2(t) = \alpha_1(t)\left[1 + \alpha_1(t) + (1/b)\,\alpha_1(t)\right]. \tag{19}$$

Let us now investigate the function $f(t)$ given by (10). We prove that $f(t) = 0$ for $t = 0$ and $0 < f(t) < 1$ for $t > 0$ for every $c > 0$.

The above assertions are evident from (10) with the exception of the case $c \neq 1$ and $t > 0$. However, using

$$e^{\varrho(t)} - 1 \equiv (c - 1) \int_0^t \lambda(\tau)\, e^{\varrho(\tau)}\, \mathrm{d}\tau\,, \quad c \neq 1\,, \tag{20}$$

we may rewrite the expression for $f(t)$ successively as follows:

(21)
$$f(t) = \frac{e^{\varrho(t)} - 1}{ce^{\varrho(t)} - 1} = \frac{1}{1 + \dfrac{(c-1)\, e^{\varrho(t)}}{e^{\varrho(t)} - 1}} = \frac{1}{1 + e^{\varrho(t)} \left[\displaystyle\int_0^t \lambda(\tau)\, e^{\varrho(\tau)}\, \mathrm{d}\tau\right]^{-1}} \quad \text{for} \quad c \neq 1\,.$$

Hence there also follows the inequality $0 < f(t) < 1$ for $c \neq 1$ and $t > 0$.

Therefore, in view of (18) and (19) $\alpha_1(t)$ and $\alpha_2(t)$ are positive functions for $t > 0$ and for every $c > 0$. The derivation of (17) from (18) and (19) is evident.

Corollary 1.2. *In the process $\{\lambda(t), \nu(t), \mu(t)\}$ with the properties given in Theorem 1, the relations*

$$\alpha_1(t) = \int_0^t \nu(\tau)\, \mathrm{d}\tau \qquad \textit{for} \quad c = 1\,, \tag{22}$$

$$= \frac{b\,(1 - e^{-\varrho(t)})}{c - 1} \quad \textit{for} \quad c \neq 1$$

hold.

Proof: The expressions (22) are obtained by setting (10) into (18) and using (7) for $c = 1$, and by setting (21) into (18) for $c \neq 1$, respectively.

Note. The influence of immigration on the growth of $\alpha_1(t)$ is evident from the equation

$$\alpha_1(t) = \alpha_1^*(t) \int_0^t \frac{\nu(\tau)}{\alpha_1^*(\tau)}\, \mathrm{d}\tau\,, \quad t \geqq 0\,, \tag{23}$$

where $\alpha_1^*(t)$ is the expected value of the size of the population at time t due to the nonhomogeneous birth-death process $\{\lambda(t), \mu(t)\}$. The relation (23) follows from equation (13) in [4]

$$\alpha_1^*(t) = \exp\left\{- \int_0^t [\mu(\tau) - \lambda(\tau)]\, \mathrm{d}\tau\right\},$$

and from equation (22), which we may write in a form valid for every $c > 0$

$$\alpha_1(t) = e^{-\varrho(t)} \int_0^t \nu(\tau)\, e^{\varrho(\tau)}\, \mathrm{d}\tau\,, \quad t \geqq 0\,. \tag{24}$$

Corollary 1.3. *In the process* $\{\lambda(t),\ \nu(t),\ \mu(t)\}$ *with the properties given in Theorem 1, the relations*

$$(25)\quad \begin{aligned}\lim_{t\to\infty} \alpha_1(t) &= \int_0^\infty \nu(\tau)\,\mathrm{d}\tau && \textit{for}\quad c = 1\,,\\ &= \frac{b}{c-1}\left[1 - e^{(1-c)\int_0^\infty \lambda(\tau)\mathrm{d}\tau}\right] && \textit{for}\quad c \neq 1,\quad \int_0^\infty \lambda(\tau)\,\mathrm{d}\tau < \infty\,,\\ &= \frac{b}{c-1} && \textit{for}\quad c > 1,\quad \int_0^\infty \lambda(\tau)\mathrm{d}\tau = \infty\,,\\ &= \infty && \textit{for}\quad c < 1,\quad \int_0^\infty \lambda(\tau)\,\mathrm{d}\tau = \infty\end{aligned}$$

hold.

Proof: The equations (25) follow by taking limits in (22) for $t \to \infty$.

Corollary 1.4. *In the process* $\{\lambda(t),\ \nu(t),\ \mu(t)\}$ *with the properties given in Theorem 1 the relation*

$$(26)\qquad \lim_{t\to\infty} P_0(t) < 1$$

holds.

Proof: The equation (26) follows by faking limits in (8) for $t \to \infty$, if we introduce (10) into (8) for $c = 1$ and (21) into (8) for $c \neq 1$.

2.2 We shall now investigate a further type of nonhomogeneous birth-immigration-death process, for which $\lambda(t) \equiv 0$ everywhere in T, the so called nonhomogeneous immigration-death process $\{\nu(t),\ \mu(t)\}$. We obtain the corresponding assumptions from those considered at the beginning of Section 2.1 by putting there $\lambda(t) \equiv 0$.

Theorem 2. *Let* $\lambda(t) \equiv 0$ *everywhere in* T. *Let the functions* $\nu(t)$ *and* $\mu(t)$ *be positive and continuous in* T, *and let*

$$(27)\qquad \frac{\mu(t)}{\nu(t)} = a$$

be constant everywhere in T. *Then for the probabilities* $P_x(t)$, $t \geqq 0$, *the relations*

$$(28)\qquad P_0(t) = \exp\left\{-\frac{1}{a}\left(1 - e^{-\int_0^t \mu(\tau)\mathrm{d}\tau}\right)\right\}$$

and

$$(29)\qquad P_x(t) = \frac{1}{\Gamma(x+1)}\,P_0(t)\left[\lg\frac{1}{P_0(t)}\right]^x,\quad x = 1, 2, \ldots$$

hold.

Proof: The steps in the proof are analoguous to those in Theorem 1. In the first step we shall again look for the appropriate generating function $v = \varphi(z, t)$. The partial differential equation satisfied by the function v is obtained from (5) for $\lambda(t) \equiv 0$.

The system of first integrals corresponding to the linear partial differential equation so obtained is given according to (27) by

$$C_1 = (z - 1)\, e^{-\int_0^t \mu(\tau) d\tau}$$

and

$$C_2 = v e^{-z/a}.$$

Hence we obtain the general solution. The required integral surface satisfying the condition $v = 1$ for $t = 0$ is then

$$v = \exp\left\{\frac{1}{a}(z - 1)\left(1 - e^{-\int_0^t \mu(\tau) d\tau}\right)\right\}. \tag{30}$$

In a manner similar as in Theorem 1, we obtain from (30) the expressions for the probabilities $P_x(t)$, $x = 0, 1, 2, \ldots, t \geqq 0$. This completes the proof of Theorem 2.

Corollary 2.1. *If the process $\{\lambda(t), \nu(t), \mu(t)\}$ has the properties given in Theorem 2, then*

$$\alpha_1(t) = \frac{1}{a}\left(1 - e^{-\int_0^t \mu(\tau) d\tau}\right) \tag{31}$$

and

$$\alpha_2(t) = \alpha_1(t)\left[1 + \alpha_1(t)\right], \quad t \geqq 0. \tag{32}$$

Proof: Equations (31) and (32) follow from (30) using the known relations between the generating function v and the moments $\alpha_i(t)$, $i = 1, 2$.

Corollary 2.2. *In the process $\{\lambda(t), \nu(t), \mu(t)\}$ with the properties given in Theorem 2, the relation*

$$\mathsf{D}^2[\xi(t)] = \mathsf{E}[\xi(t)] = \lg \frac{1}{P_0(t)} \tag{33}$$

holds for every $t \geqq 0$, where $\mathsf{D}^2[\xi(t)]$ is the variance of the random variable $\xi(t)$.

Proof: The equation to be proved follows immediatly from (31), (32) and (28).

Corollary 2.3. *If the process $\{\lambda(t), \nu(t), \mu(t)\}$ has the properties given in Theorem 2, then*

$$\lim_{t \to \infty} \alpha_1(t) = \frac{1}{a}\left(1 - e^{-\int_0^\infty \mu(\tau) d\tau}\right) \quad \textit{for} \quad \int_0^\infty \mu(\tau)\, d\tau < \infty, \tag{34}$$

$$= \frac{1}{a} \qquad \textit{for} \quad \int_0^\infty \mu(\tau)\, d\tau = \infty.$$

Proof: Equation (34) is the limiting case of (31) for $t \to \infty$.

Corollary 2.4. *If the process $\{\lambda(t), \nu(t), \mu(t)\}$ has the properties given in Theorem 2. then*

$$\lim_{t \to \infty} P_0(t) < 1 . \tag{35}$$

Proof: From (28) it follows that

$$\lim_{t \to \infty} P_0(t) = e^{-1/a}, \qquad \text{when} \quad \int_0^\infty \mu(\tau)\, \mathrm{d}\tau = \infty ,$$

$$= e^{-k/a}, \quad 0 < k < 1, \quad \text{when} \quad \int_0^\infty \mu(\tau)\, \mathrm{d}\tau < \infty ;$$

since $0 < a < \infty$, (35) holds.

Theorem 3. *Let $\lambda(t) \equiv 0$ everywhere in T. Let $\nu(t)$ and $\mu(t)$ be positive functions, continuous in T and satisfying identically the equality*

$$\frac{\nu(t)}{\mu(t)} = \int_0^t \nu(\tau)\, \mathrm{d}\tau \tag{36}$$

everywhere in T. Then the probabilities $P_x(t)$, $t \geqq 0$, are given by expressions

$$P_0(t) = \mathrm{e}^{-(1/2) \int_0^t \nu(\tau)\, \mathrm{d}\tau} \tag{37}$$

and

$$P_x(t) = \frac{1}{\Gamma(x+1)} P_0(t) \left[\lg \frac{1}{P_0(t)} \right]^x, \quad x = 1, 2, \ldots , \tag{38}$$

Proof: In the first step we shall again solve for the corresponding generating function $v = \varphi(z, t)$. The partial differential equation satisfied by the function v follows from (5) and, according to (36), is of the form

$$\frac{\partial v}{\partial t} + (z-1) \frac{\nu(t)}{\int_0^t \nu(\tau)\, \mathrm{d}\tau} \cdot \frac{\partial v}{\partial z} = (z-1)\, \nu(t) \,.\, v \,. \tag{39}$$

Hence we obtain the system of ordinary differential equations

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}z \int_0^t \nu(\tau)\, \mathrm{d}\tau}{(z-1)\, \nu(t)} = \frac{\mathrm{d}v}{(z-1)\, v \,.\, \nu(t)} \,.$$

Taking now a combination of the upper members, such that the sum of the lower members is equal to zero, we obtain the so called Pfaff's equation

$$Q\, \mathrm{d}t + R\, \mathrm{d}z + S\, \mathrm{d}v = 0 ; \tag{40}$$

here the functions Q, R and S are of the form

(41)
$$Q = (z - 1)\, v \frac{v(t)}{\int_0^t v(\tau)\, \mathrm{d}\tau},$$
$$R = v,$$
$$S = -2\left[\int_0^t v(\tau)\, \mathrm{d}\tau\right]^{-1}$$

and are defined in the domain $G : \{0 < t < \infty;\ 0 < z \leqq 1\ ;\ 0 < v \leqq 1\}$ containing no singular point, i.e. no point at which $Q = R = S = 0$. According to (41) and (36), it is obvious that a necessary and sufficient condition for the integrability of Pfaff's equation, in the form of a single relation namely (cf. e.g. [5])

$$Q\left(\frac{\partial R}{\partial v} - \frac{\partial S}{\partial z}\right) + R\left(\frac{\partial S}{\partial t} - \frac{\partial Q}{\partial v}\right) + S\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial t}\right) = 0$$

is fulfilled identically.

Hence it follows that the general solution of equation (40) is a oneparametric system of surfaces

$$\Phi(t, z, v) = C_{(0)}\,.$$

This equation describes the twodimensional integral variety for the equation (40).

Due to the symmetry of the form (40) of Pfaff's equation in t, z and v, we may write this equation in a form, where on the left hand side of the equation there appears either one of the differentials $\mathrm{d}t$, $\mathrm{d}z$ or $\mathrm{d}v$. In our case the expression for $\mathrm{d}z$ appears to be the most suitable for integration (according to our assumptions $v \neq 0$ everywhere in G),

$$\mathrm{d}z = \frac{2}{v \int_0^t v(\tau)\, \mathrm{d}\tau}\, \mathrm{d}v - \frac{(z - 1)\, v(t)}{\int_0^t v(\tau)\, \mathrm{d}\tau}\, \mathrm{d}t\,.$$

Hence there follows a system of two partial differential equations

(42)
$$\frac{\partial z}{\partial t} = (1 - z) \frac{v(t)}{\int_0^t v(\tau)\, \mathrm{d}\tau},$$

(43)
$$\frac{\partial z}{\partial v} = \frac{2}{v \int_0^t v(\tau)\, \mathrm{d}\tau}\,.$$

From equation (42) we find

$$z = 1 - \frac{g(v)}{\int_0^t \nu(\tau)\,\mathrm{d}\tau}. \tag{44}$$

By setting (44) into (43) we obtain

$$\frac{\mathrm{d}g}{\mathrm{d}v} = -\frac{2}{v},$$

from which

$$g = 2\lg v^{-1} + C_* .$$

Introducing now this value into (44) and putting $C_* = -2\lg C$, we obtain the general solution of the system in the form

$$z = 1 + \frac{2\lg C \,.\, v}{\int_0^t \nu(\tau)\,\mathrm{d}\tau}.$$

In view of the condition $v = 1$ for $z = 1$, we obtain

$$z = 1 + \frac{2\lg v}{\int_0^t \nu(\tau)\,\mathrm{d}\tau},$$

so that

$$v = \exp\left\{-\tfrac{1}{2}(1 - z)\int_0^t \nu(\tau)\,\mathrm{d}\tau\right\}. \tag{45}$$

The above argument was derived for the domain G, in which $0 < t < \infty$. However from (45) it is clear that the derived function v satisfies also the condition $v = 1$ for $t = 0$. Therefore the expression (45) yields the required generating function.

By the same procedure as in Theorem 1, using (45) we obtain the expressions for the probabilities $P_x(t)$, $t \geqq 0$, $x = 0, 1, 2, \ldots$, in the form (37) resp. (38). This completes the proof of Theorem 3.

Corollary 3.1. *If a process $\{\lambda(t), \nu(t), \mu(t)\}$ has the properties given in Theorem 3, then*

$$\alpha_1(t) = \tfrac{1}{2}\int_0^t \nu(\tau)\,\mathrm{d}\tau \tag{46}$$

and

$$\alpha_2(t) = \alpha_1(t)\,[1 + \alpha_1(t)], \quad t \geqq 0 . \tag{47}$$

Proof: The equations (46) and (47) are easily derived from (45).

Corollary 3.2. *If a process* $\{\lambda(t), \nu(t), \mu(t)\}$ *has the properties given in Theorem 3, then the relation* (33) *holds.*

Proof: The assertion follows immediately from (46), (47) and (37).

Corollary 3.3. *If a process* $\{\lambda(t), \nu(t), \mu(t)\}$ *has the properties given in Theorem 3, then*

$$\lim_{t \to \infty} P_0(t) < 1 . \tag{48}$$

Proof: The relation follows from (37) by taking limits for $t \to \infty$.

2.3 Bartlett devotes attention to the influence of immigration on the change of the population size in Section 3.41 of [2]. With aid of equation (1), e.c., we may obtain the expression for the generating function of the nonhomogeneous process $\{\nu(t); \mu(t)\}$, where the functions $\nu(t)$ and $\mu(t)$, positive and continuous in T, are not mutually related. In view of the assumptions given at the beginning of section 2. of the present paper, we obtain

$$\varphi(z, t) = \exp \left\{ (z - 1) \int_0^t \nu(\tau)\, e^{-\int_0^\tau \mu(s)\, \mathrm{d}s}\, \mathrm{d}\tau \right\} . \tag{49}$$

Using (49) to express the first two moments $\alpha_i(t)$, $i = 1, 2$, and the probability $P_0(t)$, $t > 0$, we find that (33) still holds. Therefore it is clear that (33) expresses a necessary condition for each nonhomogeneous process $\{\nu(t), \mu(t)\}$, $\nu(t) > 0$, $\mu(t) > 0$, $t \in T$, irrespective of an eventual relationship between the functions $\nu(t)$ and $\mu(t)$. On the other hand, equation (17) gives a necessary condition only for a nonhomogeneous process $\{\lambda(t), \nu(t), \mu(t)\}$ with the properties given in Theorem 1.

References

[1] *M. S. Bartlett:* Stochastic Population Models; John Wiley, New York 1960.

[2] *M. S. Bartlett:* An Introduction to Stochastic Processes with Special Reference to Methods and Applications; Cambridge University Press 1955.

[3] *W. Feller:* An Introduction to Probability Theory and its Applications, vol. 1; John Wiley, New York 1950.

[4] *D. G. Kendall*: On the Generalized Birth and Death Process; Ann. Math. Statistics 19 (1948), 1—15.

[5] *V. V. Stěpanov:* Kurs diferenciálních rovnic (Differential equations); (Czech translation, original in Russian), Přírodovědecké nakladatelství, Praha 1950.

Výtah

K NĚKTERÝM TYPŮM NEHOMOGENNÍCH PROCESŮ ROZENÍ – IMIGRACE – UMÍRÁNÍ

VRATISLAV HORÁLEK

V práci je uvažován nehomogenní proces rození – imigrace – umírání s intensitami rození $\lambda(t)$, imigrace $\nu(t)$ a umírání $\mu(t)$. Za běžných předpokladů jsou řešeny tyto tři speciální případy:

a) funkce $\lambda(t)$, $\nu(t)$ a $\mu(t)$ jsou positivní a spojité v intervalu $T = (0, \infty)$ a podíly $[\nu(t)/\lambda(t)] = b$ a $[\mu(t)/\lambda(t)] = c$ jsou všude v T konstantní;

b) $\lambda(t) \equiv 0$ všude v T a funkce $\nu(t)$ a $\mu(t)$ jsou positivní a spojité v T, přičemž podíl $[\mu(t)/\nu(t)] = a$ je všude v T konstantní;

c) $\lambda(t) \equiv 0$ všude v T a funkce $\nu(t)$ a $\mu(t)$ jsou positivní a spojité v T a splňují všude v T identicky rovnost

$$\frac{\nu(t)}{\mu(t)} = \int_0^t \nu(\tau)\,d\tau\,.$$

Jsou odvozeny vzorce pro pravděpodobnosti $P_x(t)$, $t \geqq 0$, $x = 0, 1, 2, \ldots$, že soubor v čase t bude tvořen právě x částicemi, dále vzorce pro prvé dva obecné momenty $\alpha_i(t)$, $i = 1, 2$, rozdělení počtu částic v čase t a je určena pravděpodobnost vymření souboru pro $t \to \infty$.

Je ukázáno, že nutnou podmínkou pro každý nehomogenní proces rození – imigrace – umírání $\{\lambda(t), \nu(t), \mu(t)\}$ s konstantními podíly intensit $[\nu(t)/\lambda(t)] = b$ a $[\mu(t)/\lambda(t)] = c$ je, aby pro každé $t \in T$

$$\mathsf{V}^2[\xi(t)] - \mathsf{E}^{-1}[\xi(t)] = b^{-1} = \text{konst}\,.$$

a dále, že nutnou podmínkou pro každý nehomogenní proces imigrace – umírání $\{\nu(t), \mu(t)\}$ bez ohledu na funkční vztah mezi intensitami $\nu(t)$ a $\mu(t)$ je, aby pro každé $t \in T$

$$\mathsf{D}^2[\xi(t)] = \mathsf{E}[\xi(t)] = \lg \frac{1}{P_0(t)}\,,$$

kde $\mathsf{E}[\xi(t)]$ je matematická naděje, $\mathsf{D}^2[\xi(t)]$ rozptyl a $\mathsf{V}[\xi(t)]$ variační koeficient náhodné proměnné $\xi(t)$, nabývající hodnot počtu částic v souboru v čase t.

Резюме

К НЕКОТОРЫМ ВИДАМ НЕОДНОРОДНЫХ ПРОЦЕССОВ РОЖДЕНИЯ—ИММИГРАЦИИ—ГИБЕЛИ

ВРАТИСЛАВ ГОРАЛЕК (Vratislav Horálek)

В работе рассматривается неоднородный процесс рождения — иммиграции — гибели с коэффициентами рождения $\lambda(t)$, иммиграции $\nu(t)$ и гибели $\mu(t)$. При нормальных предположениях решаются следующие три специальных случая:

а) функции $\lambda(t)$, $\nu(t)$ и $\mu(t)$ положительны и непрерывны в интервале $T = (0, \infty)$, и отношения $[\nu(t)/\lambda(t)] = b$ и $[\mu(t)/\lambda(t)] = c$ всюду в T постоянны;

б) $\lambda(t) \equiv 0$ всюду в T, и функции $\nu(t)$ и $\mu(t)$ положительны и непрерывны в T, причем отношение $[\mu(t)/\nu(t)] = a$ всюду в T постоянно;

в) $\lambda(t) \equiv 0$ всюду в T, и функции $\nu(t)$ и $\mu(t)$ положительны и непрерывны в T и удовлетворяют всюду в T тождественно равенству

$$\frac{\nu(t)}{\mu(t)} = \int_0^t \nu(\tau)\, \mathrm{d}\tau .$$

Далее выводятся формулы для вероятностей $P_x(t)$, $t \geqq 0$, $x = 0, 1, 2, \ldots$, что совокупность во времени t будет образована именно x частицами, потом формулы для первых двух общих моментов $\alpha_i(t)$, $i = 1, 2$, распределение количества частиц во времени t, и, наконец, установлена вероятность, что совокупность вымрет для $t \to \infty$.

Указывается, что необходимым условием для каждого неоднородного процесса рождения — иммиграции — гибели $\{\lambda(t), \nu(t), \mu(t)\}$ с постоянными отношениями коэффициентов $[\nu(t)/\lambda(t)] = b$ и $[\mu(t)/\lambda(t)] = c$ является то, чтобы для каждого $t \in T$

$$\mathsf{V}^2[\xi(t)] - \mathsf{E}^{-1}[\xi(t)] = b^{-1} = \text{konst.},$$

и далее, что необходимым условием для каждого неоднородного процесса иммиграции — гибели $\{\nu(t), \mu(t)\}$, несмотря на функциональное отношение между коэффициентами $\nu(t)$ и $\mu(t)$, является то, чтобы для каждого $t \in T$

$$\mathsf{D}^2[\xi(t)] = \mathsf{E}[\xi(t)] = \lg \frac{1}{P_0(t)},$$

где $\mathsf{E}[\xi(t)]$ — математическое ожидание, $\mathsf{D}^2[\xi(t)]$ — дисперсия и $\mathsf{V}[\xi(t)]$ — коэффициент вариации случайной величины $\xi(t)$, принимающей значения количества частиц в совокупности во времени t.

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