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ON THE EXISTENCE OF GENERALIZED SOLUTIONS OF NONLINEAR FIRST ORDER PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS IN TWO INDEPENDENT VARIABLES

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1. INTRODUCTION

Let $a_0 > 0$, $B = [-b_0, 0] \times [-b_1, b_1]$, where $b_0, b_1 \in R_+$, $R_+ = [0, +\infty)$. For any function $z: [-b_0, a_0] \times R \to R$ and for a fixed $(x, y) \in [0, a_0] \times R$ we define the function $z_{(x,y)}: B \to R$ by $z_{(x,y)}(t, s) = z(x + t, y + s)$, $(t, s) \in B$. For any metric spaces X, Y by $C(X, Y)$ we denote the set of all continuous functions defined on X and taking values in Y.

Suppose that $f: [0, a_0] \times R \times R \times C(B, R) \times R \to R$, $\varphi: [-b_0, 0] \times R \to R$ and let us consider the following Cauchy problem for the nonlinear differential-functional equation of the first order

(1) $$D_x z(x, y) = f(x, y, z(x, y), z_{(x,y)}, D_y z(x, y)),$$

(2) $$z(x, y) = \varphi(x, y), \quad (x, y) \in [-b_0, 0] \times R.$$

A function $u \in C([-b_0, a_0] \times R, R)$ is a generalized solution of (1), (2) if

(i) u satisfies the Lipschitz condition on $[0, a_0] \times R$,

(ii) there exists a function $\lambda \in C((0, a_0], R_+)$ such that $l^{-2}[u(x, y + l) - 2u(x, y) + u(x, y - l)] \leqq \lambda(x)$, for $(x, y) \in (0, a_0] \times R$, $l \in R$, $l \neq 0$,

(iii) u satisfies (1) a.e. ("almost everywhere") on $[0, a_0] \times R$ and the initial condition (2) for all $(x, y) \in [-b_0, 0] \times R$.

Remark 1. If we omit the condition (ii) in the above definition, then the solution is not unique. An adequate example for f without a functional argument is given in [16].

Generalized solutions of nonlinear first order partial differential equations have been investigated in a large number of papers by various authors. Theorems of existence, uniqueness and continuous dependence upon Cauchy or boundary data for quasilinear systems have been given by L. Cesari [7], [8], P. Bassanini [1]–[3] and P. Pucci [18]. Quasilinear differential-integral systems and systems with a retarded argument are considered in [4], [13], [14]. Nonlinear differential equations have been studied by M. Cinquini-Cibrario, S. Cinquini [9]. Generalized solutions of quasilinear and nonlinear equations with operators of the Volterra type are investigated in [20]–[22]. Additional bibliographical information may be found in [19].

Generalized solutions of nonlinear equations are also investigated in the case when assumptions for given functions are sufficient for existence of classical solutions (of class C^1). For classical solutions we can prove only a local existence and therefore to obtain theorems of a global existence we need generalized solutions. Theorems of this type have been given by S. N. Kruzhkov [16] and for equations with a retarded argument by Z. Kamont, S. Zacharek [15].

Classical solutions of nonlinear differential-functional equations or equations with a retarded argument are discussed in [5], [6], [10]–[12].

In this paper we prove the global existence of generalized solutions of (1), (2) extending the results of paper [16]. The proof is based on the difference method (see also [17]).

2. ASSUMPTIONS AND DEFINITIONS

We denote by $C_{0+L}(B, R)$ a set of all continuous functions from B to R which satisfy the Lipschitz condition on B. Furthermore, for any $t \in R_+$ let $C_{0+L}(B, R, t) = \{u \in C_{0+L}(B, R): \|u\|_{0+L} = \|u\|_0 + \|u\|_L \leqq t\}$, where

$$\|u\|_0 = \sup \{|u(s, v)|: (s, v) \in B\},$$

$$\|u\|_L = \sup \{[|s - \bar{s}| + |v - \bar{v}|]^{-1} |u(s, v) - u(\bar{s}, \bar{v})|: (s, v), (\bar{s}, \bar{v}) \in B\}.$$

Suppose that the function $f: [0, a_0] \times R \times R \times C(B, R) \times R \to R$ of the variables (x, y, p, w, q) is of class C^2. By $D_y f$, $D_p f$, $D_q f$, $D^2_{yy} f$, $D^2_{yq} f$, $D^2_{yp} f$, $D^2_{pp} f$, $D^2_{qp} f$, $D^2_{qq} f$ we denote first or second order partial derivatives of f. $D_w f$ is the Frechet derivative of f i.e. $D_w f(x, y, p, w, q) \in \mathscr{L}(C(B, R), R)$, where $\mathscr{L}(X, Y)$ denotes a set of all linear operators from X to Y. Symbols $D^2_{yw} f$, $D^2_{pw} f$, $D^2_{qw} f$ have the same meaning as $D_w f$ while $D^2_{ww} f$ denotes the second order Frechet derivative i.e. $D^2_{ww} f(x, y, p, w, q) \in \mathscr{L}(C(B, R), \mathscr{L}(C(B, R), R))$.

Assumption H. *Suppose that*

1° $\varphi \in C([-b_0, 0] \times R, R)$ *and there are constants* $\tilde{M}, \tilde{L} \in R_+$ *such that for all* $(x, y), (x, \bar{y}) \in [-b_0, 0] \times R$ *we have*

$$|\varphi(x, y)| \leqq \tilde{M}, \quad |\varphi(x, y) - \varphi(x, \bar{y})| \leqq \tilde{L}|y - \bar{y}|;$$

2° *if* $b_0 > 0$, *then there is a constant* $\tilde{K} \in R_+$ *such that for all* $(x, y) \in [-b_0, 0] \times R$, $l \in R$, $l \neq 0$ *we have* $l^{-2}[\varphi(x, y + l) - 2\varphi(x, y) + \varphi(x, y - l)] \leqq \tilde{K}$;

3° $f: [0, a_0] \times R \times R \times C(B, R) \times R \to R$ *is of class* C^2;

4° *there are a constant* $N \geqq \tilde{M}$ *and a nondecreasing function* $V \in C([\tilde{M}, N], R_+)$, $\int_{\tilde{M}}^N \mathrm{d}t/V(t) \geqq a_0$, *such that for all* $t \in R_+$, $(x, y, p, w, q) \in [0, a_0] \times R \times [-t, t] \times C(B, R, t) \times R$ *we have* $|f(x, y, p, w, q)| \leqq V(t)$;

5° *there are constants* $N_1 \geqq L$, $A > 0$ *and a nondecreasing function* $W \in C([\tilde{L}.N_1], R_+)$, $\int_{\tilde{L}}^{N_1} dt/[(2t+1)W(N+3t)] \geqq a_0$ *such that for all* $t \in R_+$, $(x, y, p, w, q) \in [0, a_0] \times R \times [-N, N] \times C_{0+L}(B, R, t) \times R$ *we have*

$$|D_q f(x, y, p, w, q)| \leqq A, \qquad |D_y f(x, y, p, w, q)| \leqq W(t),$$

$$|D_p f(x, y, p, w, q)| \leqq W(t), \quad \|D_w f(x, y, p, w, q)\| \leqq W(t);$$

6° *for all* $(x, y, p, w, q) \in [0, a_0] \times R \times [-N, N] \times C_{0+L}(B, R, N + 3N_1) \times [-N_1, N_1]$, $\bar{w} \in C(B, R)$, $\bar{w} \geqq 0$, *we have* $D_w f(x, y, p, w, q)(\bar{w}) \geqq 0$;

7° *the derivatives* $D^2_{yy}f$, $D^2_{yp}f$, $D^2_{yw}f$, $D^2_{yq}f$, $D^2_{pp}f$, $D^2_{pw}f$, $D^2_{pq}f$, $D^2_{ww}f$, $D^2_{wq}f$ *are bounded and* $D^2_{qq}f \leqq 0$ *on* $[0, a_0] \times R \times [-N, N] \times C_{0+L}(B, R, N + 3N_1) \times [-N_1, N_1]$;

8° *if* $b_0 = 0$, *then there are constants* $\delta \in (0, a_0]$, $\mu > 0$ *such that* $D^2_{qq}f \leqq -\mu$ *on* $[0, \delta] \times R \times [-N, N] \times C_{0+L}(B, R, N + 3N_1) \times [-N_1, N_1]$.

Let Z be a set of all integers, and let $2Z$ be a set of all even numbers. Now we introduce a difference scheme for (1), (2). For $h, k > 0$ we define $x^{(i)} = ih$, $i = 0, 1, \ldots, n_0$, $n_0 h = a_0$ and $y^{(j)} = jk$, $j \in Z$. If $b_0 > 0$, then there is an integer $n_1 > 0$ such that $-n_1 h \leqq -b_0 < (-n_1 + 1)h$. We define $x^{(i)} = ih$, $i = -n_1 + 1, \ldots, -1$, and $x^{(-n_1)} = -b_0$.

Let $U = \{(h, k)\colon A < k/h\}$, $E^* = \{(x^{(i)}, y^{(j)})\colon i = 0, \ldots, n_0,\ j \in Z\}$. For $i = 0, \ldots, n_0 - 1$, $j \in 2Z$ we write $P_{ij} = [x^{(i)}, x^{(i+1)}] \times [y^{(j-1)}, y^{(j+1)}]$, $Q_i = \bigcup_{j \in 2Z} P_{ij}$. If $v\colon E^* \to R$, then we denote $v^{(i,j)} = v(x^{(i)}, y^{(j)})$, $i = 0, \ldots, n_0$, $j \in Z$. Let Δ_0, Δ_1 be operators defined by

$$\Delta_0 v^{(i,j)} = \frac{1}{h}\left[v^{(i+1,j)} - v^{(i,j)}\right], \quad \Delta_1 v^{(i,j)} = \frac{1}{2k}\left[v^{(i,j+1)} - v^{(i,j-1)}\right].$$

Furthermore, let $\Delta^2_{i'j'} v^{(i,j)} = \Delta_{i'}(\Delta_{j'} v^{(i,j)})$, $i', j' = 0, 1$.

Let $\varphi_{hk}\colon [-b_0, 0] \times R \to R$ be a function defined in the following way:

(i) If $b_0 > 0$, then for each $(x, y) \in [-b_0, 0] \times R$ there are i, $-n_1 \leqq i < 0$ and $j \in 2Z$ such that $(x, y) \in [x^{(i)}, x^{(i+1)}] \times [y^{(j-1)}, y^{(j+1)}]$. Then we write $\varphi_{hk}(x, y) = \varphi(x^{(i)}, y^{(j-1)}) + (x - x^{(i)})\Delta_0 \varphi(x^{(i)}, y^{(j-1)}) + (y - y^{(j-1)})\Delta_1 \varphi(x^{(i)}, y^{(j)}) + (x - x^{(i)})(y - y^{(j-1)})\Delta^2_{01}\varphi(x^{(i)}, y^{(j)})$.

(ii) If $b_0 = 0$, then for each $y \in R$ there is $j \in 2Z$ such that $y \in [y^{(j-1)}, y^{(j+1)}]$. Then we write $\varphi_{hk}(y) = \varphi(y^{(j-1)}) + (y - y^{(j-1)})\Delta_1 \varphi(y^{(j)})$.

For any $(h, k) \in U$ let us define the function $u_{hk}\colon [-b_0, a_0] \times R \to R$. We use the mathematical induction in the following way:

(i) Let $v^{(0,j)} = \varphi(0, y^{(j)})$, $j \in Z$ and $u_{hk}(x, y) = \varphi_{hk}(x, y)$ for $(x, y) \in [-b_0, 0] \times R$.

(ii) If for some i, $0 \leqq i \leqq n_0 - 1$ we have defined $v^{(i,j)}$, $j \in Z$ and u_{hk} on $([-b_0, 0] \times R) \cup Q_0 \cup \ldots \cup Q_{i-1}$, then

(3) $$v^{(i+1,j)} = \tfrac{1}{2}(v^{(i,j+1)} + v^{(i,j-1)}) + hf(x^{(i)}, y^{(j)}, \tfrac{1}{2}(v^{(i,j+1)} + v^{(i,j-1)}), (u_{hk})_{(x^{(i)}, y^{(j)})}, \Delta_1 v^{(i,j)}),$$ where $j \in Z$,

(4) $$u_{hk}(x, y) = v^{(i,j-1)} + (x - x^{(i)})\,\Delta_0 v^{(i,j-1)} + (y - y^{(j-1)})\,\Delta_1 v^{(i,j)} + \\ + (x - x^{(i)})(y - y^{(j-1)})\,\Delta_{01}^2 v^{(i,j)}, \quad \text{where} \quad (x, y) \in P_{ij}, \quad j \in 2Z.$$

It is easy to see that (4) defines a continuous function on $[-b_0, a_0] \times R$. In the sequel we will write $(u_{hk})_{(i,j)}$ instead of $(u_{hk})_{(x^{(i)},y^{(j)})}$.

3. PROPERTIES OF A SOLUTION OF THE DIFFERENCE EQUATION

Lemma 1. *If $f \in C([0, a_0] \times R \times R \times C(B, R) \times R, R)$, and conditions* 1°, 4° *of Assumption* H *are satisfied, then for all $i = 0, \ldots, n_0$, $j \in Z$ we have*

(5) $$|v^{(i,j)}| \leqq N.$$

Proof. It follows from (3) that for $i = 0, \ldots, n_0 - 1$, $j \in Z$ we have

$$|v^{(i+1,j)}| \leqq \tfrac{1}{2}|v^{(i,j+1)} + v^{(i,j-1)}| + \\ + h|f(x^{(i)}, y^{(j)}, \tfrac{1}{2}(v^{(i,j+1)} + v^{(i,j-1)}), (u_{hk})_{(i,j)}, \Delta_1 v^{(i,j)})|.$$

Let $\bar{v}^{(i)} = \sup\{|v^{(\tau,j)}|: -n_1 \leqq \tau \leqq i,\ j \in Z\}$. The boundness of φ implies that $\bar{v}^{(i)} < +\infty$. From the condition 4° of Assumption H we have $\bar{v}^{(i+1)} \leqq \bar{v}^{(i)} + hV(\bar{v}^{(i)})$, and hence

(6) $$\frac{1}{h}[\bar{v}^{(i+1)} - \bar{v}^{(i)}] \leqq V(\bar{v}^{(i)}), \quad i = 0, \ldots, n_0 - 1.$$

Let us consider the Cauchy problem

(7) $$D_x w(x) = V(w(x)), \quad w(0) = \tilde{M}.$$

If w is a solution of (7), then it is a nondecreasing function and hence $D_x w$ is a composition of two nondecreasing functions. Thus w is a convex function and from (6) we see that $\bar{v}^{(i)} \leqq w(x^{(i)})$, $i = 0, \ldots, n_0$. We also have that w satisfies

$$\int_{\tilde{M}}^{w(x)} \frac{dt}{V(t)} = x.$$

From the condition 4° of Assumption H we have then $w(x) \leqq N$, $x \in [0, a_0]$. Therefore we have $\bar{v}^{(i)} \leqq N$, $i = 0, \ldots, n_0$, which is equivalent to (5).

Lemma 2. *If $f: [0, a_0] \times R \times R \times C(B, R) \times R \to R$ is of class C^1, $(h, k) \in U$ and conditions* 1°, 4°, 5° *of Assumption* H *are satisfied, then for all $i = 0, \ldots, n_0$, $j \in Z$ we have*

(8) $$|\Delta_1 v^{(i,j)}| \leqq N_1.$$

Proof. It follows from (3) that for $i = 0, \ldots, n_0 - 1$, $j \in Z$ we have

$$\Delta_1 v^{(i+1,j)} = \tfrac{1}{2}(\Delta_1 v^{(i,j+1)} + \Delta_1 v^{(i,j-1)}) + \\ + \frac{h}{2k}[f(x^{(i)}, y^{(j+1)}, \tfrac{1}{2}(v^{(i,j+2)} + v^{(i,j)}), (u_{hk})_{(i,j+1)}, \Delta_1 v^{(i,j+1)}) - \\ - f(x^{(i)}, y^{(j-1)}, \tfrac{1}{2}(v^{(i,j)} + v^{(i,j-2)}), (u_{hk})_{(i,j-1)}, \Delta_1 v^{(i,j-1)})].$$

Using the Lagrange theorem we obtain

$$\Delta_1 v^{(i+1,j)} = \Delta_1 v^{(i,j+1)} \left[\frac{1}{2} + \frac{h}{2k} D_q f(P^{(i,j)})\right] +$$
$$+ \Delta_1 v^{(i,j-1)} \left[\frac{1}{2} - \frac{h}{2k} D_q f(P^{(i,j)})\right] +$$
$$+ h D_y f(P^{(i,j)}) + \frac{h}{2} (\Delta_1 v^{(i,j+1)} + \Delta_1 v^{(i,j-1)}) D_p f(P^{(i,j)}) +$$
$$+ h D_w f(P^{(i,j)}) (r_{i,j}),$$

where $P^{(i,j)}$ is an intermediate point, and $r_{i,j}$ is defined by

$$r_{i,j} = \frac{1}{2k} [(u_{hk})_{(i,j+1)} - (u_{hk})_{(i,j-1)}].$$

Let $z^{(i)} = \sup \{|\Delta_1 v^{(\tau,j)}|: -n_1 \leqq \tau \leqq i, j \in Z\}$. From (5) it follows that $z^{(i)} < +\infty$. Since $|D_q f(P^{(i,j)})| \leqq A$ and $(h, k) \in U$, we have

$$\left|\Delta_1 v^{(i,j+1)} \left[\frac{1}{2} + \frac{h}{2k} D_q f(P^{(i,j)})\right] + \Delta_1 v^{(i,j-1)} \left[\frac{1}{2} - \frac{h}{2k} D_q f(P^{(i,j)})\right]\right| \leqq z^{(i)}.$$

Thus the above inequality, (5), the condition 5° of Assumption H and $\|r_{i,j}\|_0 \leqq z^{(i)}$ yield

$$z^{(i+1)} \leqq (1 + 2hW(N + 3z^{(i)})) z^{(i)} + hW(N + 3z^{(i)}),$$

and hence

$$\frac{1}{h} [z^{(i+1)} - z^{(i)}] \leqq (2z^{(i)} + 1) W(N + 3z^{(i)}), \quad i = 0, \ldots, n_0 - 1.$$

Taking into consideration the Cauchy problem

$$D_x w(x) = (2w(x) + 1) W(N + 3 w(x)), \quad w(0) = \tilde{L},$$

and using the same arguments as in the proof of Lemma 1 we obtain (8).

Lemma 3. *Suppose that Assumption* H *is satisfied and that* $(h, k) \in U$. *Then for sufficiently small* k, h *there is a constant* $N_2 \in R_+$ *such that*

(9) $$\Delta_{11}^2 v^{(i,j)} \leqq \frac{N_2}{x^{(i)}}, \quad \text{for} \quad b_0 = 0, \quad i = 1, \ldots, n_0, \quad j \in Z,$$

(10) $$\Delta_{11}^2 v^{(i,j)} \leqq N_2, \quad \text{for} \quad b_0 > 0, \quad i = 0, \ldots, n_0, \quad j \in Z.$$

Proof. We will first prove (9). From (3) it follows that for $i = 0, \ldots, n_0 - 1$, $j \in Z$ we have

$$\Delta_{11}^2 v^{(i+1,j)} = \tfrac{1}{2}(\Delta_{11}^2 v^{(i,j+1)} + \Delta_{11}^2 v^{(i,j-1)}) +$$
$$+ \frac{h}{(2k)^2} [f(x^{(i)}, y^{(j+2)}, \tfrac{1}{2}(v^{(i,j+3)} + v^{(i,j+1)}), (u_{hk})_{(i,j+2)}, \Delta_1 v^{(i,j+2)}) +$$
$$- 2f(x^{(i)}, y^{(j)}, \tfrac{1}{2}(v^{(i,j+1)} + v^{(i,j-1)}), (u_{hk})_{(i,j)}, \Delta_1 v^{(i,j)}) +$$
$$+ f(x^{(i)}, y^{(j-2)}, \tfrac{1}{2}(v^{(i,j-1)} + v^{(i,j-3)}), (u_{hk})_{(i,j-2)}, \Delta_1 v^{(i,j-2)})].$$

Let $r_{i,j}$ have the same meaning as in the proof of Lemma 2 and let

$$q_{i,j} = \frac{1}{(2k)^2}\left[(u_{hk})_{(i,j+2)} - 2(u_{hk})_{(i,j)} + (u_{hk})_{(i,j-2)}\right],$$

$$Q^{(i,j)} = \left(x^{(i)}, y^{(j)}, \tfrac{1}{2}\left(v^{(i,j+1)} + v^{(i,j-1)}\right), (u_{hk})_{(i,j)}, \Delta_1 v^{(i,j)}\right).$$

From the relations

$$v^{(i,j'+1)} + v^{(i,j'-1)} - \left(v^{(i,j'-1)} + v^{(i,j'-3)}\right) = 2k\left(\Delta_1 v^{(i,j')} + \Delta_1 v^{(i,j'-2)}\right),$$

$$\Delta_1 v^{(i,j'+1)} - \Delta_1 v^{(i,j'-1)} = 2k\, \Delta_{11}^2 v^{(i,j')},$$

and from Taylor's formula we obtain

$$\begin{aligned}
\Delta_{11}^2 v^{(i+1,j)} &= \Delta_{11}^2 v^{(i,j+1)}\Bigg[\frac{1}{2} + \frac{h}{2k}\, D_q f(Q^{(i,j)}) + h D_{yq}^2 f(Q_1^{(i,j)}) + \\
&+ \frac{h}{2}\left(\Delta_1 v^{(i,j+2)} + \Delta_1 v^{(i,j)}\right) D_{qp}^2 f(Q_1^{(i,j)}) + h D_{qw}^2 f(Q_1^{(i,j)})\left(r_{i,j+1}\right)\Bigg] + \\
&+ \Delta_{11}^2 v^{(i,j-1)}\Bigg[\frac{1}{2} - \frac{h}{2k}\, D_q f(Q^{(i,j)}) + h D_{yq}^2 f(Q_2^{(i,j)}) + \\
&+ \frac{h}{2}\left(\Delta_1 v^{(i,j)} + \Delta_1 v^{(i,j-2)}\right) D_{qp}^2 f(Q_2^{(i,j)}) + h D_{qw}^2 f(Q_2^{(i,j)})\left(r_{i,j-1}\right)\Bigg] + \\
&+ \frac{h}{2}\left(\Delta_{11}^2 v^{(i,j+1)} + \Delta_{11}^2 v^{(i,j-1)}\right) D_p f(Q^{(i,j)}) + h D_w f(Q^{(i,j)})\left(q_{i,j}\right) + R^{(i,j)},
\end{aligned}$$

where $Q_1^{(i,j)}$, $Q_2^{(i,j)}$ are intermediate points and

$$\begin{aligned}
R^{(i,j)} &= \frac{h}{2}\left[D_{yy}^2 f(Q_1^{(i,j)}) + D_{yy}^2 f(Q_2^{(i,j)})\right] + \\
&+ \frac{h}{2}\left[\tfrac{1}{4}\left(\Delta_1 v^{(i,j+2)} + \Delta_1 v^{(i,j)}\right)^2 D_{pp}^2 f(Q_1^{(i,j)}) + \right. \\
&+ \left.\left(\Delta_1 v^{(i,j+2)} + \Delta_1 v^{(i,j)}\right) D_{yp}^2 f(Q_1^{(i,j)})\right] + \\
&+ \frac{h}{2}\left[\tfrac{1}{4}\left(\Delta_1 v^{(i,j)} + \Delta_1 v^{(i,j-2)}\right)^2 D_{pp}^2 f(Q_2^{(i,j)}) + \right. \\
&+ \left.\left(\Delta_1 v^{(i,j)} + \Delta_1 v^{(i,j-2)}\right) D_{yp}^2 f(Q_2^{(i,j)})\right] + \\
&+ h\left[\tfrac{1}{2} D_{ww}^2 f(Q_1^{(i,j)})\left(r_{i,j+1}\right) + D_{yw}^2 f(Q_1^{(i,j)}) + \right. \\
&+ \left.\tfrac{1}{2}\left(\Delta_1 v^{(i,j)} + \Delta_1 v^{(i,j+2)}\right) D_{pw}^2 f(Q_1^{(i,j)})\right]\left(r_{i,j+1}\right) + \\
&+ h\left[\tfrac{1}{2} D_{ww}^2 f(Q_2^{(i,j)})\left(r_{i,j-1}\right) + D_{yw}^2 f(Q_2^{(i,j)}) + \right. \\
&+ \left.\tfrac{1}{2}\left(\Delta_1 v^{(i,j)} + \Delta_1 v^{(i,j-2)}\right) D_{pw}^2 f(Q_2^{(i,j)})\right]\left(r_{i,j-1}\right) + \\
&+ \frac{h}{2}\left[\Delta_{11}^2 v^{(i,j+1)}\right]^2 D_{qq}^2 f(Q_1^{(i,j)}) + \frac{h}{2}\left[\Delta_{11}^2 v^{(i,j-1)}\right]^2 D_{qq}^2 f(Q_2^{(i,j)}).
\end{aligned}$$

We will first estimate $\Delta_{11}^2 v^{(i,j)}$ for i such that $x^{(i)} \leqq \delta$, where δ is the constant from the condition 8° from Assumption H. For $i = 0, \ldots, n_0 - 1, j \in Z$ we write

$$A_1^{(i,j)} = D_{yq}^2 f(Q_1^{(i,j)}) + \tfrac{1}{2}(\Delta_1 v^{(i,j+2)} + \Delta_1 v^{(i,j)})\, D_{qp}^2 f(Q_1^{(i,j)}) + $$
$$+ D_{qw}^2 f(Q_1^{(i,j)})\,(r_{i,j+1}) + \tfrac{1}{2} D_p f(Q_1^{(i,j)}),$$
$$A_2^{(i,j)} = D_{yq}^2 f(Q_2^{(i,j)}) + \tfrac{1}{2}(\Delta_1 v^{(i,j)} + \Delta_1 v^{(i,j-2)})\, D_{qp}^2 f(Q_2^{(i,j)}) + $$
$$+ D_{qw}^2 f(Q_2^{(i,j)})\,(r_{i,j-1}) + \tfrac{1}{2} D_p f(Q_2^{(i,j)}).$$

There are constants $d_1, d_2 \in R_+$, $d_1 > d_2$, and

$$c = \min\left(\frac{\mu}{2}, A\,\frac{1 + d_1 h}{2N_1}\right)$$

such that

$$A_1^{(i,j)} + A_2^{(i,j)} \leqq d_1\,, \quad R^{(i,j)} \leqq d_2 h - c[\Delta_{11}^2 v^{(i,j+1)} + \Delta_{11}^2 v^{(i,j-1)}]\, h\,.$$

Let h, k be sufficiently small so that

$$\frac{1}{2} + \frac{h}{2k}\, D_q f(Q^{(i,j)}) + hA_1^{(i,j)} \geqq 0\,, \quad \frac{1}{2} - \frac{h}{2k}\, D_q f(Q^{(i,j)}) + hA_2^{(i,j)} \geqq 0\,.$$

We introduce the following notations

$$S^{(i,j)} = \max\{\Delta_{11}^2 v^{(i,j+1)}, \Delta_{11}^2 v^{(i,j-1)}, 0\}\,,$$
$$\tilde{S}^{(i)} = \sup\{S^{(\tau,j)}: 0 \leqq \tau \leqq i,\ j \in Z\}\,.$$

From (5) it follows that $\tilde{S}^{(i)} < +\infty$. Therefore we have

$$\Delta_{11}^2 v^{(i+1,j)} \leqq S^{(i,j)}[1 + d_1 h] + $$
$$+ W(N + 3N_1)\,\tilde{S}^{(i)} h + d_2 h - c(S^{(i,j)})^2\, h\,.$$

From the inequality

$$\Delta_{11}^2 v^{(i,j)} = \frac{1}{2k}\,(\Delta_1 v^{(i,j+1)} - \Delta_1 v^{(i,j-1)}) \leqq \frac{N_1}{k}\,,$$

and from conditions $A < k/h$ and $c \leqq A(1 + d_1 h)/2N_1$ we obtain

$$\Delta_{11}^2 v^{(i,j)} \leqq \frac{1 + d_1 h}{2ch}\,. \tag{11}$$

Let us consider the polynomial $H(y) = y(1 + d_1 h) + W(N + 3N_1)\,\tilde{S}^{(i)} h + d_2 h - chy^2$. It is easy to see that $D_y H(y) = 1 + d_1 h - 2chy \geqq 0$ for $y \leqq (1 + d_1 h)/2ch$. By force of (11) we derive then

$$\Delta_{11}^2 v^{(i+1,j)} \leqq \tilde{S}^{(i)}[1 + (d_1 + W(N + 3N_1))\, h] + d_2 h - c(\tilde{S}^{(i)})^2\, h\,.$$

The right hand side of the above inequality is positive, which gives

$$\tilde{S}^{(i+1)} \leqq \tilde{S}^{(i)}[1 + (d_1 + W(N + 3N_1))\, h] + d_2 h - c(\tilde{S}^{(i)})^2\, h\,.$$

Hence for $\hat{S}^{(i)} = \tilde{S}^{(i)} + 1$ we have

$$\hat{S}^{(i+1)} \leqq \hat{S}^{(i)}[1 + (d_1 + W(N + 3N_1) + 2c)\, h] +$$
$$+ d_2 h - (d_1 + W(N + 3N_1) + c)\, h - c(\hat{S}^{(i)})^2\, h\,.$$

Using the condition $d_1 > d_2$ and putting $\tilde{d} = d_1 + W(N + 3N_1) + 2c$ we obtain

$$\hat{S}^{(i+1)} \leqq \hat{S}^{(i)}(1 + \tilde{d}h) - ch(\hat{S}^{(i)})^2\,.$$

Let h be so small that $\tilde{d}h \leqq \frac{1}{2}$. Then multiplying the last inequality by $(1 - \tilde{d}h)^{i+1}$ and putting $W^{(i)} = (1 - \tilde{d}h)^i\, \hat{S}^{(i)}$ we get

$$W^{(i+1)} \leqq W^{(i)} - ch(W^{(i)})^2\,(1 - \tilde{d}h)^{-i+1} \leqq W^{(i)} - ch(W^{(i)})^2\,, \quad i > 0\,,$$

$$W^{(1)} \leqq W^{(0)} - ch(W^{(0)})^2\,(1 - \tilde{d}h) \leqq W^{(0)} - \frac{c}{2}\,(W^{(0)})^2\,,$$

and hence

(12) $$\frac{1}{h}\,[W^{(i+1)} - W^{(i)}] \leqq -\frac{c}{2}\,(W^{(i)})^2\,.$$

Let us consider the following Cauchy problem

$$D_x\, w(x) = -\frac{c}{2}\, w^2(x)\,, \quad w(0) = W^{(0)}\,.$$

The solution of this problem is given by

$$w(x) = \frac{1}{\dfrac{c}{2}\, x + \dfrac{1}{W^{(0)}}}\,.$$

We see that w is convex and then by force of (12) we have

$$W^{(i)} \leqq w(x^{(i)}) = \frac{1}{\dfrac{c}{2}\, x^{(i)} + \dfrac{1}{W^{(0)}}} < \frac{2}{cx^{(i)}}\,, \quad i > 0\,.$$

From the above inequality we derive

$$(1 - \tilde{d}h)^i\,(\tilde{S}^{(i)} + 1) \leqq \frac{2}{cx^{(i)}}\,,$$

and hence from the condition $\tilde{d}h \leqq \frac{1}{2}$ we have

$$\tilde{S}^{(i)} \leqq \frac{1}{x^{(i)}}\left[\frac{2}{c}\,(1 - \tilde{d}h)^{-i} - ih\right] \leqq \frac{1}{x^{(i)}}\left\{\frac{2}{c}\,[(1 - \tilde{d}h)^{-1/\tilde{d}h}]^{\tilde{d}ih} - ih\right\} \leqq \frac{\tilde{N}_2}{x^{(i)}}\,,$$

where

$$\tilde{N}_2 = \frac{2}{c}\,\exp\,(\tilde{d}\delta \ln 4) - \delta\,.$$

For i such that $x^{(i)} > \delta$ we analogously derive

$$\tilde{S}^{(i+1)} \leqq \tilde{S}^{(i)}[1 + (d_1 + W(N + 3N_1))\, h] + d_2 h\,.$$

By force of the mathematical induction we have

$$\tilde{S}^{(i)} \leqq \exp(da_0)\, \tilde{S}^{([\delta/h])} + \frac{d_2}{\hat{d}}\,[\exp(da_0) - 1]\,, \tag{13}$$

where $\hat{d} = d_1 + W(N + 3N_1)$ and $[t]$ denotes the integral part of t. Supposing that h is so small that $h \leqq \delta/2$ and putting

$$N_2 = \max\left\{\tilde{N}_2, \frac{2\tilde{N}_2}{\delta}\exp(da_0) + \frac{d_2}{\hat{d}}\,[\exp(da_0) - 1] \quad \text{we get (9)}\,.\right.$$

In order to prove (10), let us adopt the previous notations $r_{i,j}$, $q_{i,j}$, $Q^{(i,j)}$, $Q_1^{(i,j)}$, $Q_2^{(i,j)}$, $R^{(i,j)}$, $A_1^{(i,j)}$, $A_2^{(i,j)}$ and let h, k be so small that

$$\frac{1}{2} + \frac{h}{2k}\, D_q f(Q^{(i,j)}) + hA_1^{(i,j)} \geqq 0\,, \quad \frac{1}{2} - \frac{h}{2k}\, D_q f(Q^{(i,j)}) + hA_2^{(i,j)} \geqq 0\,.$$

There are constants d_1, d_2 such that

$$A_1^{(i,j)} + A_2^{(i,j)} \leqq d_1\,, \quad R^{(i,j)} \leqq d_2 h\,.$$

Using the previous arguments we prove that

$$\tilde{S}^{(i+1)} \leqq \tilde{S}^{(i)}[1 + (d_1 + W(N + 3N_1))\, h] + d_2 h\,, \quad i = 0, \ldots, n_0 - 1\,,$$

where $\tilde{S}^{(i)} = \sup\{\Delta_{11}^2 v^{(\tau,j)}: -n_1 \leqq \tau \leqq i, j \in Z\}$. Now, analogously like (13) we obtain (10) with

$$N_2 = \left(\tilde{K} + \frac{d_2}{\hat{d}}\right)\exp(da_0) - \frac{d_2}{\hat{d}}\,,$$

where $\tilde{K}$ is the constant from the condition 2° of Assumption H. This ends the proof.

Let B be some constant such that $B > A$. By $\tilde{U}$ we define the set $\{(h, k): A < k/h \leqq B\}$.

Lemma 4. *Suppose that Assumption* H *is satisfied, $(h, k) \in \tilde{U}$, α, $Y > 0$, $0 \leqq i \leqq n_0$ and additionally $i > \alpha/h$ in the case $b_0 = 0$. Then there is a constant $C = C(Y, \alpha)$ for $b_0 = 0$, or $C = C(Y)$ for $b_0 > 0$ such that*

$$\sum_{|j| \leqq Y/k} 2k|\Delta_{11}^2 v^{(i,j)}| \leqq C\,. \tag{14}$$

Proof. Let $b_0 = 0$, $\bar{A} \geqq N_2/\alpha$ and $u^{(i,j)} = \Delta_1 v^{(i,j)} - \bar{A}jk$. Using (9) for any $i > \alpha/h$, $j \in Z$ we have

$$u^{(i,j+1)} - u^{(i,j-1)} = \Delta_1 v^{(i,j+1)} - \Delta_1 v^{(i,j-1)} - 2\bar{A}k =$$

$$= 2k(\Delta_{11}^2 v^{(i,j)} - \bar{A}) \leqq 2k\left(\frac{N_2}{ih} - \bar{A}\right) \leqq 0\,.$$

If $b_0 > 0$, then we take $\bar{A} \geqq N_2$ and by force of (10) we obtain the same estimate for $i = 0, \ldots, n_0, j \in Z$. From this we have

$$\sum_{|j| \leqq Y/k} 2k|\Delta_1 u^{(i,j)}| = \sum_{|j| \leqq Y/k} [u^{(i,j+1)} - u^{(i,j-1)}] \leqq$$

$$\leqq 4 \max_{|j| \leqq Y/k} |u^{(i,j)}| \leqq 4(N_1 + \bar{A}Y),$$

and

$$\sum_{|j| \leqq Y/k} 2k|\Delta_{11}^2 v^{(i,j)}| = \sum_{|j| \leqq Y/k} |\Delta_1 v^{(i,j+1)} - \Delta_1 v^{(i,j-1)}| =$$

$$= \sum_{|j| \leqq Y/k} |u^{(i,j+1)} - u^{(i,j-1)} + 2\bar{A}k| \leqq \sum_{|j| \leqq Y/k} 2k|\Delta_1 u^{(i,j)}| +$$

$$+ 4\bar{A}Y + 2\bar{A}Ba_0 .$$

Thus (14) is satisfied with $C = 4N_1 + 8\bar{A}Y + 2\bar{A}Ba_0$.

Remark 2. The analogous properties to that proved in Lemmas 2–4 we may obtain also for the operator $\tilde{\Delta}_1$ defined by $\tilde{\Delta}_1 v^{(i,j)} = (1/k)[v^{(i,j)} - v^{(i,j-1)}]$.

For any $1 \leqq i \leqq n_0$, $j \in Z$, $0 \leqq n \leqq i - 1$ let $U_{ij}(n) = \{s \in Z: s - j + i - n \in 2Z\}$.

Lemma 5. *If* $f: [0, a_0] \times R \times R \times C(B, R) \times R \to R$ *is of class* C^1, $(h, k) \in U$ *and conditions* 1°, 4°, 5° *of Assumption* H *are satisfied, then there are constants* $a_{i,j}^{n,s} \geqq 0$, $\eta_{i,j}^n$, $i = 1, \ldots, n_0$, $j \in Z$, $n = 0, \ldots, i - 1$, $s \in U_{ij}(n)$, $j - (i - n) \leqq s \leqq j + (i - n)$, *such that*

$$\Delta_1 v^{(i,j)} = \sum_{\substack{s=j-(i-n) \\ s \in U_{ij}(n)}}^{j+(i-n)} a_{i,j}^{n,s} \Delta_1 v^{(n,s)} + \eta_{i,j}^n , \qquad \sum_{\substack{s=j-(i-n) \\ s \in U_{ij}(n)}}^{j+(i-n)} a_{i,j}^{n,s} = 1 ,$$

$$|\eta_{i,j}^n| \leqq (i - n) hc_1 ,$$

where $c_1 = W(N + 3N_1)(1 + 2N_1)$.

Proof. Analogously like in Lemma 2 we get

$$\Delta_1 v^{(i,j)} = \Delta_1 v^{(i-1,j+1)} \left[\frac{1}{2} + \frac{h}{2k} D_q f(P^{(i-1,j)})\right] +$$

$$+ \Delta_1 v^{(i-1,j-1)} \left[\frac{1}{2} - \frac{h}{2k} D_q f(P^{(i-1,j)})\right] + \eta_{i,j}^{i-1} ,$$

where

$$\eta_{i,j}^{i-1} = hD_y f(P^{(i-1,j)}) +$$

$$+ \frac{h}{2} (\Delta_1 v^{(i-1,j+1)} + \Delta_1 v^{(i-1,j-1)}) D_p f(P^{(i-1,j)}) +$$

$$+ hD_w f(P^{(i-1,j)}) (r_{i-1,j}) , \quad |\eta_{i,j}^{i-1}| \leqq hc_1 .$$

Thus the lemma holds for $n = i - 1$. Assume that the lemma holds for some $n < i$, we will prove it for $n - 1$. For $i = 1, \dots, n_0$, $j \in Z$ we have

$$\Delta_1 v^{(i,j)} = \sum_{\substack{s=j-(i-n)\\ s\in U_{ij}(n)}}^{j+(i-n)} a_{i,j}^{n,s} \Delta_1 v^{(n,s)} + \eta_{i,j}^n =$$

$$= \sum_{\substack{s=j-(i-n)\\ s\in U_{ij}(n)}}^{j+(i-n)} a_{i,j}^{n,s} \left\{ \Delta_1 v^{(n-1,s+1)} \left[\frac{1}{2} + \frac{h}{2k} D_q f(P^{(n-1,s)})\right] + \right.$$

$$\left. + \Delta_1 v^{(n-1,s-1)} \left[\frac{1}{2} - \frac{h}{2k} D_q f(P^{(n-1,s)})\right] + \tilde{\eta}_{i,j}^s \right\} + \eta_{i,j}^n ,$$

where

$$\tilde{\eta}_{i,j}^s = hD_y f(P^{(n-1,s)}) + \frac{h}{2}(\Delta_1 v^{(n-1,s+1)} + \Delta_1 v^{(n-1,s-1)}) D_p f(P^{(n-1,s)}) +$$

$$+ hD_w f(P^{(n-1,s)})(r_{n-1,s}) .$$

We define constants $a_{i,j}^{n-1,s}$, $\eta_{i,j}^{n-1}$ in the following way:

$$a_{i,j}^{n-1,s} = a_{i,j}^{n,s-1} \left[\frac{1}{2} + \frac{h}{2k} D_q f(P^{(n-1,s-1)})\right] +$$

$$+ a_{i,j}^{n,s+1} \left[\frac{1}{2} - \frac{h}{2k} D_q f(P^{(n-1,s+1)})\right],$$

$$s = j - (i - n) + 1, \dots, j + (i - n) - 1 ,$$

$$a_{i,j}^{n-1,j-(i-n)-1} = a_{i,j}^{n,j-(i-n)} \left[\frac{1}{2} - \frac{h}{2k} D_q f(P^{(n-1,j-(i-n))})\right],$$

$$a_{i,j}^{n-1,j+(i-n)+1} = a_{i,i}^{n,j+(i-n)} \left[\frac{1}{2} + \frac{h}{2k} D_q f(P^{(n-1,j+(i-n))})\right],$$

$$\eta_{i,j}^{n-1} = \sum_{\substack{s=j-(i-n)\\ s\in U_{ij}(n)}}^{j+(i-n)} a_{i,j}^{n,s} \tilde{\eta}_{i,j}^s + \eta_{i,j}^n .$$

For these constants we have

$$\Delta_1 v^{(i,j)} = \sum_{\substack{s=j-(i-n)-1\\ s\in U_{ij}(n-1)}}^{j+(i-n)+1} a_{i,j}^{n-1,s} \Delta_1 v^{(n-1,s)} + \eta_{i,j}^{n-1} ,$$

$$\sum_{\substack{s=j-(i-n)-1\\ s\in U_{ij}(n-1)}}^{j+(i-n)+1} a_{i,j}^{n-1,s} = 1 ,$$

$$|\eta_{i,j}^{n-1}| \leqq (i - n + 1) hc_1 ,$$

which completes the proof of Lemma 5.

Lemma 6. *Suppose that Assumption* H *is satisfied,* $(h, k) \in \tilde{U}$, α, $Y > 0$, $1 \leqq i \leqq n_0$, $0 \leqq n \leqq i - 1$ *and additionally* $i > \alpha/h$, $n > \alpha/h$ *in the case* $b_0 = 0$. *Then* *there is a constant $L = L(Y, \alpha)$ for $b_0 = 0$, or $L = L(Y)$ for $b_0 > 0$ such that*

$$(15) \qquad \sum_{|j| \leqq Y/k} 2k|\Delta_1 v^{(i,j)} - \Delta_1 v^{(n,j)}| \leqq Lh(i - n).$$

Proof. We will first prove (15) for the case $i - n \in 2Z$. By force of Lemma 5 we have

$$\sum_{|j| \leqq Y/k} 2k|\Delta_1 v^{(i,j)} - \Delta_1 v^{(n,j)}| \leqq$$

$$\leqq \sum_{|j| \leqq Y/k} \sum_{\substack{s=j-(i-n) \\ s \in U_{ij}(n)}}^{j+(i-n)} a_{i,j}^{n,s}|\Delta_1 v^{(n,s)} - \Delta_1 v^{(n,j)}| \, 2k +$$

$$+ (4Y + 2Ba_0) \, c_1(i - n) \, h \leqq$$

$$\sum_{|j| \leqq Y/k} \sum_{\substack{s=j-(i-n) \\ s \in U_{ij}(n)}}^{j+(i-n)} a_{i,j}^{n,s} \sum_{r=j-(i-n)}^{j+(i-n)} |\Delta_1 v^{(n,r+1)} - \Delta_1 v^{(n,r-1)}| \, 2k +$$

$$+ (4Y + 2Ba_0) \, c_1(i - n) \, h \leqq$$

$$\leqq [2(i - n) + 1] \, 2k \sum_{|r| \leqq Y/k + (i-n)} |\Delta_1 v^{(n,r+1)} - \Delta_1 v^{(n,r-1)}| +$$

$$+ (4Y + 2Ba_0) \, c_1(i - n) \, h .$$

Using (14) with $C = C(Y + Ba_0, \alpha)$ for $b_0 = 0$, or $C = C(Y + Ba_0)$ for $b_0 > 0$ we get (15) with the constant $L = 6BC + 4Yc_1 + 2Ba_0c_1$. From Remark 2 we obtain (15) for $i - n \notin 2Z$.

4. THE SEQUENCE OF APPROXIMATE SOLUTIONS

Lemma 7. *Suppose that $f: [0, a_0] \times R \times R \times C(B, R) \times R \to R$ is of class C^1 and that conditions 1°, 4°, 5° of Assumption* H *are satisfied. Then there is a sequence $\{(h_\nu, k_\nu)\}_{\nu=1}^{\infty}$, $(h_\nu, k_\nu) \in \tilde{U}$, $\lim_{\nu \to \infty} h_\nu = \lim_{\nu \to \infty} k_\nu = 0$, and a function $\tilde{u} \in C([0, a_0] \times R, R)$ such that $\lim_{\nu \to \infty} u_{h_\nu k_\nu}(x, y) = \tilde{u}(x, y)$ almost uniformly on $[0, a_0] \times R$.*

This lemma follows from Lemmas 1, 2 and from Remark 2.

Let us define sequences $\{u^{(\nu)}\}_{\nu=1}^{\infty}$, $\{V^{(\nu)}\}_{\nu=1}^{\infty}$, $\{W^{(\nu)}\}_{\nu=1}^{\infty}$. We put $u^{(\nu)} = u_{h_\nu k_\nu}$. If $(x, y) \in [0, a_0] \times R$, then there are i, j, $0 \leqq i \leqq n_0 - 1$, $j \in 2Z$ such that $(x, y) \in [x^{(i)}, x^{(i+1)}) \times [y^{(j-1)}, y^{(j+1)})$. Let

$$V^{(\nu)}(x, y) = \Delta_1 v^{(i,j)} + (x - x^{(i)}) \, \Delta_{01}^2 v^{(i,j)} ,$$

$$W^{(\nu)}(x, y) = \Delta_0 v^{(i,j-1)} + (y - y^{(j-1)}) \, \Delta_{01}^2 v^{(i,j)} ,$$

where the difference operators are defined for $h = h_\nu$, $k = k_\nu$. If $i = n_0 - 1$, then we replace in the above definitions the interval $[x^{(i)}, x^{(i+1)})$ by $[x^{(i)}, x^{(i+1)}]$. Thus we have $V^{(\nu)}, W^{(\nu)}: [0, a_0] \times R \to R$ and $V^{(\nu)}(x, y) = D_y u^{(\nu)}(x, y)$, $W^{(\nu)}(x, y) = D_x u^{(\nu)}(x, y)$ a.e. on $[0, a_0] \times R$.

By $\mathrm{Var}_Y[V^{(v)}(x, \cdot)]$ we denote the variation of the function $V^{(v)}(x, \cdot)$ on $[-Y, Y]$, $Y > 0$.

Lemma 8. *If Assumption* H *is satisfied and* α, $Y > 0$, *then for each* $x \in (\alpha, a_0]$, ($x \in [0, a_0]$ *in the case* $b_0 > 0$) *and for any integer* v *we have* $\mathrm{Var}_Y[V^{(v)}(x, \cdot)] \leqq 3C$, *where* C *is the constant from* (14).

Proof. For any $x \in [0, a_0]$ there is i, $0 \leqq i \leqq n_0 - 1$ such that $x \in [x^{(i)}, x^{(i+1)})$. Since $V^{(v)}(x, \cdot)$ is a constant function on intervals $[y^{(j-1)}, y^{(j+1)})$, $j \in 2Z$, we have

$$\mathrm{Var}_Y[V^{(v)}(x, \cdot)] = \sum_{j \in 2Z, |jk_v| \leqq Y} |V^{(v)}(x, y^{(j+1)}) - V^{(v)}(x, y^{(j-1)})| \leqq$$
$$\leqq \sum_{|j| \leqq Y/k_v} |\Delta_1 v^{(i,j+1)} + (x - x^{(i)}) \Delta_{01}^2 v^{(i,j+1)} - \Delta_1 v^{(i,j-1)} -$$
$$- (x - x^{(i)}) \Delta_{01}^2 v^{(i,j-1)}| \leqq \sum_{|j| \leqq Y/k_v} [|\Delta_1 v^{(i,j+1)} - \Delta_1 v^{(i,j-1)}| +$$
$$+ \frac{x - x^{(i)}}{h_v} |\Delta_1 v^{(i+1,j+1)} - \Delta_1 v^{(i,j+1)} - \Delta_1 v^{(i+1,j-1)} + \Delta_1 v^{(i,j-1)}|] \leqq$$
$$\leqq \sum_{|j| \leqq Y/k_v} 2k_v |\Delta_{11}^2 v^{(i,j)}| + \frac{x - x^{(i)}}{h_v} (2k_v |\Delta_{11}^2 v^{(i+1,j)}| + 2k_v |\Delta_{11}^2 v^{(i,j)}|)] .$$

The above inequality and (14) complete the proof of Lemma 8.

By $L(Y)$ we denote a set of all Lebesgue integrable functions ψ: $[0, a_0] \times [-Y, Y] \to R$ with the norm $\|\psi\|_{L(Y)} = \int_0^{a_0} \int_{-Y}^{Y} |\psi(x, y)| \, dx \, dy$.

Lemma 9. *If Assumption* H *is satisfied, then there is a sequence* $\{v_s\}_{s=1}^{\infty}$ *and a measurable function* $\bar{v}$: $[0, a_0] \times R \to R$ *such that* $\lim_{s \to \infty} \|V^{(v_s)} - \bar{v}\|_{L(Y)} = 0$.

Proof. Let $Y > 0$, $\alpha \in (0, a_0)$ and let $\{x_r\}_{r=1}^{\infty}$ be a sequence of all rational numbers from the interval $[\alpha, a_0]$. It is easy to see that for any integer v we have $|V^{(v)}(x, y)| \leqq 3N_1$, $(x, y) \in [0, a_0] \times R$. From this and from Lemma 8 it follows that assumptions of Helly's theorem are satisfied. Hence for any $x \in [\alpha, a_0]$ there is a subsequence of the sequence $\{V^{(v)}(x, \cdot)\}_{v=1}^{\infty}$ which is convergent on $[-Y, Y]$. If we apply the diagonal process, then we obtain a subsequence of $\{V^{(v)}\}_{v=1}^{\infty}$, which is convergent on the set $\{(x, y) \in [\alpha, a_0] \times R: x = x_r$ for some $r\}$. We denote this sequence by $\{V^{(v)}\}_{v=1}^{\infty}$ again.

We will prove that

$$\lim_{v,s \to \infty} \int_{-Y}^{Y} |V^{(s)}(x, y) - V^{(v)}(x, y)| \, dy = 0 , \tag{16}$$

uniformly with respect to $x \in [\alpha, a_0]$.

For each $\varepsilon > 0$ there is a finite subset $\{x_{r_1}, \ldots, x_{r_m}\}$ of the sequence $\{x_r\}_{r=1}^{\infty}$ such that the distance between any two successive elements of this set is less then $\varepsilon/5L$, where L is the constant from inequality (15). For sufficiently large v, s we have

$$\int_{-Y}^{Y} |V^{(v)}(x_{r_l}, y) - V^{(s)}(x_{r_l}, y)| \, dy < \varepsilon/5 , \quad l = 1, \ldots, m .$$

For each $x \in [\alpha, a_0]$ there is l, $1 \leqq l \leqq m$ such that $0 \leqq x - x_{r_l} < \varepsilon/5L$. For any v, s

we have then

$$\int_{-Y}^{Y} |V^{(s)}(x, y) - V^{(v)}(x, y)| \, dy \leqq \int_{-Y}^{Y} |V^{(s)}(x, y) - V^{(s)}(x_{r_l}, y)| \, dy +$$
$$+ \int_{-Y}^{Y} |V^{(s)}(x_{r_l}, y) - V^{(v)}(x_{r_l}, y)| \, dy +$$
$$+ \int_{-Y}^{Y} |V^{(v)}(x_{r_l}, y) - V^{(v)}(x, y)| \, dy .$$

Using the definition of $V^{(v)}$ we get

$$\int_{-Y}^{Y} |V^{(v)}(x, y) - V^{(v)}(x_{r_l}, y)| \, dy \leqq$$
$$\leqq \sum_{j \in 2Z, |jk_v| \leqq Y} 2k_v \left\| \left[\Delta_1 v^{(i,j)} + \frac{x - x^{(i)}}{h_v} (\Delta_1 v^{(i+1,j)} - \Delta_1 v^{(i,j)}) \right] - \right.$$
$$\left. - \left[\Delta_1 v^{(i',j)} + \frac{x_{r_l} - x^{(i')}}{h_v} (\Delta_1 v^{(i'+1,j)} - \Delta_1 v^{(i',j)}) \right] \right\| ,$$

where $i = [x/h_v]$, $i' = [x_{r_l}/h_v]$. Hence by force of (15) we have

$$\int_{-Y}^{Y} |V^{(v)}(x, y) - V^{(v)}(x_{r_l}, y)| \, dy \leqq \sum_{|j| \leqq Y/k_v} 2k_v [|\Delta_1 v^{(i,j)} - \Delta_1 v^{(i',j)}| +$$
$$+ |\Delta_1 v^{(i+1,j)} - \Delta_1 v^{(i,j)}| + |\Delta_1 v^{(i'+1,j)} - \Delta_1 v^{(i',j)}|] \leqq$$
$$\leqq Lh_v \left(\left[\frac{x}{h_v} \right] - \left[\frac{x_{r_l}}{h_v} \right] \right) + 2Lh_v \leqq L(x - x_{r_l}) + 3Lh_v < \tfrac{2}{5}\varepsilon ,$$

for v sufficiently large. Finally, we obtain

$$\int_{-Y}^{Y} |V^{(s)}(x, y) - V^{(v)}(x, y)| \, dy < \varepsilon ,$$

for s, v sufficiently large. This ends the proof of (16).

Since the convergence in (16) is uniform on $[\alpha, a_0]$ for any $\alpha \in (0, a_0)$ we obtain the almost uniform convergence on $[0, a_0]$. From this we have $\lim_{v,s \to \infty} \| V^{(s)} - V^{(v)} \|_{L(Y)} = 0$. The completeness of $L(Y)$ completes the proof of Lemma 9.

Remark 3. If $b_0 > 0$, then it is not necessary to consider the interval $[\alpha, a_0]$, $\alpha > 0$, because (16) holds uniformly with respect to $x \in [0, a_0]$.

5. THE MAIN THEOREM

Theorem 1. *If Assumption* H *is satisfied, then there is a function* $u \in C([-b_0, a_0] \times R, R)$ *which is a generalized solution of* (1), (2).

Proof. It follows from Lemma 7 that there is a sequence $\{(h_v, k_v)\}_{v=1}^{\infty}$, $(h_v, k_v) \in \tilde{U}$ such that the sequence $\{u^{(v)}\}_{v=1}^{\infty}$, $u^{(v)} = u_{h_v k_v}$ is uniformly convergent to a function $\tilde{u}$ on $[0, a_0] \times R$. By force of Lemma 9 there is a subsequence of $\{V^{(v)}\}_{v=1}^{\infty}$ which is convergent in the $L(Y)$ norm to $\bar{v}$. The sequence and its subsequence we denote by the same symbol for simplicity. Let $\tilde{u}(x, y) = \varphi(x, y)$ for $(x, y) \in [-b_0, 0] \times R$. Then the sequence $\{\varphi_{h_v k_v}\}_{v=1}^{\infty}$ is uniformly convergent to $\tilde{u}$ on $[-b_0, 0] \times R$. For

$(x, y) \in [0, a_0] \times R$ we write

$$\bar{f}^{(\nu)}(x, y) = f(x, y, u^{(\nu)}(x, y), u^{(\nu)}_{(x,y)}, V^{(\nu)}(x, y)),$$
$$\bar{w}(x, y) = f(x, y, \tilde{u}(x, y), \tilde{u}_{(x,y)}, \bar{v}(x, y)).$$

We will prove that $\lim_{\nu \to \infty} \|W^{(\nu)} - \bar{w}\|_{L(Y)} = 0$, $Y > 0$. From

$$\|W^{(\nu)} - \bar{w}\|_{L(Y)} \leqq \|W^{(\nu)} - \bar{f}^{(\nu)}\|_{L(Y)} + \|\bar{f}^{(\nu)} - \bar{w}\|_{L(Y)},$$
$$\lim_{\nu \to \infty} \|\bar{f}^{(\nu)} - \bar{w}\|_{L(Y)} = 0,$$

we see that it is sufficient if we prove

$$\lim_{\nu \to \infty} \|W^{(\nu)} - \bar{f}^{(\nu)}\|_{L(Y)} = 0. \tag{17}$$

If $\alpha > 0$, then for each $x \in [\alpha, a_0]$ there is i, $0 \leqq i \leqq n_0 - 1$ such that $x \in \in [x^{(i)}, x^{(i+1)})$. (If $b_0 > 0$, then we take $x \in [0, a_0]$.) We have then

$$\int_{-Y}^{Y} |W^{(\nu)}(x, y) - \bar{f}^{(\nu)}(x, y)| \, dy \leqq$$
$$\sum_{j \in 2Z, |jk_\nu| \leqq Y} \int_{y^{(j-1)}}^{y^{(j+1)}} |\Delta_0 v^{(i,j-1)} - \bar{f}^{(\nu)}(x, y) +$$
$$+ \frac{y - y^{(j-1)}}{h_\nu} [\Delta_1 v^{(i+1,j)} - \Delta_1 v^{(i,j)}] \, dy \leqq$$
$$\leqq \sum_{|j| \leqq Y/k_\nu} \int_{y^{(j-1)}}^{y^{(j+1)}} |f(x^{(i)}, y^{(j-1)}, \tfrac{1}{2}(v^{(i,j)} + v^{(i,j-2)}), u^{(\nu)}_{(x^{(i)}, y^{(j-1)})}, \Delta_1 v^{(i,j-1)}) -$$
$$- \bar{f}^{(\nu)}(x, y)| \, dy + \sum_{|j| \leqq Y/k_\nu} \frac{2k_\nu}{h_\nu} |\tfrac{1}{2}(v^{(i,j)} + v^{(i,j-2)}) - v^{(i,j-1)}| +$$
$$+ \sum_{|j| \leqq Y/k_\nu} \frac{(2k_\nu)^2}{h_\nu} |\Delta_1 v^{(i+1,j)} - \Delta_1 v^{(i,j)}|.$$

Using the Lipschitz condition for f, Lemmas 1–3 and Remark 2 we see that the first component of the right hand side of the above inequality tends to zero if $\nu \to \infty$. From Remark 2 we obtain that there is a constant C_0 such that

$$\sum_{|j| \leqq Y/k_\nu} 2k_\nu |\tilde{\Delta}^2_{11} v^{(i,j)}| \leqq C_0,$$

and hence

$$\sum_{|j| \leqq Y/k_\nu} \frac{2k_\nu}{h_\nu} |\tfrac{1}{2}(v^{(i,j)} + v^{(i,j-2)}) - v^{(i,j-1)}| \leqq BC_0 k_\nu.$$

Futhermore, from Lemma 6 we have

$$\sum_{|j| \leqq Y/k_\nu} \frac{(2k_\nu)^2}{h_\nu} |\Delta_1 v^{(i+1,j)} - \Delta_1 v^{(i,j)}| \leqq 2BLh_\nu.$$

Finally, we obtain

$$\lim_{\nu \to \infty} \int_{-Y}^{Y} |W^{(\nu)}(x, y) - \bar{f}^{(\nu)}(x, y)| \, dy = 0,$$

almost uniformly with respect to $x \in [0, a_0]$, from which we have (17).

From Lemmas 1, 2 and from Remark 2 we obtain that $\tilde{u}$ satisfies the Lipschitz condition on $[0, a_0] \times R$, and hence the derivatives $D_x\tilde{u}$, $D_y\tilde{u}$ exist a.e. on $[0, a_0] \times \times R$. Since the sequences $\{V^{(v)}\}_{v=1}^{\infty}$, $\{W^{(v)}\}_{v=1}^{\infty}$ are equibounded and convegent in the $L(Y)$ norm to $\bar{v}$, $\bar{w}$ respectively, it follows that $D_x\tilde{u} = \bar{w}$, $D_y\tilde{u} = \bar{v}$ a.e. on $[0, a_0] \times \times [-Y, Y]$ for any $Y > 0$. Thus $\tilde{u}$ satisfies (1) a.e. on $[0, a_0] \times R$. From the definition of $\tilde{u}$ we see that the initial condition (2) holds. Furthermore, it is easy to prove the existence of a constant $M > 0$ such that for any $(x, y) \in [0, a_0] \times R$, $l \in R$, $l \neq 0$ we have

$$l^{-2}[\tilde{u}(x, y + l) - 2\tilde{u}(x, y) + \tilde{u}(x, y - l)] \leqq M \qquad \text{if} \quad b_0 = 0\,,$$

$$l^{-2}[\tilde{u}(x, y + l) - 2\tilde{u}(x, y) + \tilde{u}(x, y - l)] \leqq M/x \quad \text{if} \quad b_0 > 0\,.$$

Hence u is a generalized solution of (1), (2), which ends the proof.

Remark 4. Using the same methods as in the proof of Theorem 1 we obtain the existence of a generalized solution of the following differential-functional system of first order partial equations

$$D_x z_i(x, y) = f_i(x, y, z(x, y), z_{(x,y)}, D_y z_i(x, y))\,,$$

$$z_i(x, y) = \varphi_i(x, y)\,, \quad (x, y) \in [-b_0, 0] \times R\,,$$

where $i = 1, \ldots, m$, $z = (z_1, \ldots, z_m)$, $\varphi = (\varphi_1, \ldots, \varphi_m) : [-b_0, 0] \times R \to R^m$, $f = = (f_1, \ldots, f_m) : [0, a_0] \times R \times R^m \times C(B, R^m) \times R \to R^m$.

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