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N_2-LOCALLY CONNECTED GRAPHS AND THEIR UPPER EMBEDDABILITY

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0. Let G be a graph in the sense of [1] with vertex set $V(G)$ and edge set $E(G)$. Let $u \in V(G)$; we denote by $V(u, G)$ the set of all $u' \in V(G)$ such that u and u' are adjacent; we denote by $E(u, G)$ the set of all $e \in E(G)$ such that e is not incident with u and at least one of the vertices incident with e is adjacent to u; if $V(u, G) \neq \emptyset$, then we denote by $N_1(u, G)$ the subgraph of G induced by $V(u, G)$; if $E(u, G) \neq \emptyset$, then we denote by $N_2(u, G)$ the subgraph of G induced by $E(u, G)$. We say that G is locally connected if $V(v, G) \neq \emptyset$ and $N_1(v, G)$ is connected, for each $v \in V(G)$; see [2] and [12]. We say that G is N_2-locally connected if $E(w, G) \neq \emptyset$ and $N_2(w, G)$ is connected, for each $w \in V(G)$; see [11] and [10]. As was shown in [10], if G is N_2-locally connected, then every edge of G which belongs to a cycle of length 3 or 4.

1. Let G be a graph, and let $\mathscr{P}$ be a partition of $V(G)$. If $\mathscr{R} \subseteq \mathscr{P}$, then we denote by $E_{\mathscr{R}}$ the set of all $e \in E(G)$ such that the vertices incident with e belong to distinct elements of $\mathscr{R}$, and moreover, we denote by $G(\mathscr{R})$ the subgraph of G induced by

$$\bigcup_{R \in \mathscr{R}} R .$$

We shall say that $\mathscr{P}$ is a C-partition of G if $|P| \geqq 2$ and $G(\{P\})$ is connected, for each $P \in \mathscr{P}$.

The following theorem gives the first of the two main results of the present paper:

Theorem 1. *Let G be a 2-connected, N_2-locally connected graph. Then*

(1) $$|E_{\mathscr{P}}| \geqq 2(|\mathscr{P}| - 1),$$

for every C-partition $\mathscr{P}$ of G.

Before proving Theorem 1 we shall prove the following lemma:

Lemma 1. *Let G be a 2-connected, N_2-locally connected graph, and let $\mathscr{P}$ be a C-partition of G such that $|\mathscr{P}| \geqq 2$. Then there exists $\mathscr{R} \subseteq \mathscr{P}$ such that $|\mathscr{R}| \geqq 2$, $G(\mathscr{R})$ is connected and*

(2) $$|E_{\mathscr{R}}| \geqq 2(|\mathscr{R}| - 1).$$

Proof. We first assume that there exist distinct P^*, $P^{**} \in \mathscr{P}$ such that $|E_{\{P^*,P^{**}\}}| \geqq 2$. If we put $\mathscr{R} = \{P^*, P^{**}\}$, we can see that $|\mathscr{R}| \geqq 2$, $G(\mathscr{R})$ is connected and (2) holds.

We now assume that

$$(3) \qquad |E_{\{P^\#,P^\flat\}}| \leqq 1\,, \quad \text{for any distinct} \quad P^\#, P^\flat \in \mathscr{P}\,.$$

Since G is 2-connected and $\mathscr{P}$ is a C-partition of G, we can see that

$$(4) \qquad V(N_2(u, G) \cap P \neq \emptyset\,, \quad \text{for every} \quad P \in \mathscr{P} \quad \text{and every} \quad u \in P\,.$$

Since G is N_2-locally connected, it follows from (3) and (4) that

(5) if $P \in \mathscr{P}$ and $u \in P$ such that u is incident with an edge in $E_{\mathscr{P}}$, then there exist distinct $P^{(1)}$, $P^{(2)} \in \mathscr{P} - \{P\}$ and $u^{(1)} \in P^{(1)}$, $u^{(2)} \in P^{(2)}$, $v \in P - \{u\}$ such that $uu^{(1)}$, $u^{(1)}u^{(2)}$, $u^{(2)}v \in E_{\mathscr{P}}$.

We shall define a sequence $\mathscr{R}_1, \ldots$ as follows.

Consider an arbitrary $P_1 \in \mathscr{P}$. Since G is connected and $|\mathscr{P}| \geqq 2$, there exists $u_1 \in P_1$ such that u_1 is incident with an edge in $E_{\mathscr{P}}$. According to (5), there exist distinct $P_1^{(1)}$, $P_1^{(2)} \in \mathscr{P} - \{P_1\}$ and $u_1^{(1)} \in P_1^{(1)}$, $u_1^{(2)} \in P_1^{(2)}$, $v_1 \in P_1 - \{u_1\}$ such that $u_1u_1^{(1)}$, $u_1^{(1)}u_1^{(2)}$, $u_1^{(2)}v_1 \in E_{\mathscr{P}}$. Denote $\mathscr{R}_1 = \{P_1, P_1^{(1)}, P_1^{(2)}\}$ and $E_1 = \{u_1u_1^{(1)}, u_1^{(1)}u_1^{(2)}, u_1^{(2)}v_1\}$. Obviously, $G(\mathscr{R}_1)$ is connected.

Let $n \geqq 1$. Assume that the members $\mathscr{R}_1, \ldots, \mathscr{R}_n$ of the sequence were constructed. We denote by $\tilde{\mathscr{R}}_n$ the set of all $R \in \mathscr{R}_n$ with the properties that exactly one vertex of R is incident with an edge in E_n. If $\tilde{\mathscr{R}}_n = \emptyset$, we put $\mathscr{R}_{n+1} = \mathscr{R}_n$ and $E_{n+1} = E_n$. Let $\mathscr{R}_n \neq \emptyset$. Let us choose an arbitrary $P_{n+1} \in \tilde{\mathscr{R}}_n$. There exists exactly one $u_{n+1} \in P_{n+1}$ such that u_{n+1} is incident with an edge in E_n. As follows from (5), there exist distinct $P_{n+1}^{(1)}$, $P_{n+1}^{(2)} \in \mathscr{P} - \{P_{n+1}\}$ and $u_{n+1}^{(1)} \in P_{n+1}^{(1)}$, $u_{n+1}^{(2)} \in P_{n+1}^{(2)}$, $v_{n+1} \in P_{n+1} - \{u_{n+1}\}$ such that $u_{n+1}u_{n+1}^{(1)}$, $u_{n+1}^{(1)}u_{n+1}^{(2)}$, $u_{n+1}^{(2)}v_{n+1} \in E_{\mathscr{P}}$. Denote $\mathscr{R}_{n+1} = \mathscr{R}_n \cup \{P_{n+1}^{(1)}, P_{n+1}^{(2)}\}$ and $E_{n+1} = E_n \cup \{u_{n+1}u_{n+1}^{(1)}, u_{n+1}^{(1)}u_{n+1}^{(2)}, u_{n+1}^{(2)}v_{n+1}\}$. Clearly, $G(\mathscr{R}_{n+1})$ is connected.

It is easy to see that there exists exactly one $m \geqq 2$ such that $E_{m-1} \neq E_m = E_{m+1}$. We now prove that

$$(6) \qquad |E_k| \geqq 2|\mathscr{R}_k| - |\tilde{\mathscr{R}}_k| - 1\,,$$

for each $k \in \{1, \ldots, m\}$.

We proceed by the induction on k. The case when $k = 1$ is obvious. Let $k \geqq 2$. According to the induction assumption, $|E_{k-1}| \geqq 2|\mathscr{R}_{k-1}| - |\tilde{\mathscr{R}}_{k-1}| - 1$. Obviously, $P_k \in \tilde{\mathscr{R}}_{k-1} - \tilde{\mathscr{R}}_k$. Denote $e^{(1)} = u_ku_k^{(1)}$, $e^{(2)} = u_k^{(2)}v_k$ and $f = u_k^{(1)}u_k^{(2)}$. Let $i \in \{1, 2\}$. It is clear that if $P_k^{(i)} \notin \mathscr{R}_{k-1}$, then $P_k^{(i)} \in \tilde{\mathscr{R}}_k - \tilde{\mathscr{R}}_{k-1}$ and $e^{(i)}$, $f \in E_k - E_{k-1}$. Moreover, it is not difficult to see that if $P_k^{(i)} \in \tilde{\mathscr{R}}_{k-1} - \tilde{\mathscr{R}}_k$, then $P_k^{(i)} \in \mathscr{R}_{k-1} - \mathscr{R}_k$ and $e^{(i)}$, $f \in E_k - E_{k-1}$. These observations imply (6).

Recall that $m \geqq 2$. Since $\mathscr{R}_m = \emptyset$, it follows from (6) that $|E_m| \geqq 2|\mathscr{R}_m| - 1$. Put $\mathscr{R} = \mathscr{R}_m$. Since $E_m \subseteq E_{\mathscr{R}}$, we have that (2) holds. Since $|\mathscr{R}| \geqq 2$ and $E(\mathscr{R})$ is connected, the proof of the lemma is complete.

Proof of Theorem 1. There exists a C-partition $\mathscr{P}^*$ of G such that

(7) $$2(|\mathscr{P}^*| - 1) - |E_{\mathscr{P}^*}| \geqq 2(|\mathscr{P}'| - 1) - |E_{\mathscr{P}'}|, \quad \text{for every } C\text{-partition } \mathscr{P}' \text{ of } G$$

and

(8) $$2(|\mathscr{P}^*| - 1) - |E_{\mathscr{P}^*}| > 2(|\mathscr{P}''| - 1) - |E_{\mathscr{P}''}|, \quad \text{for every } C\text{-partition } P'' \text{ of } G \text{ with the property that } |\mathscr{P}''| < |\mathscr{P}^*|.$$

Let first $|\mathscr{P}^*| \geqq 2$. According to Lemma 1, there exists $\mathscr{R} \subseteq \mathscr{P}^*$ such that $|\mathscr{R}| \geqq 2$, $G(\mathscr{R})$ is connected and $|E_{\mathscr{R}}| \geqq 2(|\mathscr{R}| - 1)$. Denote

$$P^\# = \bigcup_{R \in \mathscr{R}} R$$

and $\mathscr{P}^\# = (\mathscr{P}^* - \mathscr{R}) \cup \{P^\#\}$. Clearly, $\mathscr{P}^\#$ is a C-partition of G and $|\mathscr{P}^\#| < |\mathscr{P}^*|$. Since $E_{\mathscr{P}^*} = E_{\mathscr{P}^\#} \cup E_{\mathscr{R}}$ and $E_{\mathscr{P}^*} \cap E_{\mathscr{R}} = \emptyset$, we have that

$$\begin{aligned} &2(|\mathscr{P}^*| - 1) - |E_{\mathscr{P}^*}| = \\ &= 2(|\mathscr{P}^\#| - 1) - |E_{\mathscr{P}^\#}| + 2(|\mathscr{R}| - 1) - |E_{\mathscr{R}}| \leqq \\ &\leqq 2(|\mathscr{P}^\#| - 1) - |E_{\mathscr{P}^\#}|, \end{aligned}$$

which is a contradiction with (8).

Let now $|\mathscr{P}^*| = 1$. Then $|E_{\mathscr{P}^*}| = 0 = 2(|\mathscr{P}^*| - 1)$. It follows from (7) that (1) holds for every C-partition $\mathscr{P}$ of G. Thus, the proof of the theorem is complete.

2. The upper embeddability belongs to important notions of the theory of embedding (pseudo)graphs into surfaces; cf. [13] or Chapter 5 in [1]. A connected pseudograph G with p vertices and q edges is said to be upper embeddable if there exists a 2-cell embedding of G into the orientable surface of genus

$$[(q - p + 1)/2].$$

If F is a pseudograph, then we denote by $b(F)$ the number of components H of F such that $|E(H)| - |V(H)|$ is even, and we denote by $c(F)$ the number of all components of F.

Theorem A. *Let G be a connected pseudograph. Then the following statements are equivalent:*

(α) *G is upper embeddable;*

(β) *there exists a spanning tree T of G such that for at most one component H of $G - E(T)$, $|E(H)|$ is odd;*

(γ) *$|A| \geqq b(G - A) + c(G - A) - 2$, for every $A \subseteq E(G)$.*

The equivalence $(\alpha) \Leftrightarrow (\beta)$ was proved independently in [5], [6] and [14] (but the result in [5] was formulated rather differently). The equivalence $(\beta) \Leftrightarrow (\gamma)$ was proved independently in [4] and [8] (the result in [4] was formulated rather differently).

It was proved in [7] that if G is connected, locally connected graph, then G is upper embeddable; the proof in [7] was based on the equivalence $(\alpha) \Leftrightarrow (\beta)$. We shall now prove that if G is connected, N_2-locally connected, then G is upper embeddable; for the case when G is 2-connected, the proof will be based on the equivalence $(\alpha) \Leftrightarrow (\gamma)$.

Lemma 2. *Let G be a 2-connected, N_2-locally connected graph. Then G is upper embeddable.*

Proof. Obviously, there exists $A_0 \subseteq E(G)$ such that

$$(9) \qquad \begin{aligned} & b(G - A_0) + c(G - A_0) - 2 - |A_0| \geqq \\ & \geqq b(G - A') + c(G - A') - 2 - |A'|, \quad \text{for every} \quad A' \subseteq E(G) \end{aligned}$$

and

$$\begin{aligned} & b(G - A_0) + c(G - A_0) - 2 - |A_0| > b(G - A'') + \\ & + c(G - A'') - 2 - |A''| \end{aligned}$$

for every $A'' \subseteq E(G)$ with the property that $|A''| < |A_0|$. It is not difficult to see that

(10) $\quad |V(H)| \geqq 2$, for each component H of $G - A_0$, and

(11) $\quad e$ is incident with vertices of distinct components of $G - A_0$, for each $e \in A_0$.

According to (10) and (11), there exists a C-partition $\mathscr{P}$ of G such that $P \in \mathscr{P}$ if and only if P is the vertex set of a component of $G - A_0$. Thus $E_{\mathscr{P}} = A_0$. Theorem 1 implies that $|A_0| \geqq 2(c(G - A_0) - 1)$. As follows from (9), the proof of the lemma is complete.

As was shown in [10], if G is connected, N_2-locally connected graph, then at most one block of G is cyclic. Therefore, if we combine Lemma 2 with Theorem 1, we easily get the following result:

Theorem 2. *If G is connected, N_2-locally connected graph, then G is upper embeddable.*

Theorem 2 is a generalization of the theorem in [7]. Another results more general

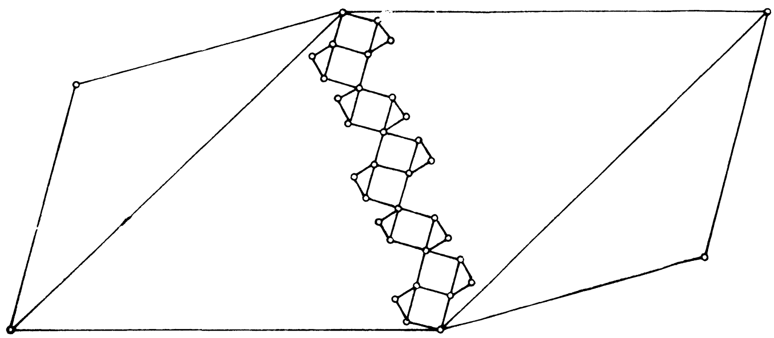

Fig. 1

than the theorem in [7] were proved in [3] and [9]. In [3] Glukhov proved that if G is a 2-connected multigraph such that each edge of G belongs to a cycle of length at most 3, then G is upper embeddable. In [9] the present author proved that if G is a connected graph with the property that there exists $i \in \{1, 2\}$ such that $V(u_i, G) \neq \emptyset$ and $N_1(u_i, G)$ is connected, for every pair of adjacent vertices u_1 and u_2 of G, then G is upper embeddable.

Fig. 1 presents a 2-connected graph, say a graph G_1, such that each edge of G_1 belongs to a cycle of length 3 or 4. We can see that there exists $j \in \{1, 2\}$ such that $E(v_j, G_1) \neq \emptyset$ and $N_2(v_j, G_1)$ is connected, for every pair of adjacent vertices v_1 and v_2 of G_1. It is easy to show that G_1 is not upper embeddable.

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