Jan Fischer; Pavel Kolář; Bohdan Maslowski; Jan Seidler; Štefan Schwabik

Ivo Vrkoč sexagenarian

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 4, 737–750

Persistent URL: http://dml.cz/dmlcz/102506

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

NEWS AND NOTICES

IVO VRKOČ SEXAGENARIAN

JAN FISCHER, PAVEL KOLÁŘ, BOHDAN MASLOWSKI, JAN SEIDLER, ŠTEFAN SCHWABIK, Praha

Unbelievable as it may seem, Ivo Vrkoč, DrSc., was born sixty years ago, on June 10, 1931 in Kladno. However, while the enthusiasm and vigor with which he attacks any mathematical problem belie the inexorable time, the results of his research are much more than adequate to his real age and their extent and importance would do honour to any world-famous mathematician.

After graduating from the Faculty of Science (now Faculty of Mathematics and Physics) of Charles University in Prague, Ivo Vrkoč came to the Mathematical Institute of the Czechoslovak Academy of Sciences. Here he found himself in a group of young mathematicians who obeyed an advice of the older ones (in this case it was Professor Vladimír Knichal) and with enthusiasm entered a field with no historical background in the country, namely the qualitative theory of ordinary differential equations. During a relatively short period the group worked their way to the world èlite, and I. Vrkoč was one of its prominent members. In 1955 he published his first mathematical paper [1] which immediately found its place in the world context. The same can be told about his second paper [2] written together with J. Kurzweil. These papers dealt with conversion of theorems on stability or instability, which were deduced by means of Lyapunov functions. The paper [1] concerned the Četayev instability theorem that had been published for the first time in 1934 and in a more detailed form in N. G. Četayev's monograph *Stability of Motion* (Gostechizdat, Moscow 1946). Vrkoč showed that if the trivial solution of a system

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

with a function f continuously differentiable in x, is not stable, then there exists a function $V(t, x)$ exhibiting the properties required by Četayev's theorem. To this end he used a sophisticated technique whose complexity consists in a thorough investigation of the properties of solutions in a neighbourhood of the instable equilibrium of the system and in the construction of a function $V(t, x)$ exploiting these properties. The problem of characterization of stability of the steady state of the system (1) in terms of a Lyapunov function for continuous right hand sides had been solved by T. Yoshizawa; however, the Lyapunov function constructed by him was not continuous. A construction of a continuous Lyapunov function is given in [2]. Here it is proved that it is not always possible to characterize stability or uniform stability of the equilibrium by a smooth Lyapunov function, and necessary and sufficient conditions are given for the right hand side of the system (1) to guarantee the existence of a smooth Lyapunov function. The results are "sharp" and soon after their discovery they rightly entered monographs which are still authoritative in stability theory. Only few Czechoslovak mathematicians succeeded in entering the history by their mathematical debuts, as was the case with Ivo Vrkoč. And it should be pointed out that the Prague school of the qualitative theory of ordinary differential equations owes its lasting renown particularly to these early works of I. Vrkoč and J. Kurzweil.

In [4] I. Vrkoč offerred to the Czechoslovak mathematical community a brief report on the topic of his dissertation for the CSc (Candidate of Science) degree. It was then published in extenso [5] and like the previous works, it soon attracted the attention of world specialists in the stability theory. Fifteen years later S.-N. Chow and J. A. Yorke (Lyapunov theory and perturbation of stable and asymptotically stable systems. J. Differential Equations, 15 (1974), 308–321) characterized the work as "a monumental paper", being fully aware of the meaning of these words.

In [5] Vrkoč dealt with the system (1) on the assumption that the right hand side of (1) satisfies the Carathéodory conditions and $f(t, 0) = 0$. Then the system has the trivial solution $x = 0$ which Vrkoč called integrally stable if for every $\delta > 0$ there is $B(\delta) > 0$ such that

(i) $\lim_{\delta \to 0} B(\delta) = 0$;

(ii) if a function $\eta(t, x)$ satisfies the condition

$$\int_{t_0}^{\infty} \sup_{\|x\| \leqq B(\delta)} \|\eta(t, x)\| \, \mathrm{d}t < \delta ,$$

then every solution $x(t)$ of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) + \eta(t, x)$$

with $\|x(t_0)\| < \delta$ can be extended to $t \geqq t_0$, and $\|x(t)\| < \delta$ holds for $t \geqq t_0$ (t_0 can differ for various solutions $x(t)$).

Adding the attractivity of the trivial solution Vrkoč introduced the notion of asymptotic variational stability and studied this concept in the relation to other notions then known (stability with respect to permanently acting perturbations, variational stability, strong stability). The monumentality of Vrkoč's work consists in his starting with the premonition that a large perturbation acting for a short time period cannot throw a reasonable system (1) off balance, and in his transforming this premonition into the form of an established fact.

An extraordinary achievement of this work is the characterization of integral stability and asymptotic integral stability by means of a suitable Lyapunov function. This requires a very fine technique, especially when the Lyapunov function corresponding to the stable solution is constructed. No wonder the paper [5] was soon included in monographs as a fundamental result in the field of qualitative theory of ordinary differential equations. (A. Halanay – 1966, W. Hahn – 1967, T. Yoshizawa – 1966.) Even if I. Vrkoč finished his mathematical career by writing the paper [5], his name would still be well known to mathematicians working in the field.

In 1962 Vrkoč published his paper [7] in which he investigated stability of the trivial solution of the system of ordinary differential equations with respect to permanently acting perturbations from the viewpoint of its characterization by means of a Lyapunov function. By this paper, in which he again employed his sophisticated analytical methods, he ended one period of his mathematical career, which brought general respect and fame to the Prague school of the qualitative theory of ordinary differential equations. It was an extremely fruitful period stemming from the classical Russian and Soviet traditions going back to A. M. Lyapunov, and it actually built a bridge to the modern analytic approach to Lyapunov functions by taking into account that in this century it is the Lebesgue integral that stirs up the development in mathematical analysis. Meanwhile, in 1961, Vrkoč published his paper [6]. Here he turned to topological aspects of stability of solutions of the system (1) and proved that, if the system satisfies some conditions guaranteeing the existence of a solution (e.g. the Carathéodory conditions), then the family of all initial states at $t = 0$ from which a stable (equistable, uniformly stable) solution starts, is a G_δ set. This is a nice result by itself. Nonetheless, Vrkoč also proved that, given a G_δ set $M \subset \boldsymbol{R}^2$, there exists a system (1) in $\boldsymbol{R}^2$ such that each of its solutions starting at $t = 0$ from a point $x \in M$ is uniformly stable, and each of its solution starting at $t = 0$ from a point x lying in the complement of M, is unstable. Moreover, he showed that the system (1) can be chosen in such a way that its right hand side has continuous partial derivatives of all orders. The main ideas of the proof of the assertion are valid even for the general case of $\boldsymbol{R}^n$. However, the proof is presented for $n = 2$, its technique being complicated enough even so. For the case of the one-dimensional autonomous ordinary differential equation (i.e. for dynamical systems) Vrkoč presented an ingenious description of the set from which unstable trajectories start. This paper of Ivo Vrkoč is well known to specialists in topological dynamics, since it is a paragon of what can be achieved if deep understanding of methods of mathematical analysis is combined with their application to an apparently abstract branch of mathematics.

After this fruitful period of outstanding mathematical achievements Ivo Vrkoč entered another domain which had no tradition in Czechoslovakia, namely the theory of stochastic differential equations.

From the viewpoint of today's mathematics, the theory of stochastic differential equations seems rather remote from the theory of ordinary differential equations, and it is more usually included in the probability theory. Nonetheless, the way of I. Vrkoč from ordinary equations to stochastic ones has been a continuous process. His first "stochastic" papers [9] and [11] actually represent a certain continuation of his former research in stability, the perturbations of the right hand side of the equation having random character. In fact, these papers do not yet deal with the "true" Itô stochastic equations but with the so called differential equations with randomness, in which the random process appearing on the right hand side of the equation has trajectories sufficiently regular to allow to consider solutions in the classical Carathéodory sense (thus the case of "white noise" is not covered).

Let us acquaint the reader with the papers [9] and [11] by at least sketching a special case of the of the main result of [11]. Let us consider the class of equations of the form

$$\dot{x}_n = -\lambda_n x_n + S_n(t, \omega, x_n), \quad x_n(t_0) = x_0,$$

where $\lambda_n \geqq 0$ and S_n have the character of random perturbations. The author's attention is concentrated on an estimation of the asymptotic behaviour of expressions of the type

$$(2) \qquad \sup_{S_n} P[\sup_{t_0 \leqq t \leqq t_n} x_n(t, \omega) \geqq v_n]$$

for $n \to \infty$, $t_n \geqq t_0$ and v_n being given numbers. $P[\ldots]$ in (2) denotes the probability that the process x_n in the interval $[t_0, t_n]$ at least once exceeds the bound v_n. Provided the influence of the individual perturbations decrease with $n \to \infty$ and the perturbations do not cause a "systematic error", the limit of the expressions (2) can be expressed in terms of a solution of the heat equation. A precise mathematical formulation of the above vague assertion is of course fairly demanding, and in [9],

[11] it is the subject of a fine analytic elaboration. As can be guessed from the above mentioned relationship with parabolic equations, situations typical for applications of these results are those in which the random perturbations S_n in a certain sense converge to a process "without memory", that is to a process of the white-noise type. The subsequent shift of mathematical interests of I. Vrkoč to stochastic differential equations thus appears to be very natural. In this field he dealt (if we disregard several minor papers) with two topics: with a generalization of the averaging method to stochastic differential equations, and with the problem of maximal diffusion for the exit from a domain. It is the latter topic which represents a continuation of the papers [9], [11]. The essential contribution was made in [19], where the problem was solved in the most general setting.

Let us consider a domain D with a sufficiently smooth boundary in $\boldsymbol{R}^n$, let $x(t)$ be a solution of a stochastic differential equation

$$\text{(3)} \qquad \mathrm{d}x(t) = f(t, x(t))\,\mathrm{d}t + B(t, x(t))\,\mathrm{d}w(t)\,, \quad x(0) = x_0 \in D\,,$$

where $w(t)$ stands for a Wiener process and the coefficients f, B are sufficiently smooth and satisfy the usual assumptions guaranteeing the existence and uniqueness of solution. Let τ be the moment of the first exit of the process $x(t)$ from the domain D. Now let the coefficient of the local drift in the equation (3) be fixed while the coefficient of diffusion B may vary. Then the time $\tau = \tau_B$ depends on B; the aim of Vrkoč was to find the maximum of the probabilities $P[\tau_B \leqq T]$ that the process $x(t)$ reaches the boundary of the domain D before a certain (fixed) moment T, over the class of all "reasonable" coefficients B. In [19] a condition is found for the maximum probability to be reached for some matrix of diffusion $B(t, x)$. Let us consider matrices B such that BB^T is positive semidefinite in $Q = (0, T) \times D$ and the parabolic equation

$$\text{(4)} \qquad \begin{aligned} u_t(t, x) = {} & \tfrac{1}{2}\operatorname{Tr}(B(T-t, x)\,D_x^2u(t, x)\,B^T(T-t, x)) + \\ & + (f(T-t, x), D_xu(t, x)) \end{aligned}$$

with boundary conditions

$$\text{(5)} \qquad \lim_{t\to 0+} u(t, x) = 0\,, \quad x \in D\,; \quad \lim_{x\to y} u(t, x) = 1\,, \quad y \in \partial D\,, \quad t \geqq 0$$

has a unique bounded solution in Q. A matrix $\tilde{B}$ will be called maximal if it maximizes the value $P[\tau_B \leqq T]$ in the class of all matrices B with the properties just introduced, for which, moreover, $B(t, x)\,B^T(t, x) - \tilde{B}(t, x)\,\tilde{B}^T(t, x)$ is positive semidefinite for $(t, x) \in Q$. One of the main general results in [19] is a theorem asserting that a matrix $\tilde{B}$ is maximal (strongly maximal if the terminology of [19] is used) provided the bounded solution of the equation

$$\begin{aligned} u_t(t, x) = {} & \tfrac{1}{2}\operatorname{Tr}(\tilde{B}(T-t, x)\,D_x^2u(t, x)\,\tilde{B}^T(T-t, x)) + \\ & + (f(T-t, x), D_xu(t, x)) \end{aligned}$$

with the boundary conditions (5) is convex in the domain Q with respect to x. The proof of this assertion, based on an elegant application of the Itô formula, demonstrates the close link between the theory of Itô-type stochastic differential equations and the theory of parabolic partial differential equations. Of course, a natural question from the viewpoint of practicability is, when the equation (4) has a single bounded solution and, especially, when such a solution is convex. Ivo Vrkoč devoted much effort to the problem of finding explicit sufficient conditions formulated in terms of the coefficients of the equation, and his research resulted in a series of papers [23], [24], [26], [27] and [32].

A characteristic feature of the papers of this series is that, as concerns the technique of the proofs, they belong rather to the field of linear partial differential equations of parabolic type, and that they contain a number of results (presented as auxiliary ones) throwing light upon some special aspects of the behaviour of solutions of such equations. Hence they are far from being interesting merely from the viewpoint of stochastic analysis.

As we have already mentioned, another stochastic subject pivotal for I. Vrkoč is the complex of techniques known in the theory of ordinary differential equations as the averaging principle. Averaging methods, inspired by some problems of physics (especially mechanics) make it possible to approximate solutions of an equation with fast oscillating coefficients by solutions of the equation with "averaged" coefficients, without making a major error. More precisely, let us consider stochastic differential equations

$$\mathrm{d}X_\varepsilon(t) = f\left(\frac{t}{\varepsilon}, X_\varepsilon(t)\right)\mathrm{d}t + B\left(\frac{t}{\varepsilon}, X_\varepsilon(t)\right)\mathrm{d}w(t)\,, \quad X_\varepsilon(0) = x_\varepsilon\,, \tag{6}$$

where $\varepsilon > 0$ is a small parameter, $w(t)$ a Wiener process (or a more general continuous process with independent increments), and the coefficients of the local drift f and the diffusion B satisfy the classical conditions for the existence and uniqueness of a solution. The problem, whether on the assumption of existence of functions $\bar{f}$, $\bar{B}$ and an initial value $\bar{x}$ satisfying

$$\bar{f}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t, x)\,\mathrm{d}t\,, \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T |B(t, x) - \bar{B}(x)|^2\,\mathrm{d}t = 0\,,$$
$$\lim_{\varepsilon\to 0+} x_\varepsilon = \bar{x}\,,$$

the solutions of the equation (6) converge in quadratic mean to the solution of the limit equation

$$\mathrm{d}\bar{X}(t) = \bar{f}(\bar{X}(t))\,\mathrm{d}t + \bar{B}(\bar{X}(t))\,\mathrm{d}w(t)\,, \quad \bar{X}(0) = \bar{x}\,, \tag{7}$$

was solved in the affirmative by I. Vrkoč in [12] (and, independently, by I. I. Gichman in his paper Differential equations with random functions (Russian), in: Zim. shkola po teorii veroyat. mat. stat., Kiev 1964, 41–85). It was even shown that

$$\lim_{\varepsilon\to 0+} E \sup_{0\le t\le T} |X_\varepsilon(t) - \bar{X}(t)|^2 = 0 \tag{8}$$

for every $T > 0$ (E denotes the mathematical expectation). A result of the type (8) has been proved also for stochastic differential equations in a domain with an adhesive boundary. As is well known, averaging problems are a special case of the problem of continuous dependence of the equation on a parameter. However, the classical results on continuous dependence are too weak for these purposes and it is necessary to study a finer kind of continuous dependence, the so called integral continuity. The power of this approach became evident only later, in the seventies, when Z. Artstein in his work (Continuous dependence on parameters: On the best possible results, J. Differential Equations *19* (1975), 214–225) made it clear that the results of the integral-continuity type are in a certain natural sense the best possible results on continuous dependence on a parameter.

In [12] and further in [13], [15], [16] Vrkoč also intensively studied, in connection with the averaging method, the problems of stability of solutions and of existence of periodic ones. Already in [12] he solved the averaging problem for the infinite time horizon ($T = +\infty$) on the assumption of uniform asymptotic stability of the limit solution $\bar{X}$. In [15] and [16] Vrkoč investigated the general system of the form

$$\text{(9)} \qquad \mathrm{d}X_\varepsilon(t) = f(t, X_\varepsilon(t), \varepsilon)\,\mathrm{d}t + B(t, X_\varepsilon(t), \varepsilon)\,\mathrm{d}w_\varepsilon(t)\,, \quad X_\varepsilon(0) = x_0$$

depending on a parameter $\varepsilon > 0$, in its relation to the deterministic system

$$\text{(10)} \qquad \dot{y}(t) = f(t, y(t), 0)\,, \quad y(0) = x_0\,.$$

Among other he showed that if the coefficients of the equation (9) are integrally continuous in ε uniformly in a certain sense (for example, we assume

$$\lim_{\varepsilon \to 0+} \int_0^t |B(\tau, y(\tau), \varepsilon)|^2\,\mathrm{d}F_\varepsilon(\tau) = 0\,, \quad F_\varepsilon(t) = E|w_\varepsilon(t)|^2\,,$$

uniformly in x_0), then the exponential stability of a solution of the equation (10) is transferred to (9) for sufficiently small $\varepsilon > 0$. Similarly the existence of periodic solutions of the equation (9) is established, provided its coefficients are periodic in t. From the viewpoint of methodology it is remarkable that both papers [15] and [16] make frequent use of the techniques introduced shortly before by J. Kurzweil for the study of invariant manifolds of ordinary differential equations.

The interest of I. Vrkoč in the averaging method and integral continuity was restored recently in connection with the research in the theory of stochastic evolution equations. A fundamental result was achieved in the paper [52] which deals with integral continuity for the equation

$$\text{(11)} \qquad \mathrm{d}X_\alpha(t) = [AX_\alpha(t) + f_\alpha(t, X_\alpha(t))]\,\mathrm{d}t + \Phi_\alpha(t, X_\alpha(t))\,\mathrm{d}w(t)\,, \quad X_\alpha(0) = \varphi_\alpha$$

in a Hilbert space H. Here we assume that A is an infinitesimal generator of a strongly continuous semigroup $S(t)$ in H, $w(t)$ is a Wiener process with values in a Hilbert space K with a nuclear covariance operator $W, f_\alpha\colon \boldsymbol{R}_+ \times H \to H$, $\Phi_\alpha\colon \boldsymbol{R}_+ \times H \to \to \mathscr{L}(K, H)$ satisfy the usual conditions of measurability, linear growth and lipschitzianity in the space variable, $\alpha \geqq 0$. The equation (11) is interpreted in the mild sense, that is as the integral equation

$$X_\alpha(t) = S(t)\,\varphi_\alpha + \int_0^t S(t-s) f_\alpha(s, X_\alpha(s))\,\mathrm{d}s +$$
$$+ \int_0^t S(t-s)\,\Phi_\alpha(s, X_\alpha(s))\,\mathrm{d}w(s)\,,$$

where the second integral on the right hand side is the stochastic Itô integral. The main result of [52] is the following theorem: If

$$\lim_{\alpha \to 0+} \int_s^t S(t-r)\,[f_\alpha(r,x) - f_0(r,x)]\,\mathrm{d}r = 0\,, \tag{12}$$

$$\lim_{\alpha \to 0+} \int_s^t \mathrm{Tr}\{(\Phi_\alpha(r,x) - \Phi_0(r,x))\,W(\Phi_\alpha(r,x) - \Phi_0(r,x))^*\}\,\mathrm{d}r = 0 \tag{13}$$

for all $x \in H$, $0 \leqq s \leqq t$, and if $\varphi_\alpha \to \varphi_0$ in H, then

$$\lim_{\alpha \to 0+} \sup_{0 \leqq t \leqq T} E\|X_\alpha(t) - X_0(t)\|_H^2 = 0 \tag{14}$$

for all $T > 0$.

The analysis of the finite-dimensional case shows that the condition (13) is essentially a restriction that cannot be weakened, while the assumption (12) on the nonlinear component of the coefficient of local drift ought to be replaced by a more natural condition

$$\lim_{\alpha \to 0+} \int_s^t f_\alpha(r,x)\,\mathrm{d}r = \int_s^t f_0(r,x)\,\mathrm{d}r\,. \tag{15}$$

It is shown in [52] that such a change is possible provided the semigroup $S(t)$ is analytic. In this form the result is applicable to current stochastic partial differential equations of parabolic type. A surprising counterexample constructed by Ivo Vrkoč, which will appear in [56], demonstrates that this change is impossible for the stochastic wave equation if we consider the natural state space $H = W^{1,2} \oplus L^2$. On the other hand, it will be shown in [56] that it is possible to replace (12) by the weaker assumption (15) provided the topology of the state space is weakened accordingly (for example, for the wave equation this means that we pay attention only to the L^2-convergence of the solutions but not the convergence of the derivatives).

The averaging and integral continuity in the theory of stochastic evolution equations is also the topic of the paper [53], where a result of the type of (14) is derived for a problem with infinite time horizon.

Let us now give a brief survey of Vrkoč's momentous cooperation with physicists in the theory of elementary particles.

The late sixties were marked by the development of a new branch of the physics of elementary particles, the so called method of analytic extrapolations. Its basic task is to describe the results of a measurement in a (kinematic) domain by an analytic function (the exact physical amplitude describing the results of a measurement is an analytic function), and extrapolate it to another domain where no measurement at all was (or could be) done. Doing this, it must be taken into account that the measurement is burdened with a certain experimental error. Moreover, it is usually carried out on the boundary of a domain in which the function to be found is holomorphic. This problem exhibits a high degree of nonuniqueness. A minor error (the difference between the physical function and its analytic approximation) in the domain of measurement may cause a big error in the domain to which we extrapolate (an uncorrect problem in Hadamard's sense), which results in the impossibility of finding any nontrivial physical prediction in the given domain. It is necessary to restict the class of the functions considered by subjecting them to suitable and physically acceptable stabilizing conditions. Therefore the physicists concentrated on the so called stable extrapolation methods. Here stability means the following property: if $f(z)$ is holomorphic in a domain of the complex plane and uniformly bounded on a set A, $|f(z)| \leqq \varepsilon$, $z \in A$, then the extrapolation to a set B is said to be stable with respect to a given class of functions F (satisfying the stabilizing conditions) if

$$\lim_{\varepsilon \to 0} \sup_{z \in B} \sup_{f \in F} |f(z)| = 0 .$$

I. Vrkoč studied this problem in [25]. One of his most interesting results concerns the possibility of stable extrapolation of the imaginary part $\operatorname{Im} f(z)$ from one part of the boundary Γ_1 of the domain in which the function is holomorphic to another part Γ_2 provided $f(z)$ is bounded and there exist bounded derivatives along the boundary Γ_2, that is

$$\left|\frac{\mathrm{d} \operatorname{Re} f(e^{i\varphi})}{\mathrm{d}\varphi}\right| \leqq N \quad \text{for} \quad \mathrm{e}^{i\varphi} \in \Gamma_2 ,$$

for the unit circle. This condition is physically acceptable in a domain in which no resonance states occur. In the above mentioned paper I. Vrkoč also proved further theorems on stable extrapolation many of which the physicists had intuitively applied before without knowing their rigorous proofs.

Another field of Vrkoč's activity in physics of elementary particles concerns the properties of the scattering matrix which follow from general physical principles. This research is important in situations when instead of a selfcontained theory there is only a set of fundamental principles available that should be satisfied by any future theory, and that lead typically to a class of functions defined by their holomorphicity domain, continuity on the boundary and behaviour at $z = \infty$.

Vrkoč proved the following theorem:

Let $f(z)$ be

(i) holomorphic in D, where D is the upper complex halfplane without a semicircle with radius r_0 and centre at the origin;

(ii) continuous in the closure of D, with the possible exception of the point $z = \infty$.

Further, let

$$\operatorname{Im} f(z) \geqq 0 \quad \text{for} \quad \operatorname{Im} z = 0 , \quad \operatorname{Re} z \geqq r_1 ,$$

$$\operatorname{Im} f(z) \leqq 0 \quad \text{for} \quad \operatorname{Im} z = 0 , \quad \operatorname{Re} z \leqq -r_1 ,$$

$$\lim_{|z| \to \infty} f(z)/z^2 = 0$$

and

$$\left(\int_{-\infty}^{-r} + \int_r^{\infty}\right) x^{-1} \operatorname{Im} f(x) \, \mathrm{d}x > \int_0^{\pi} \operatorname{Re} f(re^{i\varphi}) \, \mathrm{d}\varphi ,$$

where $r \geqq \max(r_0, r_1)$.

Then there exists R such that $\operatorname{Im}[f(z)/z] > 0$ for all $z \in D$ with $|z| > R$.

The theorem made it possible to establish a number of interesting relations between the behaviour of the real and imaginary parts of the scattering amplitude, its phase and modulus and total effective cross-section at extremely high energies. (Let us note that the total effective cross-section is proportional to the imaginary part of the amplitude for scattering with zero angle.) It is no coincidence that this theoretical study became topical in the seventies in connection with rapid progress in the construction of new accelerators for particle collisions, in particular at very high impact energies.

Vrkoč entered this process with his proverbial ability of grasping the mathematical essence of a physical problem and deduced economical theorems, that is strong assertions from weak assumptions. In a period when the theory is not completed and only the general principles are known, mathematician's abilities of this kind are a true blessing for physics. It was especially Vrkoč's condition (1) that became famous due to its generality and applicability to the total effective cross-section. Another significant generalization was that of Pomeranchuk's theorem [35], which was also obtained thanks to the generality of Vrkoč's formulations, and also Vrkoč's proof that the convergence of the integral

$$\int_{E_0}^{0} \sigma_{-}(E) \, E^{-1} \, \mathrm{d}E$$

(where $\sigma_{-}(E)$ is the difference between the total effective cross-sections of the direct and the so called cross associate reaction and E is the energy of the impact) is a sufficient condition for the existence (and vanishing) of the so called Meiman's limit $\operatorname{Lim} \sigma_{-}(E) = 0$.

In the seventies the so called derivative relations came into fashion. They were motivated by the possibility of expressing the real part of an analytic function not by the integral of the imaginary part as is the case with dispersion relations (Cauchy theorem) but in terms of a differential operator defined by the sum of an infinite series of derivatives of the imaginary part. Ivo Vrkoč proved that this operator is defined only on the class of entire functions [47]. The following theorem was then of fundamental importance for further applications: If a function $f\colon I \to R^1$ has all derivatives at each point of the interval $I \subset R^1$ (i.e. $f \in C^{\infty}(I)$) and the series of the odd derivatives

$$\sum_{n=0}^{\infty} f^{(2n+1)}(x)$$

converges for all $x \in I$, then the function f (as well as the sum of the series) is extensible to an entire function.

Vrkoč also had an inomissible share in new results concerning the generalization of Froissart's bound to the scattering amplitude for complex angles of scattering [39, 43]. Here he contributed in particular by deriving conditions under which the inequalities valid for harmonic or subharmonic functions on the boundary can be transferred to the interior of the domain.

We could continue to discuss Vrkoč's results useful for theoretical physics for a long time. However, let us just point out his vivid interest in the mathematical core of physical problems, his invariable readiness to help the physicists, and the ability to understand their nonexact language.

Besides self-contained collections of papers from the above mentioned fields of mathematics (and physics), I. Vrkoč has been engaged in many other mathematical problems. Let us mention at least a few Vrkoč's papers belonging to this category.

The paper [17] is important from the viewpoint of nonlinear functional analysis: proceeding constructively, Vrkoč gave here a complete analytical description of the general Carathéodory operator. This result is extremely important from the viewpoint of modern mathematics.

A class of functions playing an important role in the theory of invariant manifolds (which was then intensively studied in the seminar on ordinary differential equations in Prague) was investigated in [18]. The papers [33], [38] and [51] from the theory of differential inclusions also stemmed from the subjects studied in the seminar.

Vrkoč's meeting with R. C. Brown who stayed in Prague for a longer period led to papers [44] and [46] which can be included in the theory of selfadjoint boundary value problems. Here Vrkoč contributed the proof of the strictness of the Rayleigh-Ritz inequality, even using in this connection the ancient computer Hewlett-Packard that was then the only computer available in the institute. This remarkable device submitted to Vrkoč's will also during his work on [45], [54] and [55], which were the results of a long-lasting cooperation with a group concentrated around M. Katětov and P. Jedlička. This group worked on mathematical modelling and studied mechanisms governing the sclerosis multiplex. They used facts from Thom's theory of classification of singularities of mappings as well as the elements of randomness exhibited by the disease. Vrkoč not only took part in the construction of the model but also showed uncommon abilities as a programmer, eventually conjuring from the computer graphs of the disease accepted by medical specialists.

Practical problems of applications of mathematics appear also in [48] where numerical aspects of the theory of linear integral equations in connection with energetic losses in big voltage transformers are studied.

As concerns numerical mathematics, it is a not generally known fact that Ivo Vrkoč collaborated also on the famous monograph by I. Babuška, M. Práger and E. Vitásek: *Numerical Processes in Differential Equations* (SNTL, Prague 1966) in the field of stability of numerical calculations.

In the conclusion of similar occasional accounts, made necessary by the course of time, we usually find a list of non-scientific activities of the person concerned. Ivo

Vrkoč also had to engage in some of them; they were not few and not all of them were sensible. (The latter ones he did not usually choose from his free will.) However, they have never formed a characteristic feature of Vrkoč's personality, and we do not feel it necessary to deal with them in detail. Above all, Vrkoč is a modest, amiable man and a mathematician of a renaissance scope. His spontaneous inquisitiveness and playful spirit with which he approaches any mathematical problem have made him a sought-for partner and helped him to retain intellectual qualities corresponding to much younger age.

The collegues and friends of Ivo Vrkoč wish him that he could enjoy the beauty of mathematics for many years to come with the same enthusiasm and dedication as he has done till now.

LIST OF PUBLICATIONS OF IVO VRKOČ

[1] On the conversion of the Četayev theorem (Russian). Czechoslovak Math. Journal *5*, (*80*), 1955, 451–461.

[2] On the conversion of the Lyapunov theorem on stability and the Persidskij theorem on uniform stability (*with J. Kurzweil*). Czechoslovak Math. Journal 7, (*82*), 1957, 254–272.

[3] A note to the unicity of generalized differential equations. Czechoslovak Math. Journal *8*, (*83*), 1958, 510–512.

[4] On integral stability (Czech). Časopis pěst. mat. *83*, 1958, 358–362.

[5] Integral stability (Russian). Czechoslovak Math. Journal *9*, (*84*), 1959, 71–129.

[6] The topological structure of the set of stable solutions of a differential system. Czechoslovak Math. Journal *11*, (*86*), 1961, 262–305.

[7] Stability under permanently acting perturbations (Russian). Časopis pěst. mat. *87*, 1962, 326–358.

[8] On some stability problems. Proceedings of the Conference Equadiff held in Prague 1962, 217–221.

[9] Über eine bestimmte Klasse der zufälligen Prozesse mit absorbierten Barrieren. Časopis pěst. mat. *89*, (1964), 402–425.

[10] On certain properties of a system of linear differential equations. Časopis pěst. mat. *90*, (1965), 311–322.

[11] On homogeneous linear differential equations with random perturbations. Czechoslovak Math. Journal *16*, (*91*), 1966, 199–230.

[12] Extension of the averaging method to stochastic equations. Czechoslovak Math. Journal *16*, (*91*), 1966, 518–544.

[13] Extension of the averaging method to stochastic equations. Acta Facultatis R.N. Univ. Comenianae, Mathematica *XVII* — 1976, 237–242.

[14] Note on stability of control system (*with J. Kučera*). Časopis pěst. mat. *93*, (1968), 472–479.

[15] The exponential stability and periodic solutions of Itô stochastic equations with small stochastic terms. Czechoslovak Math. Journal *18*, (*93*), 1968, 301–314.

[16] The weak exponential stability and periodic solutions of Itô stochastic equations with small stochastic terms. Czechoslovak Math. Journal *18*, (*93*), 1968, 722–752.

[17] The representation of Carathéodory operators. Czechoslovak Math. Journal *19*, (*94*), 1969, 99–109.

[18] The class of functions fulfilling the inequality

$$\|f(x+z) - f(x) - f(y+z) + f(y)\| \leqq \|x-y\| \, \omega(\|z\|) .$$

Czechoslovak Math. Journal *19*, (*94*), 1969, 500—514.

[19] Some maximum principles for stochastic equations. Czechoslovak Math. Journal *19*, (*94*), 1969, 569—604.

[20] A remark on nonoscillatory differential equations. Mathematica *11* (*34*), 1969, 171—177.

[21] Note on stability of a linear homogeneous control system (*with J. Kučera*). Časopis pěst. mat. *95*, (1970), 56—61.

[22] Remark about the relation between measurable and continuous functions. Časopis pěst. mat. *96*, (1971), 225—228.

[23] Some explicit conditions for maximal local diffusions in one-dimensional case. Czechoslovak Math. Journal *21*, (*96*), 1971, 236—256.

[24] Conditions for maximal local diffusions in multi-dimensional case. Czechoslovak Math. Journal *22*, (*97*), 1972, 393—422.

[25] Continuous dependence of holomorphic functions on partly given boundary values. Czechoslovak Math. Journal *23*, (*98*), 1973, 455—466.

[26] On conditions guaranteeing the maximal probability that the solutions of an Itô equation leave a given region. Transactions of the Prague conference on information theory 1971, Academia Praha, 1973, 893—902.

[27] An optimal control problem for Itô stochastic equations. Proceedings of Equadiff III, III. Czech. Conf. on Diff. Equations 1972, J. E. Purkyně Univ. Brno 1973, 205—207.

[28] Diffusion processes and their connection to partial differential equations of parabolic type. Nonlinear evolutions and potential theory. Proceedings of a summer school 1973, Academia Praha 1975, 133—142.

[29] Rigorous phase-modulus correlations for forward scattering amplitude at high energies (*with J. Fischer and P. Kolář*). Physical Review D, Vol. *13*, 1976, 133—149.

[30] Asymptotic correlations between total cross-sections and forward scattering amplitude (*with J. Fischer and P. Kolář*). Czech. J. Phys., Vol. B26, 1976, 86—91.

[31] New asymptotic theorems for the amplitude of forward scattering of hadrons (*with J. Fischer and P. Kolář*). 4th Conference of Czechoslovak Physicists Liberec 1975, Academia Praha 1976, 283—284.

[32] Conditions for strong maximality of local diffusions in multi-dimensional case. Czechoslovak Math. Journal *28*, (*103*), 1978, 200—244.

[33] A new definition and some modifications of Filippov cone. Proceedings of Equadiff IV. Lecture Notes in Math. 703, Springer-Verlag 1979, 433—441.

[34] Liouville formula for system of linear homogeneous Itô stochastic differential equations. CMUC *19*, *1* (1978), 141—146.

[35] Pomeranchuk-type theorems for total and elastic cross sections (*with J. Fischer and R. Šály*). Physical Review D, Vol. *18*, No. 11, 1978, 4271—4281.

[36] Some new results in Pomeranchuk-like theorems (*with J. Fischer and R. Šály*). Proceedings of the Hadron Structure '77 Conference, High Tatras 1977, Physics and Applications Vol. *4*, 1979, Veda Bratislava, 315—320.

[37] Exercises in Stochastic Analysis (*with P. Mandl and V. Lánská*). Appendix to Kybernetika, Kybernetika *14* (1978), 3—35.

[38] Scorza-Dragoni property of Filippov mappings. Tôhoku Math. Journal. The second series, Vol. *32*, (1980), 309—316.

[39] A new expression for the high-energy upper bound of the scattering amplitude (*with M. Chaichian, J. Fischer, S. M. Molodensky and N. F. Nelipa*). J. Math. Phys. *22* (*2*), 1981, 380—384.

[40] Extension of the Froissart bound to unphysical scattering angles (*with M. Chaichain, J. Fischer and N. F. Nelipa*). Czech. Journal of Physics Vol. *B31*, 1981, 1324—1333.
[41] Froissart bound for complex scattering angles (Czech) (*with M. Chaichian, J. Fischer and N. F. Nelipa*). 7th Conference of Czechoslovak Physicists, Praha 24.—28. 8. 1981, Academia.
[42] Strongly maximal matrix functions in regions containing stable solutions. Proc. of the Equadiff conference held in Bratislava 1981, Teubner Texte zur Math. 47, 352—355.
[43] Improvement of the Froissart-Martin bound for a complex scattering angle (with *M. Chaichian, J. Fischer and N. F. Nelipa*). J. Math. Phys. *24* (2), 1983, 291—299.
[44] On generalization of Rayleigh-Ritz inequality (*with R. C. Brown*). Workshop on Diff. Equations and Control Theory, INCREST, Univ. of Bucharest, 1983, 39—42.
[45] Mathematical modelling of the course of sclerosis multiplex using the principles of Thom's catastrophe theory (Czech) (*with P. Jedlička, M. Katětov and J. Bočková*). Čs. neurologie a neurochirurgie *46*/79, 1983, 41—50.
[46] Some multipoint extensions of the Rayleigh-Ritz inequality (*with R. C. Brown*). Proc. of the Royal Soc. of Edinburgh, *100A*, (1985), 157—174.
[47] Holomorphic extension of a function whose odd derivatives are summable. Czechoslovak Math. Journal *36*, (110), 1985, 59—65.
[48] Analysis of integral equations attached to skin effect. Aplikace matematiky *30*, (1985), 361—374.
[49] Sixty years of Jaroslav Kurzweil (*with J. Jarník, Š. Schwabik and M. Tvrdý*). Czechoslovak Math. Journal *36*, (*111*), 1986, 147—166; the same in Czech in Časopis pěst. mat. *111* (1986), 91—111.
[50] Introduction to the theory of Itô's stochastic differential equations (Czech). Fyzika a matematica, sborník prací semináře Souš 1987, JČSMF 1987.
[51] Absolutely continuous selections from absolutely continuous set valued map (*with V. Křivan*). Czechoslovak Math. Journal *40*, (*115*), 1990, 503—513.
[52] An averaging principle for stochastic evolution equations (*with J. Seidler*). Časopis pěst. mat. *115*, (1990), 240—263.
[53] An averaging principle for stochastic evolution equations II (*with B. Maslowski and J. Seidler*). Mathematica Bohemica *116*, (1991), 191—224.
[54] Some results of modelling of the course of sclerosis multiplex using the fundamental notions of catastrophe theory (Czech) (*with J. Bočková*). Research Report MÚ ČSAV and MFF UK, 1981.
[55] Mathematical model of sclerosis multiplex based on the conjecture of random distribution plaques (Czech). Research Report MÚ ČSAV and MFF UK, 1985.
[56] Integral continuity and stability for stochastic hyperbolic systems (*with B. Maslowski and J. Seidler*). Preprint Math. Inst. of the Czech. Acad. of Sci., No. 65, 1991.