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LOCAL SOLVABILITY OF DIAGONAL SEMILINEAR PARABOLIC SYSTEMS

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INTRODUCTION

In this note we want to present a more direct extension of our results, reported recently in [3], concerning the solvability of a single equation onto diagonal semilinear parabolic systems of the form:

$$u_t^\nu = \sum_{i,j} (a_{ij}^\nu(t, x)\, u_{x_j}^\nu)_{x_i} + f^\nu(t, x, \boldsymbol{u}, u_x^\nu) =: P^\nu u^\nu + f^\nu \,, \tag{1}$$

where $i, j = 1, \ldots, n$, $\nu = 1, \ldots, N$, $\boldsymbol{u} = (u^1, \ldots, u^N)$ and $u_x^\nu = (u_{x_1}^\nu, \ldots, u_{x_n}^\nu)$ is a space gradient of u^ν. The form (1) is more restrictive than that considered recently in [1] or [2], however our proofs are more elementary and allow a more precise estimate (see the estimate of the life time of solution in [3]) in this special case. There are also important examples taken from applications covered by (1) (compare with the end of this note). We complete (1) by the initial condition

$$u^\nu(0, x) = u_0^\nu(x) \,, \quad x \in \Omega \subset R^n \,, \quad \nu = 1, \ldots, N \,, \tag{2}$$

with bounded, smooth domain Ω and boundary conditions of one of the two following types:

$$\varphi^\nu(x)\, u^\nu + \psi^\nu(x) \frac{\partial u^\nu}{\partial N^\nu} = 0 \quad \text{on} \quad \partial\Omega \,, \tag{3}$$

where

$$\frac{\partial}{\partial N^\nu} := \sum_{i,j} a_{ij}^\nu(t, x) \frac{\partial}{\partial x_j} \cos(n, x_i) \,,$$

n is the inward normal vector to $\partial\Omega$, and one (and only one for each ν) of the additional requirements on the functions φ^ν, ψ^ν is assumed to hold:

(3a) $\psi^\nu(x) = 0$ and $\varphi^\nu(x) \geqq \varphi_0 > 0$ on $\partial\Omega$ (the Dirichlet condition for u^ν),

(3b) $\varphi^\nu(x) \leqq 0$ and $\psi^\nu(x) \geqq \psi_0 > 0$ (the third boundary condition for u^ν).

1. ASSUMPTIONS

The following conditions concerning the data in (1)–(3) are assumed to hold throughout this note; for a fixed $\alpha \in (0, 1)$

(A1) The equation is parabolic:

$$\underset{\tau>0}{\forall}\ \underset{a_0>0}{\exists}\ \underset{\substack{x\in\bar{\Omega}\\ t\in[0,\tau]}}{\forall}\ \underset{\substack{\zeta\in R^n\\ 1\leqq\nu\leqq N}}{\forall}\ \sum_{i,j} a_{ij}^{\nu}(t, x)\,\zeta_i\zeta_j \geqq a_0|\zeta|^2 .$$

(A2) $a_{ij}^{\nu}, (a_{ij}^{\nu})_{x_i}$ are Holder continuous in x (exponent α), $(a_{ij}^{\nu})_{x_i}$ are locally Holder continuous in t (exponent $\frac{1}{2}\alpha$) and a_{ij}^{ν} are locally Lipschitz continuous in t, all this in the set $[0, \infty) \times \bar{\Omega}$.

(A3) f^{ν} are locally Lipschitz continuous with respect to t, u^{μ} and $u_{x_i}^{\nu}$ are Holder continuous (exponent α) in x, the Lipschitz, Holder constants are valid in sets $[0, \tau] \times \bar{\Omega} \times [-r_1, r_1]^N \times [-r_2, r_2]^n$ ($r_1, r_2 > 0$ arbitrary).

(A4) $\varphi^{\nu}, \psi^{\nu} \in C^{1+\alpha}(\partial\Omega)$, $\partial\Omega \in C^{2+\alpha}$.

(A5) $u_0^{\nu} \in C^{2+\alpha}(\bar{\Omega})$ and necessary compatibility conditions are satisfied;

$$u_0^{\nu} = 0 = P^{\nu}u_0^{\nu} + f^{\nu}(0, x, \boldsymbol{u}_0, (u_0^{\nu})_x)$$

on $\partial\Omega$ if (3a) holds,

$$\varphi^{\nu}(x)\, u_0^{\nu} + \psi^{\nu}(x) \frac{\partial u_0^{\nu}}{\partial N^{\nu}} = 0$$

on $\partial\Omega$ under the condition (3b).

The conditions (A1)–(A5) mentioned above are sufficient for local solvability of the problem (1)–(3) as shown below.

By a $C^{1,2}$ solution $\boldsymbol{u}$ of our problem we mean its classical solution with derivatives appearing in (1) continuous on compact subsets of $[0, T_{ex}) \times \bar{\Omega}$, where $T_{ex} \leqq +\infty$ is the life time of such a solution. We set $\|v\|_p$ for the $L^p(\Omega)$ norm of v, $\|v\|_{2,p}$ for the $W^{2,p}(\Omega)$ norm of v.

Analogously as in [3] we introduce a set

(4) $$X := \{(t, x, \boldsymbol{u}, \boldsymbol{p}) \in R^+ \times \bar{\Omega} \times R^N \times R^n ;\quad t \in [0, T_0] ,\quad x \in \bar{\Omega} ,$$
$$|\boldsymbol{u}| \leqq M_1 ,\quad |\boldsymbol{p}| \leqq M_2\} ,$$

where $T_0 > 0$ is fixed, M_1 and M_2 are two positive numbers and $|\boldsymbol{u}| = \sum_{\nu} |u^{\nu}|$. Inside X the Lipschitz constants for a_{ij}^{ν}, f^{ν} are fixed and denoted as follows; A is a Lipschitz constant for all a_{ij}^{ν} with respect to t, L_t for all f^{ν} with respect to t, L_u for all f^{ν} with respect to u^{μ}, $\mu = 1, \ldots, N$, L_{∇} for all f^{ν} with respect to $u_{x_i}^{\nu}$, $i = 1, \ldots, n$, $\nu = 1, \ldots, N$.

We are ready to formulate the introductory lemma of our note.

Lemma 1. *As long as a $C^{1,2}$ solution $\boldsymbol{u}$ of* (1)–(3) *remains in* X, *the following estimates for sufficiently small* δ $(0 < \delta \leqq \delta_0)$ *hold*:

$$(5) \qquad \underset{C_\delta > 0}{\exists} \sum_\nu \sum_i \|u^\nu_{x_i}(t,\cdot)\|_\infty \leqq \delta \sum_\nu (\|u^\nu_t(t,\cdot)\|_p + NM|\Omega|^{1/p}) + $$
$$+ C_\delta \|\boldsymbol{u}(t,\cdot)\|_p,$$

where $p > n$, $M \geqq \|f(t,\cdot,\mathbf{0},\mathbf{0})\|_\infty$ *for* $t \in [0, T_0]$, *and* $\|\boldsymbol{u}(t,\cdot)\|_Y := \sum_\nu \|u^\nu(t,\cdot)\|_Y$ *as usual*, $C_\delta = \text{const.}\, \delta^{-(p+n)/(p-n)}$.

Outline of the proof. The proof is based on the following three estimates:

(i) Since the equation (1) is fulfiled and inside X global Lipschitz constants are valid, then:

$$\sum_\nu \|P^\nu u^\nu(t,\cdot)\|_p \leqq \sum_\nu \|u^\nu_t(t,\cdot)\|_p + \sum_\nu \sum_i L_\nabla \|u^\nu_{x_i}(t,\cdot)\|_p +$$
$$+ L_u \|\boldsymbol{u}(t,\cdot)\|_p + NM|\Omega|^{1/p}.$$

(ii) As a consequence of the Calderon-Zygmund estimates (compare e.g. [8], Chapt. III, § 11) for solutions of linear elliptic equations, we have:

$$\sum_\nu \|u^\nu(t,\cdot)\|_{2,p} \leqq \text{const.} \sum_\nu (\|P^\nu u^\nu(t,\cdot)\|_p + \|u^\nu(t,\cdot)\|_p),$$

the const. above being valid while $\boldsymbol{u}$ remains in X.

(iii) As a consequence of the Nirenberg-Gagliardo estimates, for arbitrary $\delta_1 > 0$ and every fixed $\nu \in \{1, \ldots, N\}$:

$$\sum_i \|u^\nu_{x_i}(t,\cdot)\|_\infty \leqq \delta_1 \|u^\nu(t,\cdot)\|_{2,p} + C_{\delta_1} \|u^\nu(t,\cdot)\|_p,$$

whenever $p > n$.

The three estimates together give as the result (5), provided a sufficiently small δ_1 in (iii) is taken.

An estimate analogous to (5) for the (smooth) $C^{1,2}$ solution $\boldsymbol{u}$ remains still valid for $t = 0$ and $\boldsymbol{u}(t,\cdot)$ replaced by $\boldsymbol{u}_0$ with the only evident change of $u^\nu_t(t,\cdot)$ for $P^\nu u^\nu_0 + f^\nu(0, x, \boldsymbol{u}_0, (u^\nu_0)_x)$. The proof is completed.

We are now able to formulate the a priori estimates fundamental in the proof of local solvability of (1)–(3).

Theorem 1. *For two arbitrary pairs of positive numbers* (m_1, m_2) *and* (M_1, M_2), *such that* $m_1 < M_1$ *and* $m_2 < M_2$, *there exists a time* $T \in (0, T_0]$, *that every* $C^{1,2}$ *solution* $\boldsymbol{u}$ *of* (1)–(3) *corresponding to the initial function* $\boldsymbol{u}_0$ *with*

$$(6) \qquad \|\boldsymbol{u}_0\|_\infty \leqq m_1 \quad \text{and} \quad \delta \sum_\nu (\|P^\nu u^\nu_0 + f^\nu(0,\cdot,\boldsymbol{u}_0,(u^\nu_0)_x)\|_p +$$
$$+ NM|\Omega|^{1/p}) + C_\delta \|\boldsymbol{u}_0\|_p \leqq m_2$$

(*with* δ, C_δ *the same as in Lemma* 1), *satisfies, at least for* $t \leqq T$:

$$\|u(t,\cdot)\|_\infty \leqq M_1 \quad \textit{and} \quad \sum_\nu \sum_i \|u^\nu_{x_i}(t,\cdot)\|_\infty \leqq M_2 . \tag{7}$$

We present here only a part of the proof devoted to the a priori estimates of the time derivatives $u^\nu_t(t,\cdot)$ in $L^p(\Omega)$, p being an even number grater than $\max\{n; 2\}$, differing in details from its one-dimensional analogon. The above mentioned estimates together with an $L^\infty(\Omega)$ a priori estimate of $u(t,\cdot)$ (which is omitted in the present note) leads, in the presence of Lemma 1, to the second conclusion in (7). We have:

Lemma 2. *Under the assumptions of Theorem* 1, *whenever* u *remains in* X, *then*:

$$\|u_t(t,\cdot)\|_p^2 \leqq \left[\|u_t(0,\cdot)\|_p^2 + \frac{c_1}{c}\left(1 - \exp\left(-\frac{2c}{p}t\right)\right)\right]\exp\left(\frac{2c}{p}t\right), \tag{8}$$

with c *independent of* p (*here* p *is an even number greater than* $\max\{n; 2\}$).

Proof. As a consequence of (1) we get an equation for difference quotients (for fixed $h > 0$ we set

$g_h(t,x) := h^{-1}(g(t+h,x) - g(t,x)))$:

$$u^\nu_{ht} = \sum_{i,j} (a^\nu_{ij}(t,x)\, u^\nu_{x_j})_{x_i h} + f^\nu_h(t,x,u,u^\nu_x) . \tag{9}$$

Multiplying (9) by $(u^\nu_h)^{p-1}$, integrating over Ω and by parts and summing with respect to ν, we obtain:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \sum_\nu (u^\nu_h)^p \,\mathrm{d}x = \tag{10}$$

$$= \sum_\nu \left\{\int_{\partial\Omega}\left[-\frac{\partial u^\nu}{\partial N^\nu}\right]_h (u^\nu_h)^{p-1}\,\mathrm{d}s - \int_\Omega \sum_{i,j}\left[a^\nu_{ijh}(t,x)\, u^\nu_{x_j}(t+h,x) + a^\nu_{ij}(t,x)\, u^\nu_{hx_j}(t,x)\right] .\right.$$

$$\left. . \left[(u^\nu_h)^{p-1}\right]_{x_i}\mathrm{d}x + \int_\Omega f^\nu_h (u^\nu_h)^{p-1}\,\mathrm{d}x\right\} =: \sum_\nu \{R_1 - [R_2 + R_3] + R_4\} .$$

Due to (3a) or (3b) the boundary integral is non-positive ($R_1 \leqq 0$), the remaining right side components are estimated as follows:

$$|R_2| \leqq \frac{2(p-1)}{p} A \sum_i \|[(u^\nu_h)^{p/2}]_{x_i}\|_2 \, \|(u^\nu_h)^{(p-2)/2}\|_{2p/(p-2)} \times \tag{11}$$

$$\times \left(\sum_j \|u^\nu_{x_j}(t+h,\cdot)\|_p\right),$$

where the Holder inequality was used,

$$R_3 \leqq -\frac{4(p-1)}{p^2} a_0 \int_\Omega \sum_i [(u^\nu_h)^{p/2}]^2_{x_i}\,\mathrm{d}x ,$$

further

$$|R_4| \leqq L_t \int_\Omega \sum_i |u_h^\nu|^{p-1}\, dx + L_u \int_\Omega \sum_\mu |u_h^\mu (u_h^\nu)^{p-1}|\, dx +$$

$$+ L_\nabla \frac{2}{p} \int_\Omega \sum_i |[(u_h^\nu)^{p/2}]_{x_i} (u_h^\nu)^{p/2}|\, dx \leqq$$

$$\leqq L_t \int_\Omega [(u_h^\nu)^{p-2} + (u_h^\nu)^p]\, dx + L_u \int_\Omega \sum_\mu \left[\frac{p-1}{p}(u_h^\nu)^p + \frac{1}{p}(u_h^\mu)^p\right] dx +$$

$$+ L_\nabla \frac{\varepsilon}{p} \int_\Omega \sum_i [(u_h^\nu)^{p/2}]_{x_i}^2\, dx + L_\nabla \frac{n}{\varepsilon} \int_\Omega (u_h^\nu)^p\, dx ,$$

where the Cauchy and Young inequalities and an estimate $|a|^{p-1} \leqq a^{p-2} + a^p$ were used. The final, following from (10) estimate has thus the form:

$$\frac{1}{p}\frac{d}{dt}\int_\Omega \sum_\nu (u_h^\nu)^p\, dx \leqq \sum_\nu \left\{\left[-\frac{4(p-1)}{p^2} a_0 + \frac{2(p-1)}{p} A \frac{\bar\varepsilon}{2} + L_\nabla \frac{\varepsilon}{p}\right] \times\right.$$

$$\times \int_\Omega \sum_i [(u_h^\nu)^{p/2}]_{x_i}^2\, dx + \frac{(p-1)\, An}{p\bar\varepsilon} \|(u_h^\nu)^{(p-2)/2}\|_{2p/(p-2)}^2 \big(\sum_j \|u_{x_j}^\nu(t+h,\cdot)\|_p\big)^2 +$$

$$\left. + L_t |\Omega|^{2/p} \left(\int_\Omega (u_h^\nu)^p\, dx\right)^{(p-2)/p}\right\} +$$

$$+ \left[L_\nabla \frac{n}{\varepsilon} + L_t + L_u \left(\frac{p-1}{p} N + \frac{1}{p} N\right)\right] \int_\Omega \sum_\nu (u_h^\nu)^p\, dx .$$

Estimating the second right side component with the use of (5), choosing ε, $\bar\varepsilon$ small enough, so that the first square bracket at the right hand side becomes non-positive, letting $h \to 0^+$, we arrive at the estimate:

$$\frac{d}{dt}\int_\Omega \sum_\nu (u_t^\nu)^p\, dx \leqq c \int_\Omega \sum_\nu (u_t^\nu)^p\, dx + c_1 \left(\int_\Omega \sum_\nu (u_t^\nu)^p\, dx\right)^{(p-2)/p}$$

generating (8) directly. The proof is completed.

The remaining part of the proof of Theorem 1 is left to the reader, as it is analogous to that presented in [3], Theorem 1. Note, that in the same way as in [3], the estimation of the life time T_{ex} of the solution $\boldsymbol{u}$ to (1)–(3) is possible.

We now have the fundamental:

Theorem 2. *Under the assumptions* (A1)–(A5) *there exists unique local solution* $\boldsymbol{u} \in (C^{1+(\gamma/2),2+\gamma})^N$ *of the problem* (1)–(3).

Idea of the proof. From now on any particular equation in (1) will be treated separately (just as in [3], Theorem 2) and we can find the a priori estimates of the Holder norms of $\boldsymbol{u}$ in the following spaces:

(12) $$\forall_{1 \leqq \nu \leqq N} \; u^\nu \in C^{1/2,1/2}([0,T] \times \bar\Omega),$$

and

$$\forall_{\substack{1 \leq \nu \leq N \\ 1 \leq i \leq n}} u^\nu_{x_i} \in C^{\delta/2,\delta}([0, T] \times \bar{\Omega}) \tag{13}$$

with $\delta = \frac{1}{2} s/(s + 1)$ and arbitrary $s \in (0, 1 - (n/p))$. Later, it is a familiar consequence of the classical Schauder type estimates in Holder norms (compare [4] for the Dirichlet boundary condition and [5] for the third boundary condition) and the Leray-Schauder Principle (see e.g. [8]), that the solution of (1)–(3) exists; $u^\nu \in C^{1+(\gamma/2),2+\gamma}([0, T] \times \bar{\Omega})$ with $\gamma = \min\{\alpha; \delta\}$. Uniqueness of this solution follows easily from the Lipschitz continuity of f^ν with respect to the functional arguments.

3. EXAMPLES

We will close our considerations with some examples of systems of the form covered by (1).

Example 1. Consider first a problem studied by A. A. Kiselev and O. A. Ladyzenskaja in [6]:

$$\boldsymbol{v}_t - \nu\Delta\boldsymbol{v} + \sum_{k=1}^{3} v_k \boldsymbol{v}_{x_k} = \boldsymbol{f}(t, x), \tag{14}$$

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega, \qquad \boldsymbol{v}(0, x) = \boldsymbol{a}(x), \qquad x = (x_1, x_2, x_3) \in \Omega \subset R^3,$$

similar in nature to the famous Navier-Stokes system in dimension three. All our assumptions are satisfied, hence local existence of the solution $\boldsymbol{v}$ is justified.

Example 2. The system considered in [10] by F. Rothe:

$$\boldsymbol{u}_t = D\,\Delta\boldsymbol{u} + F(t, x, \boldsymbol{u}), \tag{15}$$

$\boldsymbol{u} = (u^1, \dots, u^N)$, subjected to boundary conditions of the type (3a) or (3b) is given as a special case in (1). Compare also the monograph [9] by the same author, for other special examples taken from various applications.

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