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LATTICE ORDERED GROUPS WITH UNIQUE ADDITION MUST BE ARCHIMEDEAN

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In the present paper we will give an affirmative solution to a question proposed in a recent paper by Conrad and Darnel.

The linearly ordered groups with unique addition were identified by Ohkuma in [4]. Lattice ordered groups with unique addition were studied by Conrad and Darnel [1] and by the author [2]. The analogous notion for cyclically ordered groups was investigated in [3].

In [1], p. 19 it is remarked that every example thus far of a lattice ordered group with unique addition is an archimedean lattice ordered group, and that it is an open question whether or not a lattice ordered group with unique addition must be archimedean. The same question was proposed in [5], p. 266 (Problem 16).

It will be shown below that the answer to this question is 'YES'.

1. PRELIMINARIES

We recall the following definition (cf. [1], [2]):

A lattice ordered group $G_1 = (G; \leqq, +_1)$ is said to have a *unique addition* if, whenever $G_2 = (G; \leqq, +_2)$ is a lattice ordered group such that the neutral element of the group $(G; +_1)$ is the same as the neutral element of the group $(G; +_2)$, then the operation $+_1$ coincides with the operation $+_2$.

1.1. Lemma. (Cf. [1], Corollary 2.) *If $G_1 = (G; \leqq, +_1)$ is a lattice ordered group with unique addition, then each automorphism τ of the lattice $(G; \leqq)$ with $0\tau = 0$ is a group automorphism.*

The following assertion is easy to verify (cf. also [1]).

1.2. Lemma. *Let $G_1 = (G; \leqq, +_1)$ be a lattice ordered group with unique addition. Then the group $(G; +_1)$ is abelian.*

Let $G_1 = (G; \leqq, +)$ be a lattice ordered group. As usual, we put

$$G^+ = \{g \in G: g \geqq 0\}.$$

In this section some auxiliary results on the automorphisms of the lattices $(G; \leqq)$ and $(G^+; \leqq)$ will be established.

If $x, y \in G$ and $x \wedge y = 0$, then the elements x and y are called disjoint.

The following lemma can be verified by a routine calculation.

2.1. Lemma. *Let x and y be disjoint elements of G. Then the element $x - y$ is the (uniquely determined) relative complement of the element 0 in the interval $[-y, x]$.*

2.2. Lemma. *Let φ be an automorphism of the lattice $(G^+; \leqq)$. For each $x \in G$ put $\psi(x) = \varphi(x \vee 0) - \varphi(-(x \wedge 0))$. Then ψ is an automorphism of the lattice $(G; \leqq)$ with $\psi(0) = 0$.*

Proof. Since $\varphi(0) = 0$, the relation $\psi(0) = 0$ obviously holds.

Let $x, y \in G$, $x \leqq y$. Then we have $x \vee 0 \leqq y \vee 0$, whence $\varphi(x \vee 0) \leqq \varphi(y \vee 0)$. Analogously we obtain that $-\varphi(-(x \wedge 0)) \leqq -\varphi(-(y \wedge 0))$. Thus $\psi(x) \leqq \psi(y)$. Hence ψ is isotone.

Let $t_1, t_2 \in G$, $\psi(t_1) \leqq \psi(t_2)$. It is well-known that the elements $t_1 \vee 0$ and $-(t_1 \wedge 0)$ are disjoint. Since φ is an automoprhism of $(G^+, \leqq)$, the elements $\varphi(t_1 \vee 0)$ and $\varphi(-(t_1 \wedge 0))$ are disjoint as well. Hence in view of 2.1, the element $\psi(t_1)$ is a relative complement of the element 0 in the interval $[-\varphi(-(t_1 \wedge 0)), \varphi(t_1 \vee 0)]$. An analogous assertion holds for $\psi(t_2)$. Hence

$$\psi(t_1) \vee 0 = \varphi(t_1 \vee 0), \quad \psi(t_1) \wedge 0 = -\varphi(-(t_1 \wedge 0)),$$
$$\psi(t_2) \vee 0 = \varphi(t_2 \vee 0), \quad \psi(t_2) \wedge 0 = -\varphi(-(t_2 \wedge 0)).$$

Since $\psi(t_1) \leqq \psi(t_2)$ we obtain that the relations

$$\varphi(t_1 \vee 0) \leqq \varphi(t_2 \vee 0), \quad -\varphi(-(t_1 \wedge 0)) \leqq -\varphi(-(t_2 \wedge 0))$$

are valid. Thus $t_1 \vee 0 \leqq t_2 \vee 0$ and $t_1 \wedge 0 \leqq t_2 \wedge 0$. Therefore $t_1 \leqq t_2$. In particular, if $\psi(t_1) = \psi(t_2)$, then $t_1 = t_2$ and hence ψ is a monomorphism.

Let $x \in G$. Denote $\varphi^{-1}(x \vee 0) = x_1$, $\varphi^{-1}(-(x \wedge 0)) = x_2$. The elements x_1 and x_2 are disjoint. Put $t = x_1 - x_2$. Then in view of 2.1 we have $t \vee 0 = x_1$ and $t \wedge 0 = -x_2$, whence

$$\psi(t) = \varphi(x_1) - \varphi(x_2) = (x \vee 0) + (x \wedge 0) = x.$$

Thus ψ is an epimorphism.

In the remaining part of this section we assume that $G_1 = (G; \leqq, +_1)$ is an abelian lattice ordered group which fails to be archimedean. Thus there are elements a and b_1 in G such that

$$0 < na < b_1$$

is valid for each positive integer n; this situation will be denoted by writing $a \ll b_1$. Let an element a with the described property be fixed. Denote

$$B_0 = \{b \in G: a \ll b\}.$$

Next, for each positive integer n we define by induction a subset B_n of G^+ as follows:

B_n is the set of all elements $y \in G^+$ having the property that there exists $z \in B_{n-1}$ with $z < y \vee z \leqq z + a$.

Put $B = \bigcup B_n$ $(n = 0, 1, 2, \ldots)$. From the definition of B we immediately obtain:

2.3. Lemma. *$B \neq \emptyset$, and the element 0 does not belong to B.*

Let us consider the mapping φ: $G^+ \to G^+$ which is defined by

$$\varphi(x) = x + a \quad \textit{if} \quad x \in B, \quad \textit{and}$$

$$\varphi(x) = x \quad \textit{otherwise}.$$

2.4. Lemma. *Let $n \in \{0, 1, 2, \ldots\}$, $y \in B_n$, $y_1 \in G$, $y < y_1$. Then $y_1 \in B_n$.*

Proof. We proceed by induction on n. The assertion obviously holds for $n = 0$. Let $n > 0$ and suppose that the assertion is valid for B_{n-1}. Let z be as in the definition of B_n. Put

$$z' = z + (y_1 - y).$$

Then in view of the induction assumption we have $z' \in B_{n-1}$. Next,

$$z' < y_1 \vee z' \leqq z' + a,$$

whence $y_1 \in B_n$.

2.5. Corollary. *Let $y \in B$, $y_1 \in G$, $y < y_1$. Then $y_1 \in B$.*

2.6. Lemma. *The mapping φ is isotone.*

Proof. Let $y_1, y_2 \in G^+$, $y_1 \leqq y_2$. If either both y_1 and y_2 belong to B or both y_1 and y_2 belong to $G^+ \setminus B$, then clearly $\varphi(y_1) \leqq \varphi(y_2)$. If $y_1 \in B$, then in view of 2.5 the relation $y_2 \in B$ is valid. Thus it suffices to consider the case $y_1 \notin B$ and $y_2 \in B$; hence

$$\varphi(y_1) = y_1 < y_2 < y_2 + a = \varphi(y_2).$$

2.7. Lemma. *Let $n \in \{0, 1, 2, \ldots\}$, $y \in B_n$. Then $y - a \in B_n$.*

Proof. The assertion obviously holds for $n = 0$. Let $n > 0$ and assume that the assertion is valid for B_{n-1}. Let z be as in the definition of B_n. Put $z' = z - a$ and $y' = y - a$. Then $z' \in B_{n-1}$ and

$$z' < y' \vee z' \leqq z' + a,$$

whence $y' \in B_n$.

2.8. Corollary. *Let $y \in B$. Then $y - a \in B$.*

2.9. Lemma. *The mapping φ is a surjective monomorphism.*

Proof. Let $y \in G^+$. If $y \notin B$, then $\varphi(y) = y$. Next, suppose that y belongs to B. Then in view of 2.8 the relation $y - a \in B$ is valid. Thus $\varphi(y - a) = y$. Hence φ is surjective.

Let $y_1, y_2 \in G^+$, $y_1 \neq y_2$. If either $y_1, y_2 \in B$ or $y_1, y_2 \in G^+ \setminus B$, then we obviously have $\varphi(y_1) \neq \varphi(y_2)$. Assume that $y_1 \in B$ and $y_2 \notin B$. Then in view of 2.5, $\varphi(y_1) = y_1 + a \in B$. Next, $\varphi(y_2) = y_2 \notin B$. Thus $\varphi(y_1) \neq \varphi(y_2)$.

2.10. Lemma. *Let $y_1, y_2 \in G^+$, $\varphi(y_1) < \varphi(y_2)$. Then $y_1 < y_2$.*

Proof. If either $y_1, y_2 \in B$ or $y_1, y_2 \in G^+ \setminus B$, then $y_1 < y_2$. Next, 2.5 and 2.7 yield that for each $t \in G^+$ the relation

$$t \in B \Leftrightarrow \varphi(t) \in B$$

is valid. Hence in view of 2.5 it suffices to consider the case $y_1 \notin B$ and $y_2 \in B$. Thus there exists $n \in \{0, 1, 2, \dots\}$ such that $y_2 \in B_n$ and

$$\varphi(y_1) = y_1 , \quad \varphi(y_2) = y_2 + a .$$

Therefore $y_1 < y_2 + a$. Clearly $y_1 \neq y_2$.

Assume that $y_1 \nless y_2$. Then

$$y_2 < y_1 \vee y_2 \leqq y_2 + a .$$

Therefore $y_1 \in B_{n+1} \subseteq B$, which is a contradiction.

2.11. Lemma. *The mapping φ is an automorphism of the lattice $(G^+; \leqq)$.*

Proof. This is a consequence of 2.6, 2.9 and 2.10.

3. UNIQUE ADDITION

3.1. Proposition. *Let $G_1 = (G; \leqq, +)$ be an abelian lattice ordered group. Assume that G_1 fails to be archimedean. Then there is an automorphism ψ of the lattice $(G; \leqq)$ with $\psi(0) = 0$ such that ψ is not a group automorphism.*

Proof. Let φ be as in Section 2. By means of φ we construct the automorphism ψ as in 2.2. In view of this construction,

$$\psi(x) = \varphi(x) \quad \text{for each} \quad x \in G^+ .$$

Next, let the element $a \in G$ be as in Section 2. In view of 2.3, there exists $b \in B$. Then according to 2.5 we have $2b \in B$ and thus

$$\psi(2b) = 2b + a \neq 2b + 2a = 2\,\psi(b) ;$$

hence ψ is not a group automorphism.

3.2. Theorem. *Let $G_1 = (G; \leqq, +)$ be a lattice ordered group with unique addition. Then G_1 is archimedean.*

Proof. According to 1.2, G_1 is abelian. By way of contradiction, assume that G_1 fails to be archimedean. Let ψ be as in 3.1. By applying 1.1 we arrive at a contradiction.

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