Czechoslovak Mathematical Journal

Ján Jakubík

Mixed product decompositions of a directed group

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 3, 429–435

Persistent URL: http://dml.cz/dmlcz/102477

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

MIXED PRODUCT DECOMPOSITIONS OF A DIRECTED GROUP

JÁN JAKUBÍK, Košice

(Received July 26, 1989)

The notion of mixed product of partially ordered groups is a common generalization of the notions of direct product and lexicographic product.

To each directed group G there corresponds a lattice $L(G)$ which is constructed by means of the set of all mixed product decompositions of G (cf. [3]).

By constructing the lattice $L(G)$ the results on isomorphic reffinements of any two mixed product decompositions are applied. The fundamental theorem on the existence of such reffinements (in the case of lexicographic decompositions of a linearly ordered group) were proved by A. I. Maltsev [6]. Generalizations of this theorem were established by L. Fuchs [2] (for lexicographic product decompositions of directed groups) and by the author [3] (for mixed product decompositions with directed factors of a directed group). For analogous questions concerning directed grupoids cf. [5].

Roughly speaking, $L(G)$ is the system of all mixed product decompositions of a directed group G with directed factors (certain pairs of mixed product decompositions which behave in the same way with respect of forming isomorphic reffinements are identified); the partial order on $L(G)$ is defined in a natural way by means of properties of isomorphic reffinements).

If G is a linearly ordered group, then each mixed product decomposition of G is, in fact, a lexicographic product decomposition of G. In the paper [4] it was proved that if G is linearly ordered, then the lattice $L(G)$ has the following properties:

(A) Each principal filter of $L(G)$ is a complete lattice.

(B) $L(G)$ is distributive (moreover, each principal filter of $L(G)$ is completely distributive).

In the present paper it will be shown that the assertion (A) remains valid for directed groups as well. The notion of regular partition of a partially ordered set will be applied in the proof.

On the other hand, for the case of directed groups (even for the case of lattice ordered groups) the assertion (B) fails to be valid in general. In connection with this negative result the following question arises: does there exist a nontrivial lattice identity which is satisfied in the lattice $L(G)$ for each directed group G? It will be proved that the answer is "No".

1. PRELIMINARIES

The standard denotations for partially ordered groups will be used (cf. Fuchs [2]). The group operation will be denoted additively, the commutativity of this operation will not be assumed. Each subgroup of a partially ordered group is supposed to be partially ordered by the induced partial order.

We recall the basic notions concerning mixed products of directed groups and we introduce the relevant denotations.

Let I be a partially ordered set and for each $i \in I$ let G_i be a directed group. Let A be the (complete) direct product of the groups G_i. For $f \in A$ we denote by $f(i)$ the i-th component of f. We put $I(f) = \{i \in I: f(i) \neq 0\}$. Let A_1 be the set of all elements of A such that either $I(f) = \emptyset$ or $I(f)$ satisfies the descending chain condition. Then A_1 is a subgroup of G.

For $f \in A_1$ let $\min I(f)$ be the set of all minimal elements of $I(f)$. Let $f, g \in A_1$. We put $f \leqq g$ if $f(i) < g(i)$ is valid for each $i \in \min(f - g)$. In this way we obtain a directed group $(A_1; \leqq)$ which will be denoted by the symbol $\Gamma_{i \in I} G_i$.

Let G be a directed group and let α bean isomorphism of G onto $\Gamma_{i \in I} G_i$. Then α is said to be a *mixed product representation of* G. The partially ordered groups G_i are called factors of G under the representation α.

If I is linearly ordered, then α is said to be a lexicographic product representation of G. In the case that G is linearly ordered, each mixed product representation of G is a lexicographic product representation of G.

In what follows we assume that

(i) G is a nonzero directed group;

(ii) all factors of G under consideration are nonzero and directed.

Let α be as above and let $I(0) \subseteq I$. We denote by $G_{I(0)}$ the subgroup of G consisting of all elements $g \in G$ such that $i \in I \setminus I(0)$ implies $\alpha(g)(i) = 0$. If $I(0) = \{i(0)\}$ is a one-element set, then we denote $G_{I(0)} = G^0_{i(0)}$. Next we set

$$G^1_{i(0)} = \{g \in G: i \in I \text{ and } i \not> i(0) \text{ implies } \alpha(g)(i) = 0\},$$

$$G^2_{i(0)} = \{g \in G: i \in I \text{ and } i \ngeqq i(0) \text{ implies } \alpha(g)(i) = 0\}.$$

Let us have another mixed product representation of G

$$\beta: G \to \Gamma_{j \in J} G_j .$$

Without loss of generality we can assume that $I \cap J = \emptyset$. The representations α and β will be said to be equivalent if the following conditions are satisfied:

(i) there exists an isomorphism φ of the partially ordered set I onto partially ordered set J;

(ii) for each $i \in I$ there exists an isomorphism φ_i of the directed group G_i onto the directed group $G_{\varphi(i)}$;

(iii) if α_i and $\beta_{\varphi(i)}$ are the natural morphisms of G onto G_i or onto $G_{\varphi(i)}$, respectively, then the following diagram

$$\begin{array}{ccc} & G & \\ {}^{\alpha_i}\swarrow & & \searrow^{\beta_{\varphi(i)}} \\ G_i & \xrightarrow{\varphi_i} & G_{\varphi(i)} \end{array}$$

is commutative (i.e., for each $g \in G$ and each $i \in I$ the relation

$$\varphi_i(\alpha(g)(i)) = \beta(g)(\varphi(i))$$

is valid.)

Roughly speaking, equivalent mixed product representations of G cannot be considered as to be essentially different.

The mixed product representations α and β are said to be isomorphic if the conditions (i) and (ii) above hold.

Let M_0 be the class of all mixed product representations of G. Let α and β be arbitrary elements of M_0. We put $\alpha \geqq \beta$, if for each $i \in I$ there exists $j \in J$ such that

$$(1) \qquad G_i^1 \subseteq G_j^1 \subset G_j^2 \subseteq G_i^2 .$$

From the relation $G_i \neq \{0\}$ it follows that the element j is uniquely determined by the element i. It is easy to verify that $\leqq$ is a quasiorder on the class M_0. For $\alpha \in M_0$ we denote by α^- the class of all $\alpha_1 \in M_0$ such that $\alpha \leqq \alpha_1$ and $\alpha_1 \leqq \alpha$. If α^- and β^- are distinct, α_1 and α_2 belong to α, β_1 and β_2 belong to β and if $\alpha_1 \leqq \beta_1$, then $\alpha_2 \leqq \beta_2$. If $\gamma, \delta \in M_0$ and if γ is equivalent to δ, then $\gamma^- = \delta^-$. For α and β in M_0 we put $\alpha^- \leqq \beta^-$ if $\alpha \leqq \beta$. Then the class $M_0^- = \{\alpha^- : \alpha \in M_0\}$ is partially ordered by the relation $\leqq$.

An element α of M_0 will be said to be a mixed product decomposition of G if the following conditions are valid:

(i) for each $i \in I$, G_i is a subgroup of G;

(ii) whenever $i \in I$ and $g \in G_i$, then $\alpha(g)(i) = g$ and $\alpha(g)(i(1)) = 0$ for each $i(1) \in I \setminus \{i\}$.

We denote by M the set of all mixed product decompositions of G. We put

$$L(G) = \{\alpha^- \cap M : \alpha \in M_0\} .$$

For each $\alpha \in M_0$ there exists $\alpha_1 \in M$ such that α_1 is equivalent to α. In fact, let α be as above. Consider the mixed product $\Gamma_{i \in I}\, G_i^0 = H$. Then G_i^0 is a subgroup of G for each $i \in I$. Let $g \in G$. We put

$$\alpha_1(g) = \langle \ldots, h_i, \ldots \rangle_{i \in I} ,$$

where $h_i \in G_i^0$ and $\alpha(h_i)(i) = \alpha(g)(i)$. Then $\alpha_1 : G \to \Gamma_{i \in I}\, G_i^0$ is a mixed product representation of G satisfying the condition (ii) above. Hence $\alpha_1 \in M$. It is easy to verify that α_1 is equivalent to α.

We denote $\alpha^* = \alpha^- \cap M$. For $\alpha^*, \beta^* \in L(G)$ we set $\alpha^* \leqq \beta^*$ if $\alpha \leqq \beta$. Then $L(G)$ is a partially ordered set; moreover, in view of the relation between α and α_1 mentioned above we obtain that the partially ordered class M_0^- is isomorphic to the partially ordered set $L(G)$.

Let α be as above and let X be a subgroup of G. For each $i \in I$ we put $X(G_i) = \{\alpha(x)(i): x \in X\}$. The mapping α induces an isomorphism of X into $\Gamma_{i \in I} X(G_i)$; this mapping will be denoted by the same symbol α. Let us remark that some $X(G_i)$ can be zero groups; these can be omitted from the consideration.

Let α and β be elements of M. Suppose that for each $j \in J$ there exists a subset $I(j)$ of I such that

(i) α is an isomorphism of G_j into $\Gamma_{i \in I} G_j(G_i)$;

(ii) if $i \in I(j)$, then $G_j(G_i) = G_i$;

(iii) if $i \in I \setminus I(j)$, then $G_j(G_i) = \{0\}$.

Under these assumptions α is said to be a reffinement of β.

Now let α and β be arbitrary elements of M. For each $(i, j) \in I \times J$ put $G_{ij} = (G_i^2 \cap G_j^0)(G_i^0)$. We denote by K the set of all pairs $(i, j) \in I \times J$ such that $G_{ij} \neq \{0\}$. Let $k_1 = (i_1, j_1)$ and $k_2 = (i_2, j_2)$ be elements of K. We put $k_1 \leqq k_2$ if either $j_1 < j_2$, or $j_1 = j_2$ and $i_1 \leqq i_2$. Let γ be a mapping of G into $\Gamma_{k \in K} G_k$ which is defined by putting (for each $g \in G$ and each $k = (i, j) \in K$)

$$(\gamma(g))(k) = \alpha[(\beta(g))(j)](i) .$$

We shall write also $f(\alpha, \beta)$ instead of γ.

The following result (in a slightly different formulation) is contained in [3], Theorem 4.10; it is a generalization of a theorem of Maltsev [6] (concerning lexicographic product decompositions of linearly ordered groups) and of a theorem of Fuchs [5] (concerning lexicographic product decompositions of directed groups).

2.1. Theorem. *Let α and β be mixed product decomposition of G. Then*

(i) *$f(\alpha, \beta)$ is a mixed product decomposition of G;*

(ii) *the mixed product decompositions $f(\alpha, \beta)$ and $f(\beta, \alpha)$ are isomorphic;*

(iii) *$f(\alpha, \beta)$ is a reffinement of α and $f(\beta, \alpha)$ is a reffinement of β.*

From the definition of $f(\alpha, \beta)$ we also obtain that if α, α_1, β and β_1 are elements of M such that $\alpha^* = \alpha_1^*$ and $\beta^* = \beta_1^*$, then $f(\alpha, \beta)^* = f(\alpha_1, \beta_1)^*$.

2.2. Corollary. *Let α and β be elements of M. Assume that α is equivalent to β. Then α is isomorphic to β.*

Next we have (cf. [3], Theorem 6.4):

2.3. Proposition. *The partially ordered set $L(G)$ is a lattice. For each α^* and β^* from $L(G)$ we have $\alpha^* \wedge \beta^* = f(\alpha, \beta)^*$.*

A constructive description of the operation $\bigvee$ of the lattice $L(G)$ is given in [3], Section 6.

Let P be a partially ordered set. An equivalence relation ϱ on P will be said to be *regular* if it satisfies the following condition (r) and the condition (r′) dual to (r).

(r) Let p_1, p_2 and p_3 be elements of P such that $p_1 \varrho p_2$ and $p_1 < p_3$. Assume that the relation $p_1 \varrho p_3$ does not hold. Then $p_2 < p_3$.

If ϱ is a regular equivalence relation on P, then the partition of P corresponding to the equivalence ϱ will also be called regular.

Let $E(P)$ be the lattice of all equivalence relations on P and let $R(P)$ be the set of all regular equivalence relations on P.

2.4. Lemma. *$R(P)$ is a closed sublattice of $E(P)$.*

Proof. The lattice operations in $E(P)$ will be denoted by $\wedge$ and $\vee$. Let $\varrho_t (t \in T)$ be elements of $R(P)$. Then clearly $\wedge_{t\in T} \varrho_t$ also belongs to $R(P)$. Put $\varrho = \vee_{t\in T} \varrho_t$. Let p_1, p_2 and p_3 be elements of P such that $p_1 \varrho p_2$, $p_1 < p_3$. Assume that $p_1 \varrho p_3$ does not hold. We have to verify that $p_2 < p_3$ is valid.

There exist elements $q_0, q_1, q_2 \ldots, q_n$ in P and $t(1), t(2), \ldots, t(n)$ in T such that $q_0 = p_1$, $q_n = p_2$ and $q_{m-1}\, t(m)\, q_m$ is valid for $m = 1, 2, \ldots, n$. Because $p_1 \varrho p_3$ fails to hold, $q_0\, \varrho(1)\, p_3$ does not hold. Since $q_0\, \varrho(1)\, q_1$ and $\varrho(1)$ is regular, we infer that $q_1 < p_3$. By induction we obtain $q_n < p_3$. Hence ϱ satisfies the condition (r). Analogously we can verify that ϱ satisfies the condition (r′).

Let α and β be elements of M such that $\alpha \leqq \beta$. For $j \in J$ let $I_0(j)$ be the set of all $i \in I$ such that the relation (1) from Section 1 is valid. Next, for $i(1)$ and $i(2)$ from I we put $i(1)\, \varrho_\beta\, i(2)$ if there is $j \in J$ such that $i(1)$ and $i(2)$ belong to $I_0(j)$.

The following lemma is easy to verify; the proof will be omitted.

2.5. Lemma. *Under the above denotation, ϱ_β is a regular equivalence relation on I.*

Let ϱ be a regular equivalence relation on I and let P be the corresponding regular partition of I. For p_1 and p_2 from P we put $p_1 \leqq p_2$ if either $p_1 = p_2$, or $p_1 \neq p_2$ and $i_1 < i_2$ whenever $i_1 \in p_1$ and $i_2 \in p_2$. Then P is a partially ordered set under $\leqq$.

Let $p \in P$. In accordance with the denotation introduced above we denote by G_p the set of all $g \in G$ such that $\alpha(g)(i) = 0$ whenever i does not belong to p. Then G_p is subgroup of G. Let γ be the mapping of G into $\Gamma_{p\in P} G_p$ such that for each $g \in G$ and each $p \in P$ we have

$$\gamma(g)(p) = \langle \ldots, \alpha(g)(i), \ldots \rangle_{i\in p}\,.$$

2.6. Lemma. *Under the above denotation, γ is a mixed product decomposition of G and α is a reffinement of γ. Moreover, the partition of I corresponding to ϱ_γ is P.*

The proof will be omitted.

3. PRINCIPAL FILTERS OF $L(G)$

For $\alpha^* \in L(G)$ we denote by $[\alpha^*]$ the principal filter of $L(G)$ generated by α^*. Let us remark that $L(G)$ has a greatest element α_0^*, where α_0 is the trivial mixed product decomposition of G (α_0 possesses only one factor).

3.1. Lemma. *Let α and β be elements of M such that $\alpha \leqq \beta$. Then there exists β_1 in M such that $\beta_1^* = \beta^*$ and α is a reffinement of β_1.*

Proof. Let ϱ_β be the regular equivalence on I corresponding to β (cf. Lemma 2.5). Next let $P(\varrho_\beta) = K$ be the regular partition of I corresponding to ϱ_β and let $\beta_1 = \gamma$ be the element of M constructed as in Lemma 2.6. Then we have $\beta_1^* = \beta^*$ and α is a reffinement of β_1.

In view of 2.4, the system $R(I)$ of all regular equivalence relations on I is a complete lattice. For $\beta^* \in [\alpha^*]$ we denote $\varphi_1(\beta^*) = \varrho_\beta$. Then φ_1 is a correctly defined injective mapping of the set $[\alpha^*]$ into $R(I)$. If $\beta_1^* \leqq \beta_2^*$, then $\varphi(\beta_1^*) \leqq \varphi(\beta_2^*)$. If $\varrho \in R(I)$, then let γ be as in Lemma 2.6; we have $\gamma^* \in [\alpha^*]$ and $\varphi_1(\gamma^*) = \varrho$. Let $\varrho_1, \varrho_2 \in R(I)$ and suppose that γ_i corresponds to ϱ_i $(i = 1, 2)$ in the above manner. Then from $\varrho_1 \leqq \varrho_2$ we obtain $\gamma_1^* \leqq \gamma_2^*$. Thus φ_1 is an isomorphism of the partially ordered set $[\alpha^*]$ onto the complete lattice $R(I)$. Therefore we have

3.2. Theorem. *Each principal filter of $L(G)$ is a complete lattice.*

3.3. Proposition. *Let $f_1(x_1, \ldots, x_n)$ and $f_2(x_1, \ldots, x_n)$ be lattice polynomials. Assume that for each directed group G and for each $\alpha^* \in L(G)$ the lattice $[\alpha^*]$ satisfies the identity $f_1(\beta_1^*, \ldots, \beta_n^*) = f_2(\beta_1^*, \ldots, \beta_n^*)$. Then the identity $f_1(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_n)$ holds in each lattice.*

Proof. By means of contradiction, assume that there exists a lattice L_1 such that

$$f(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_n) \tag{1}$$

fails to be an identity in L_1. Then in view of the well-known theorem of Whitman [7] (cf. also Birkhoff [1], Chap. IV, § 9) there exists set I such that the lattice $E(I)$ does not fulfil the identity (1). Assume that $\leqq$ is the trivial partial order on I (i.e., for i_1 and i_2 from I we have $i_1 \leqq i_2$ if and only if $i_1 = i_2$). For each $i \in I$ let G_i be a nonzero linearly ordered group; put $G = \Gamma_{i \in I} G_i$. Then G is, in fact, the direct product $G = \prod_{i \in I} G_i$. Let α be the identity on G; hence $\alpha \in M$. We have $R(I) = E(I)$, thus $R(I)$ does not satisfy the identity (1). We have verified above, that φ_1 is an isomorphism of $[\alpha^*]$ onto $R(I)$. Hence $[\alpha^*]$ does not satisfy the identity (1), which is a contradiction.

The above proof also shows that in 3.3 "directed group" can be replaced by "lattice ordered group".

References

[1] *G. Birkhoff:* Lattice theory, Providence 1967.

[2] *L. Fuchs:* Partially ordered algebraic systems, Oxford 1963.

[3] *J. Jakubík:* The mixed product decompositions of partially ordered groups. Czech. Math. J. *20*, 1970, 184–206.

[4] *J. Jakubík:* Lexicographic product decompositions of a linearly ordered group. Czech. Math. J. *36*, 1986, 553–563.

[5] *J. Jakubík:* Lexicographic products of partially ordered groupoids. Czech. Math. J. *14*, 1964, 281—305. (In Russian.)
[6] *A. I. Maltsev:* On ordered groups. Izv. Akad. Nauk SSSR, serija matem. *13*, 1949, 473—482. (In Russian.)
[7] *P. M. Whitman:* Lattices, equivalence relations and subgroups. Bull. Amer. Math. Soc. *52*, 1946, 507—522.

Author's address: 040 01 Košice, Ždanovova 6, Czechoslovakia (Matematický ústav SAV, dislokované pracovisko.)