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ON ε-INVARIANT MEASURES AND A FUNCTIONAL EQUATION

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1. In this note we consider ε-invariant measures and apply the technique of functional equations to determine the densities of such measures for the diadic transformation.

Let $(X, \mathfrak{M}, m)$ be a measure space with a nonnegative σ-finite measure m and assume that $T: X \to X$ is a measurable and nonsingular transformation (i.e. T satisfies conditions $T^{-1}(A) \in \mathfrak{M}$ for all $A \in \mathfrak{M}$ and $m(T^{-1}(A)) = 0$ whenever $m(A) = 0$). For an arbittary function $f \in L^1$ ($L^1 = L^1(X, \mathfrak{M}, m)$) the set-function $\nu_f: \mathfrak{M} \to \mathbb{R}$ defined by

$$\nu_f(A) := \int_{T^{-1}(A)} f \,\mathrm{d}m\,, \quad A \in \mathfrak{M}\,,$$

is absolutely continuous with respect to m; denote by $P_T f$ its Radon-Nikodym derivative. The operator $P_T: L^1 \to L^1$ obtained in this way is called the *Frobenius-Perron operator* corresponding to T (cf. [2]). Immediately by this definition we have

$$\int_A P_T f \,\mathrm{d}m = \int_{T^{-1}(A)} f \,\mathrm{d}m$$

for all $f \in L^1$ and $A \in \mathfrak{M}$.

Let μ be a finite measure defined on $\mathfrak{M}$ and assume that $\varepsilon \geqq 0$. We say that μ is ε-invariant under T if

$$|\mu(T^{-1}(A)) - \mu(A)| \leqq \varepsilon\, m(A)$$

for all $A \in \mathfrak{M}$. In the case where $\varepsilon = 0$ this definition coincides with the known definition of measures invariant under T.

The following theorem is an useful tool for finding ε-invariant measures absolutely continuous with respect to the given measure m.

Theorem 1. *Assume that μ is a finite measure absolutely continuous with respect to m. Then μ is ε-invariant under T if and only if its Radon-Nikodym derivative $f = \mathrm{d}\mu/\mathrm{d}m$ satisfies the condition*

$$(1) \qquad |P_T f - f| \leqq \varepsilon$$

m-almost everyhwere on X.

Proof. Assume that the condition (1) is fulfilled and take a set $A \in \mathfrak{M}$. Then

$$|\mu(T^{-1}(A)) - \mu(A)| = |\int_{T^{-1}(A)} f\,dm - \int_A f\,dm| =$$
$$= |\int_A P_T f\,dm - \int_A f\,dm| \leqq \int_A |P_T f - f|\,dm \leqq \varepsilon\, m(A)\,.$$

Conversely, if μ is ε-invariant under T, then

$$|\int_A (P_T f - f)\,dm| = |\mu(T^{-1}(A)) - \mu(A)| \leqq \varepsilon\, m(A)$$

for every $A \in \mathfrak{M}$. Hence $|P_T f - f| \leqq \varepsilon$ (m-a.e.) ::

Remark 1. In the case where $\varepsilon = 0$ the above result reduces to the well known theorem saying that a measure μ (absolutely continuous with respect to m) is invariant under T if and only if its Radon-Nikodym derivative is a fixed point of the Frobenius-Perron operator P_T.

2. In this section we shall apply the above theorem to determine measures ε-invariant under the diadic transformation, i.e. the transformation $\tau\colon [0, 1] \to [0, 1]$ defined by $\tau(x) := 2x (\mathrm{mod}\ 1)$. One can easily compute that in this case the Frobenius-Perron operator P_τ corresponding to τ is given by

$$P_\tau f(x) = \tfrac{1}{2} f(\tfrac{1}{2}x) + \tfrac{1}{2} f(\tfrac{1}{2} + \tfrac{1}{2}x)\,, \quad x \in [0, 1]\,,$$

for all $f \in L^1$ (now $L^1 = L^1([0, 1])$). Therefore the inequality (1) assumes the form

$$(2) \qquad |\tfrac{1}{2} f(\tfrac{1}{2}x) + \tfrac{1}{2} f(\tfrac{1}{2} + \tfrac{1}{2}x) - f(x)| \leqq \varepsilon\,, \quad x \in [0, 1]\,.$$

It follows by theorem 1 that each integrable and positive solution of this inequality is the density of a measure ε-invariant under τ. One can easily check that inequality (2) is satisfied, for example, by functions of the form $f(x) = c + h(x)$, $x \in [0, 1]$, where c is a real constant and $|h(x)| \leqq \frac{1}{2}\varepsilon$, $x \in [0, 1]$. These functions are not, however, unique solutions of (2).

Putting $g(x) := f(x) - \frac{1}{2} f(\frac{1}{2}x) - \frac{1}{2} f(\frac{1}{2} + \frac{1}{2}x)$, $x \in [0, 1]$, we can rewrite inequality (2) as a system of two conditions

$$(3) \qquad f(x) = \tfrac{1}{2} f(\tfrac{1}{2}x) + \tfrac{1}{2} f(\tfrac{1}{2} + \tfrac{1}{2}x) + g(x)\,, \quad x \in [0, 1]$$

and

$$(4) \qquad |g(x)| \leqq \varepsilon\,, \quad x \in [0, 1]\,.$$

Now, every solution of the functional equation (3) with a given function g satisfying (4) is a solution of (2). The following theorem gives some condition under which equation (3) possesses an integrable solution. By P_τ^k we denote the k-th iteration of the operator P_τ; $P_\tau^0 := \mathrm{id}$.

Theorem 2. *Assume that $g \in L^1$. Equation (3) has a solution f belonging to L^1 if and only if the series $\sum_{k=0}^{\infty} P_\tau^k g$ is convergent in L^1. Every integrable solution of (3) is of the form $f = c + \sum_{k=0}^{\infty} P_\tau^k g$, where c is a real constant.*

Proof. Assume that the series $\sum_{k=0}^{\infty} P_\tau^k g$ is convergent in L^1 and c is a real constant.

Then, using the continuity of the operator P_τ and the fact that $P_\tau c = c$, we obtain

$$P_\tau(c + \sum_{k=0}^{\infty} P_\tau^k g) + g = c + \sum_{k=0}^{\infty} P_\tau^{k+1} g + g = c + \sum_{k=0}^{\infty} P_\tau^k g ,$$

which means that the function $f = c + \sum_{k=0}^{\infty} P_\tau^k g$ satisfies (3).

Conversely, assume that a function $f \in L^1$ is a solution of (3). Then, by the linearity of P_τ, we have

$$P_\tau^k f = P_\tau^{k+1} f + P_\tau^k g \quad \text{for all} \quad k = 0, 1, 2, \ldots ,$$

whence

$$\sum_{k=0}^{n} P_\tau^k g = f - P_\tau^{n+1} f , \quad n \in \mathbb{N} .$$

Since the sequence $(P_\tau^n f)_{n \in N}$ tends in L^1 to $\int_0^1 f(x) \, dx$, the series $\sum_{k=0}^{\infty} P_\tau^k g$ is convergent in L^1. Moreover, putting $c := \int_0^1 f(x) \, dx$, we obtain

$$f = c + \sum_{k=0}^{\infty} P_\tau^k g .$$

This finishes the proof ∷

Example 1. Fix a nonnegative constant ε and consider the function $g(x) = (2x - 1) \varepsilon$, $x \in [0, 1]$. It is easy to show by induction that $P_\tau^k g(x) = 2^{-k}(2x - 1) \varepsilon$, $k \in \mathbb{N}$, and so $\sum_{k=0}^{\infty} P_\tau^k g'x) = (4x - 2) \varepsilon$. Therefore functions of the form $f(x) = 4\varepsilon x + c$, where $c \in \mathbb{R}$, are solutions of (3) with g given above. If $c \geqq 0$, then these functions are the densities of measures ε-invariant under the diadic transformation τ. In particular, the function $f(x) = 4\varepsilon x + 1 - 2\varepsilon$ (where $\varepsilon \leqq \frac{1}{2}$) is the density of a normalized measure ε-invariant under τ.

Remark 2. It follows by theorem 2 that constant functions are unique integrable solutions of equation (3) with $g = 0$. Hence we obtain the well known theorem of Rényi saying that the Lebesgue measure is the unique normalized (and absolutely continuous with respect to the Lebesgue measure) measure invariant under the diadic transformation (cf. [3]).

3. In this section we shall give some further information concerning the set of these functions g for which equation (3) possesses an integrable solution.

Consider the sets

$$X := \{g \in L^1 : \sum_{k=0}^{\infty} P_\tau^k g \text{ is convergent in } L^1\}$$

$$Y := \{g \in L^1 : \int_0^1 g(x) \, dx = 0\} .$$

It is clear that X and Y are subspaces of the space L^1. In view of theorem 2 the space X consists of these and only these functions $g \in L^1$ for which equation (3) has an integrable solution.

If $g \in X$, then there exists an $f \in L^1$ such that $f = P_\tau f + g$. Hence $g \in Y$, because

$\int_0^1 P_\tau f(x)\,dx = \int_0^1 f(x)\,dx$. This proves that $X \subset Y$. The following example, due to A. Iwanik from Wrocław, shows that the inclusion is strict.

We will write (f, g) instead of $\int_0^1 f(x)\, g(x)\,dx$.

Example 2. Consider the functions $\chi_n: [0, 1] \to \{-1, 1\}$, $n \in \mathbb{N}$, defined by

$$\chi_n(x) := \begin{cases} -1, & \text{if } x_n = 0, \\ 1, & \text{if } x_n = 1 \end{cases}$$

where $(x_1, x_2, \ldots)$ is the diadic expansion of x. These functions form an ortonormal system in the space L^2 and $(\chi_n, 1) = 0$ for every $n \in \mathbb{N}$. Since the series $\sum_{n=1}^\infty 1/n^2$ is convergent, we infer, by the Riesz-Fischer theorem, that there exists a function $g \in L^2 \subset L^1$ such that $(g, \chi_n) = 1/n$, $n \in \mathbb{N}$, and the series $\sum_{n=1}^\infty (1/n)\,\chi_n$ is convergent to g in L^2. Then, by the continuity of the scalar product, we obtain

$$(g, 1) = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n} (\chi_n, 1) = 0,$$

which means that $g \in Y$.

Now, notice that $\chi_n \circ \tau = \chi_{n+1}$ for all $n \in \mathbb{N}$. Using these equalities and the fact that the operator P_τ^* conjugate to P_τ is given by $P_\tau^* h = h \circ \tau$, $h \in L^\infty$, we obtain

$$(P_\tau^k g, \chi_1) = (g, P_\tau^{*k} \chi_1) = (g, \chi_{k+1}) = \frac{1}{k+1}$$

for every $k \in \mathbb{N}$. Hence

$$\Big(\sum_{k=1}^{N} P_\tau^k g, \chi_1\Big) = \sum_{k=1}^{N} \frac{1}{k+1}, \quad N \in \mathbb{N},$$

which implies that the series $\sum_{k=1}^\infty P_\tau^k g$ is not weakly convergent. Consequently, it is not convergent in L^1, and so $g \notin X$.

Now we can prove the following

Theorem 3. *The space X is dense and of the first category in Y.*

Proof. Consider the operator $F: L^1 \to Y$ defined by $F(f) := f - P_\tau f$, $f \in L^1$. Evidently, F is linear and continuous. Moreover, $F(L^1) = X$. Since $X \neq Y$, we obtain by the open mapping theorem (cf. [4], 2.11) that X is of the first category in Y.

Now we shall show that every polynomial whose integral on $[0, 1]$ is equal to zero belongs to X. Let $f_n(x) := x^n$ and $g_n := f_n - P_\tau f_n$, $n \in \mathbb{N}$. Of course $g_n \in X$ for all $n \in \mathbb{N}$. Moreover, it is easy to notice that g_n is a polynomial of degree n (the coefficient at x^n is equal to $1 - 2^n$). Let $w(x) = a_n x^n + \ldots + a_1 x + a_0$, $x \in [0, 1]$, be an arbitrary polynomial (with real coefficients) such that $\int_0^1 w(x)\,dx = 0$. We can choose numbers $b_1, \ldots, b_n \in \mathbb{R}$ in such a way that the polynomial $b_n g_n + \ldots + b_1 g_1$ has the same coefficients at x^i, $i = 1, \ldots, n$, as the polynomial w. Then also the coefficients at x^0 must be the same, because

$$\int_0^1 w(x)\,dx = \int_0^1 (b_n\, g_n(x) + \ldots + b_1\, g_1(x))\,dx \quad (= 0).$$

Therefore $w = b_n g_n + \dots + b_1 g_1$, and so $w \in X$ as a linear combination if elements of X. Since the set of all polynomials is dense in L^1, we infer that the set of all polynomials whose integrals on $[0, 1]$ are equal to zero is dense in Y. Thus X is dense in Y ∴

Remark 3. Using a measure-theoretical version of the open mapping theorem due to K. Baron (cf. [1], Lemma), we can prove that X is a Borel subset of Y and there exists a probability measure ν on the family of all Borel subsets of Y such that $\nu(X + y) = 0$ for every $y \in Y$ (i.e. X is a Haar zero set). So, X is small in Y also from measure-theoretical point of view.

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