Czechoslovak Mathematical Journal

Michael G. Charalambous

Further theory and applications of covering dimension of uniform spaces

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 3, 378–394

Persistent URL: http://dml.cz/dmlcz/102472

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

FURTHER THEORY AND APPLICATIONS OF COVERING DIMENSION OF UNIFORM SPACES

M. G. CHARALAMBOUS, Nicosia

(Received July 28, 1988)

1. INTRODUCTION AND DEFINITIONS

In [2] we introduced and studied a covering dimension function for uniform spaces which in this paper we denote by Dim. In [3] we obtained the inverse limit theorem for Dim and applied it to deduce several results for dim, the covering dimension function for topological spaces. Other results for Dim were given in [4]. The purpose of this paper is to present further results showing that the theory of Dim is not only interesting in its own right but also useful as a tool for deducing results for the covering dimension of topological spaces. We have already established subset, sum, product, and inverse limit theorems for Dim in complete generality, for all uniform spaces. In section 2, we show that from these results immediately follow some useful subset and sum theorems and all existing results concerning the covering dimension of limits of inverse sequences of topological spaces. Section 3 is devoted mainly to the proof of theorem 5, a factorisation theorem for Dim from which all standard factorisation theorems for dim follow and on which subsequent developments are based. In section 4, we obtain sufficient conditions under which the inequality dim $\leqq$ Dim holds. These are used in section 5 to deduce the most general existing results for dim concerning subspaces, products and inverse limits. Further sufficient conditions for the inequality dim $\leqq$ Dim are obtained in section 6, and these have a variety of results for dim as easy corollaries. Other applications of results recorded here appear in [5, 6].

Uniform spaces in this paper are taken to be Hausdorff and topological spaces to be uniformisable, i.e., Tychonoff. N denotes the set of positive integers, I the unit interval $[0, 1]$, R the space of real numbers, βX and wX denote the Stone-Čech compactification and the weight of a topological space X, respectively, $W(X)$ the weight of a uniform space X and $|X|$ the cardinality of a set X.

For standard results in General Topology and Dimension Theory, the reader is referred to [7, 8, 22].

A subset G of a uniform space X is called *uniformly open* if there is an open set H of a metric space Y and a uniformly continuous function $f: X \to Y$ with $G = f^{-1}(H)$.

It was shown in [2] that the collection of all uniformly open sets of X is a base of X and is closed under finite intersections and countable unions. Also, the uniformly open subsets of a subspace Y of X are precisely those of the form $Y \cap G$ where G is uniformly open in X. Complements of uniformly open sets are called *uniformly closed*. Dim is defined as follows. $\operatorname{Dim} X = -1$ iff $X = \emptyset$ and, for n in $N \cup \{0\}$, $\operatorname{Dim} X \leqq n$ iff every finite uniformly open cover of X has a finite uniformly open refinement of order $\leqq n$. $\operatorname{Dim} X = n$ if $\operatorname{Dim} X \leqq n$ and $\operatorname{Dim} X \leqq n-1$ does not hold. If $\operatorname{Dim} X \leqq n$ for no n, we set $\operatorname{Dim} X = \infty$. If every cozero subset of X is uniformly open then $\operatorname{Dim} X = \dim X$. Thus this equality holds if a topological space is equipped with its Stone-Čech uniformity (i.e. that inherited from βX) [cf. 7, p. 472] or if X is a uniform space with Lindelöf topology or X is a metric space (with uniformity that induced by its metric).

We remark that the notation adopted here is slightly different from that employed in [2, 3, 4]. If $\mathscr{U}$ is the uniformity of X, the uniformly open and the uniformly closed sets of X were, respectively, called *$\mathscr{U}$-open* and *$\mathscr{U}$-closed* in X and Dim X was denoted by $\mathscr{U}$-dim X. Occassionally, it is convenient to revert to the old notation.

2. SOME APPLICATIONS

We first state the subset, sum and inverse limit theorems for Dim as we will repeatedly refer to them in the sequel.

Theorem 1. *For any subspace Y of a uniform space X,* $\operatorname{Dim} Y \leqq \operatorname{Dim} X$ [2, proposition 3].

Theorem 2. *If a uniform space X is the union of uniformly closed subspaces A_i with* $\operatorname{Dim} A_i \leqq n$, $i = 1, 2, 3, \dots$, *then* $\operatorname{Dim} X \leqq n$ [2, proposition 4].

Theorem 3. *If in the category of uniform spaces and uniformly continuous functions X is the limit of an inverse system $(X_\alpha, f_{\alpha\beta}; \Lambda)$ with* $\operatorname{Dim} X_\alpha \leqq n$ *for each of α in Λ, then* $\operatorname{Dim} X \leqq n$ [3, Theorem].

The following two results will be needed in the sequel. Recall that a subspace Y of a topological space X is said to be *z-embedded* in X if every cozero set of Y is of the form $Y \cap G$ for some cozero set G of X. Closed subspaces of normal spaces, arbitrary subspaces of perfectly normal spaces and Lindelöf or cozero subspaces of arbitrary spaces are z-embedded.

Proposition 1. *If Y is z-embedded in a topological space X, then* $\dim Y \leqq \dim X$ [14, theorems 1.1 and 1.3; 12, theorem 5.16].

Proof. Let X carry its Stone-Čech uniformity and Y the induced subspace uniformity. Then every cozero set of Y being of the form $G \cap Y$, where G is uniformly open in X, is uniformly open in Y. Hence $\dim Y = \operatorname{Dim} Y$, $\dim X = \operatorname{Dim} X$ and, by theorem 1, $\dim Y \leqq \dim X$.

Proposition 2. *Let* $\mathscr{G} = \{G_{i\alpha}: i \in N,\ \alpha \in \Lambda\}$ *be a σ-locally finite cozero cover of a topological space* X *with* $\dim G_{i\alpha} \leqq n$ *for each* $i \in N$ *and* $\alpha \in \Lambda$. *Then* $\dim X \leqq n$ [14, theorem 2.5; 12, theorem 7.3].

Proof. $\mathscr{G}$ is the inverse image of an open cover of a metric space under a continuous function and hence it has a σ-discrete refinement consisting of cozero sets of X, each of which, in view of proposition 1, has $\dim \leqq n$. We may thus assume that each $\{G_{i\alpha}: \alpha \in \Lambda\}$ is discrete from which it follows that the cozero set $G_i = \bigcup (G_{i\alpha}: \alpha \in \Lambda)$ of X has $\dim \leqq n$. Let $G_i = \bigcup_{j=1}^{\infty} F_{ij}$ where each F_{ij} is a zero set of X and equip X with its Stone-Čech uniformity. Then each F_{ij} is uniformly closed in X, by theorem 1, $\operatorname{Dim} F_{ij} \leqq \operatorname{Dim} G_i = \dim G_i \leqq n$ and, by theorem 2, $\dim X = \operatorname{Dim} X \leqq n$ since evidently $X = \bigcup_{i,j=1}^{\infty} F_{ij}$.

If a topological space X is the inverse limit of spaces X_α, $\alpha \in \Lambda$, a cozero cylinder of X is a set of form $\pi_\alpha^{-1}(G)$ where $\pi_\alpha: X \to X_\alpha$ is the canonical projection and G is a cozero set of X_α. The following result contains all cases of known results concerning the covering dimension of limits of inverse sequences of topological spaces.

Proposition 3. *Let* X *be the limit of an inverse sequence* $(X_i, f_{ij}; N)$ *of topological spaces and continuous functions such that* $\dim X_i \leqq n$ *and each cozero set of* X *is the countable union of cozero cylinders. Then* $\dim X \leqq n$.

Proof. Let each X_i carry its Stone-Čech uniformity and X the resulting inverse limit uniformity. Since evidently cozero cylinders are uniformity open in X and uniformly open sets are closed with respect to countable unions, then every cozero set of X is uniformly open so that $\dim X = \operatorname{Dim} X$ as well as $\dim X_i = \operatorname{Dim} X_i$. Theorem 3 now implies that $\dim X \leqq n$.

Nagami [14, theorem 4.1] and Pasynkov [18, theorem 1] have proved the inverse limit theorem for dim for perforable sequences of normal spaces, and this result incorporates all known such results [18, corollary 1]. In a perforable sequence of normal spaces, it is readily seen that every countable open cover of the limit space X has a countable refinement consisting of cozero cylinders (see the proof of proposition 2 in [26]). From this and the standard properties of cozero sets, it readily follows that, as noted in [18], X is normal and countably paracompact. Also, if $G = \bigcup_{j=1}^{\infty} F_i$ where G is a cozero and each F_i a zero set of X, then, for each i, there are cozero cylinders G_{ij}, $j \in N$, such that $F_i \subset \bigcup_{j=1}^{\infty} G_{ij} \subset G$. Hence $G = \bigcup_{i,j=1}^{\infty} G_{ij}$ and proposition 3 is applicable.

We conclude this section by deriving from theorem 3 Nagami's original and most useful result on inverse sequences [13].

Proposition 4. *Let* M *be the limit space of an inverse sequence* $(M_i, f_{ij}; N)$ *of continuous functions and metrisable spaces with* $\dim M_i \leqq n$ *for each* i *in* N. *Then* $\dim M \leqq n$.

Proof. Since $\dim X = \operatorname{Dim} X$ for a metric space X, theorem 3 yields the result if we inductively equip each M_i, $i \geqq 2$, with a compatible metric that makes $f_{i,i-1}$: $M_i \to M_{i-1}$ uniformly continuous. M_1 can have any metric compatible with its topology, and M is, of course, given the inverse limit metric.

3. SOME FACTORISATION THEOREMS

Lemma 1. *Every uniformly open cover $\{G_i: i \in N\}$ of a uniform space X with $\operatorname{Dim} X \leqq n$ has a uniformly open shrinking $\{H_i: i \in N\}$ of order $\leqq n$.*

Proof. It suffices to construct a uniformly open refinement $\{V_i: i \in N\}$ of $\{G_i: i \in N\}$ of order $\leqq n$. For if $\phi: N \to N$ is a function such that $V_i \subset G_{\phi(i)}$, we may let $H_i = \bigcup (V_j: \phi(j) = i)$.

For each i in N, there is a uniformly continuous function $f_i: X \to I$ with $G_i = f_i^{-1}(0, 1]$ [2]. Let $f = \Delta_{i=1}^{\infty} f_i: X \to I^N$ and $\pi_i: I^N \to I$ be the ith projection. The cozero cover $\{\pi_i^{-1}(0, 1]: i \in N\}$ of $I^N - \{0\}$ has a star-finite cozero refinement $\{U_i: i \in N\}$ [7, lemma 5.2.4]. For each i in N, $N_i = \{j \geqq i: U_j \cap U_i \neq \emptyset\}$ is finite, and $\{f^{-1}(U_i): i \in N\}$ is a uniformly open refinement of $\{G_i: i \in N\}$.

For a subspace Y of X, $\operatorname{Dim} Y \leqq \operatorname{Dim} X \leqq n$, and so every finite uniformly open cover of Y has a uniformly open shrinking of order $\leqq n$. It follows that we can construct by induction on i a uniformly open cover $\{V_{i,j}: j \in N\}$ of X such that $V_{i,j} \subset V_{i-1,j}$, $V_{0,j} = f^{-1}(U_j)$, $\{V_{i,j}: j \in N_i\}$ has order $\leqq n$ and $V_{ij} = V_{i-1,j}$ for $j \notin N_i$. Finally, letting $V_i = V_{i,i}$, $\{V_i: i \in N\}$ is a uniformly open refinement of $\{G_i: i \in N\}$ of order $\leqq n$.

Lemma 2. *Let A be a subspace of a uniform space X with $\operatorname{Dim} A \leqq n$, $f: X \to Y$ a uniformly continuous function into a metric space Y and $\mathscr{U}$ an open cover of $f(A)$. Then there exist a uniformly continuous $g: X \to Y \times I^N$ such that $\pi \circ g = f$, where $\pi: Y \times I^N \to Y$ is the canonical projection, and an open refinement $\mathscr{V}$ of order $\leqq n$ of the open cover $\pi^{-1}(\mathscr{U}) \cap g(A)$ of $g(A)$.*

Proof. Let $\{U_{i\lambda}: i \in N, \lambda \in \Lambda\}$ be an open refinement of $\mathscr{U}$ in $f(A)$ where each $\{U_{i\lambda}: \lambda \in \Lambda\}$ is discrete. Set $U_i = \bigcup (U_{i\lambda}: \lambda \in \Lambda)$. By lemma 1, the uniformly open cover $\{f^{-1}(U_i) \cap A: i \in N\}$ of A has a uniformly open shrinking $\{H_i: i \in N\}$ of order $\leqq n$. For each i in N, there is a uniformly open set V_i of X such that $H_i = A \cap V_i$ [2]. Let $g_i: X \to I$ be a uniformly continuous function such that $V_i = g_i^{-1}(0, 1]$ and set $g = f \,\Delta\, \Delta_{i=1}^{\infty} g_i$. If π_i denotes the projection of $Y \times I^N$ into its $(i + 1)$th factor, then $g^{-1}\pi_i^{-1}(0, 1] = g_i^{-1}(0, 1] = V_i$ and $\{\pi_i^{-1}(0, 1] \cap g(A): i \in N\}$ is an open shrinking of the open cover $\{\pi^{-1}(U_i) \cap g(A): i \in N\}$ of $g(A)$ of order $\leqq n$. Finally, we may let

$$\mathscr{V} = \{\pi_i^{-1}(0, 1] \cap \pi^{-1}(U_{i\lambda}) \cap g(A): i \in N, \lambda \in \Lambda\}.$$

Remark 1. In lemma 2, if A is a uniformly open subspace of X, we may very well take $V_i = H_i$ for each i in N. Then for each $i \in N$ and $x \notin A$, $g_i(X) = 0$ and hence $\pi: g(X - A) \to Y$ is a uniform embeding.

Theorem 4. *Let A be a subspace of a uniform space X with* $\operatorname{Dim} A \leqq n$ *and* $f\colon X \to Y$ *a uniformly continuous function into a metric space Y. Then there is uniformly continuous function* $g\colon X \to Y \times I^N$ *such that* $\operatorname{Dim} g(A) \leqq n$ *and* $\pi \circ g = f$, *where π denotes the projection of $Y \times I^N$ onto Y.*

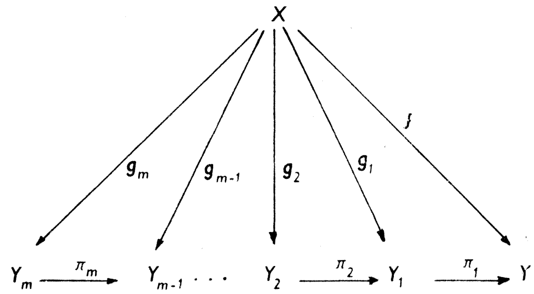

Fig. 1.

Proof. By repeated application of lemma 2, we obtain the commutative diagram of figure 1, where all functions are uniformly continuous and for each m in N, $Y_m = Y_{m-1} \times I^N$, $Y_0 = Y$, and π_m denotes canonical projection. We also obtain open covers $\mathscr{U}_m$ and $\mathscr{V}_m$ of $g_m(A)$ such that order $\mathscr{V}_m \leqq n$, mesh $\mathscr{U}_m < 1/m$, $\mathscr{V}_m$ refines $\pi_m^{-1}(\mathscr{U}_{m-1})$ and $\mathscr{U}_m$ refines $\mathscr{V}_m$ and $(\pi_m \circ \ldots \circ \pi_i)^{-1}$ $(\mathscr{S}_{im})$ for each $1 \leqq i \leqq m$, where $\mathscr{S}_{im}$ is the open cover of $g_{i-1}(A)$ consisting of open balls of diameter $1/m$, $g_0 = f$ and $\mathscr{U}_0$ is any open cover of $f(A)$.

Clearly, for each m in N, $Y_m = Y \times Z_m$, where Z_m is a copy of I^N, $g_m = f \Delta h_m$, where $h_m\colon X \to Z_m$ is uniformly continuous and $Y \times I^N$ is homeomorphic with $Y \times \prod_{m=1}^{\infty} Z_m$. Let $g = f \Delta \Delta_{m=1}^{\infty} h_m$. We assume that $Y, Z_1, Z_2, \ldots$, respectively carry metrics $d_0, d_1, d_2, \ldots$ each of which is bounded above by 1, that the metric on $Y_m = Y \times Z_m$ is given by

$$e_m(x, y) = \max \{d_0(x_1, y_1),\ d_m(x_2, y_2)\}$$

and that the metric on $Y \times \prod_{m=1}^{\infty} Z_m$ is given by

$$d(x, y) = \sup \left\{\frac{1}{i+1}\, d_i(x_i, y_i)\colon i = 0, 1, 2, \ldots\right\}.$$

Then, if σ_m denotes the natural projection from $Y \times \prod_{m=1}^{\infty} Z_m$ onto $Y_m = Y \times Z_m$, it is readily checked that $\pi_m \circ \sigma_m \circ g = \sigma_{m-1} \circ g = g_{m-1}$ and that $\{\sigma_m^{-1}(\mathscr{V}_m) \cap \cap\, g(A)\colon m \in N\}$ is a sequence of open covers of $g(A)$ each of which has order $\leqq n$ and refines its predecessor and, moreover, $\lim_{m\to\infty} \operatorname{mesh} \sigma_m^{-1}(\mathscr{V}_m) \cap g(A) = 0$. This suffices to conclude that $\dim g(A) \leqq n$ [25].

Remark 2. As in remark 1, if A is a uniformly open subspace of X in theorem 4, we may assume that π: $g(X - A) \to Y$ is a uniform embedding.

Lemma 3. *Let M be a subset of N, A a subspace of a uniform space X with* $\operatorname{Dim} A \leqq n$ *and, for each m in M, $f_m: X \to Y_m$ a uniformly continuous function into a metric space Y_m. Then there is a uniformly continuous $g: X \to (\prod_{m \in M} Y_m) \times I^N$ such that* $\operatorname{Dim} g(A) \leqq n$ *and $\pi_m \circ g = f_m$ for each m in M, where π_m denotes the natural projection from $(\prod_{m \in M} Y_m) \times I^N$ onto Y_m.*

Proof. This is a straightforward application of Theorem 4, where we take $Y = \prod_{m \in M} Y_m$ and $f = \Delta_{m \in M} f_m$.

Theorem 5. *Let $f: X \to Y$ be a uniformly continuous function, τ a cardinal number with $W(Y) \leqq \tau$ and $\{X_\lambda: \lambda < \tau\}$ a collection of subspaces of X. Then there exists a uniformly continuous $g: X \to Y \times I^\tau$ such that $\pi \circ g = f$, where π is the projection of $Y \times I^\tau$ onto Y, and* $\operatorname{Dim} g(X_\lambda) \leqq \operatorname{Dim} X_\lambda$ *for each $\lambda < \tau$.*

Proof. We may assume that τ is infinite and that $Y = \prod_{\lambda < \tau} M_\lambda$ where each M_λ is a metric space [cf. 7, remark 8.2.4], for if τ is finite, then Y is discrete and the result is evident. Let $\{J_\lambda: \lambda < \tau\}$ be a partition of J, the set of all ordinals less than τ, into τ disjoint cofinal classes. The set K of all finite and non-empty subsets of J becomes a directed set if we define $\alpha < \beta$ to mean that α is a proper subset of β. Furthermore for each $\lambda < \tau$, $K_\lambda = \{\alpha \in K: \max \alpha \in J_\lambda\}$ is a cofinal directed subset of K. For $\alpha \in K$, let $M_\alpha = \prod_{\lambda \in \alpha} M_\lambda$ and for $\beta < \alpha$ let $\pi_{\alpha\beta}$ denote the canonical projection from M_α onto M_β. In the category of uniform spaces and uniformly continuous functions, $(M_\alpha, \pi_{\alpha\beta}; K)$ is an inverse limit system with limit Y.

For α, β in K with $\beta < \alpha$ we construct the commutative diagram of figure 2, where all the functions are uniformly continuous, each π_α, σ_α, ϱ_α, $\sigma_{\alpha\beta}$ and $\varrho_{\alpha\beta}$ denotes

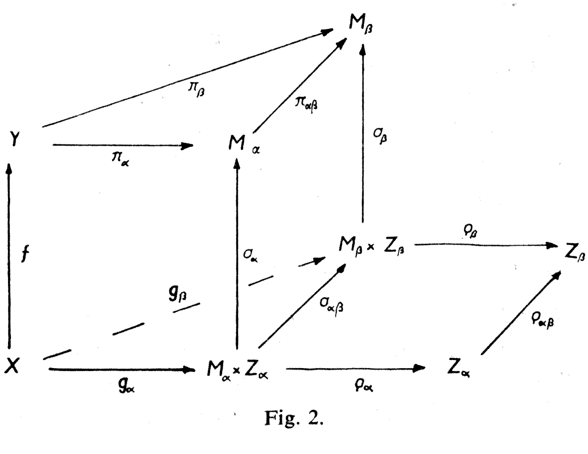

Fig. 2.

projection from a product onto a subproduct, each Z_α is a copy of I^N with $Z_\alpha = (\prod_{\gamma<\alpha} Z_\gamma) \times I^N$ and $\operatorname{Dim} g_\alpha(X_\lambda) \leqq \operatorname{Dim} X_\lambda$ when $\alpha \in K_\lambda$. The construction is by induction on $|\alpha|$. For $|\alpha| = 1$, the construction is a straightforward application of theorem 4, where we take "f" to be $\pi_\alpha \circ f$ and „A" to be the subspace X_λ of X for which $\alpha \in K_\lambda$. Now suppose that the construction has been completed for all β with $|\beta| < |\alpha|$, in particular, for all β with $\beta < \alpha$. Lemma 3 applied to $\pi_\alpha \circ f$ and $\varrho_\beta \circ g_\beta$, $\beta < \alpha$, gives a uniformly continuous $g_\alpha: X \to M_\alpha \times Z_\alpha$ such that $\operatorname{Dim} g_\alpha(X_\lambda) \leqq \operatorname{Dim} X_\lambda$ if $\alpha \in K_\lambda$, $\sigma_\alpha \circ g_\alpha = \pi_\alpha \circ f$ and, for $\beta < \alpha$, $\varrho_\beta \circ \sigma_{\alpha\beta} \circ g_\alpha = \varrho_\beta \circ g_\beta$. It is readily checked that g_α has the required properties and the construction is complete.

Clearly, we have inverse limit systems $(Z_\alpha, \varrho_{\alpha\beta}; K)$ and $(M_\alpha \times Z_\alpha, \sigma_{\alpha\beta}; K)$ with respective limits Z, a closed subspace of I^τ, and $Y \times Z$. Also, the g_α's induce a uniformly continuous $g: X \to Y \times Z$ such that $\pi \circ g = f$. Finally, for $\lambda < \tau$, $(g_\alpha(X_\lambda), \sigma_{\alpha\beta}; K_\lambda)$ is an inverse limit system with limit a subspace A_λ of $Y \times Z$ containing $g(X_\lambda)$. By the inverse limit theorem for Dim, $\operatorname{Dim} A_\lambda \leqq \operatorname{Dim} X_\lambda$ and, by the subset theorem for Dim, $\operatorname{Dim} g(X_\lambda) \leqq \operatorname{Dim} X_\lambda$.

Remark 3. If for each $\lambda < \tau$, $X_\lambda = A$, a uniformly open subset of X, it can be seen that, as in remark 1 and 2, we may assume each σ_α to be uniform embedding on $g_\alpha(X - A)$. Hence π may be taken to be a uniform isomorphism on $g(X - A)$.

Theorem 5 is a common generalisation of several known factorisation theorems for dim. We first deduce a result that generalises theorem 3 of [1] and theorem 1 of [15], where only normal spaces are considered.

Proposition 5. *Let $X_1, X_2, \dots$ be z-embedded subspaces of a topological space X. Let $f: X \to Y$ be a continuous function into a metric space Y. Then there exist a metric space Z and continuous functions $g: X \to Z$ and $h: Z \to Y$ such that $h \circ g = f$, $\dim g(X_i) \leqq \dim X_i$ for each i in N and $w\, g(A) \leqq w\, f(A)$ for every subspace A of X with $f(A)$ infinite.*

Proof. We turn X into a uniform space, equipping it with its Stone-Čech uniformity. Theorem 5 with $\tau = \aleph_0$ provides uniformly continuous functions $g: X \to Y \times I^N$ and $\pi: Y \times I^N \to Y$ such that $f = \pi \circ g$ and $\operatorname{Dim} g(X_i) \leqq \operatorname{Dim} X_i$ for each i in N. A cozero set of X_i, $i \in N$, is of the form $G \cap X_i$ for some uniformly open set G of X. Hence every cozero set of X_i is uniformly open and $\operatorname{Dim} X_i = \dim X_i$. Since $g(X_i)$ is evidently metric, $\operatorname{Dim} g(X_i) = \dim g(X_i)$, and the result follows if we let $Z = Y \times I^N$ and $h = \pi$.

A similar argument proves the following result.

Proposition 6. *Let $\{X_\alpha: \alpha < \tau\}$ be a collection of Lindelöf subspaces of a space X, where τ is a cardinal number. Let $f: X \to Y$ be a continuous function into a space Y with $w(Y) \leqq \tau$. Then there exist a space Z with $w(Z) \leqq \tau$ and continuous functions $g: X \to Z$ and $h: Z \to Y$ such that $h \circ g = f$, and $\dim g(X_\alpha) \leqq \dim X_\alpha$ for $\alpha < \tau$.*

If in proposition 5 we take X to be compact and each X_α closed in X, then we obtain theorem 2 of [1]. Some applications of theorem 5, proposition 4 and 5, and theorem 9 of section 6 can be found in [5].

4. THE INEQUALITY dim $\leqq$ Dim

It is evidently true that $\dim X \leqq \operatorname{Dim} X$ if X is metric or if the topology of X is Lindelöf or, more generally, if it has the monotonicity property with respect to dim [3]. However, in general this inequality is false. For if X, Y are topological spaces with $X \subset Y$ and $\dim Y < \dim X$, and Y is given its Stone-Čech uniformity, then by the subset theorem for Dim, $\operatorname{Dim} X \leqq \operatorname{Dim} Y = \dim Y$ and hence $\operatorname{Dim} X < < \dim X$. The following result provides a sufficient and useful condition under which the inequality above holds. Fot this result, we adopt the notation that for a given set Λ, Λ^* denotes the set of finite non-empty subsets of Λ directed by strict set inclusion.

Theorem 6. *Let X be a uniform space satisfying the condition that every finite cozero cover of X can be refined by a σ-locally finite cover consisting of uniformly open sets of X with* Dim $\leqq n$. *Then* $\dim X \leqq n$.

Proof. Let $\mathscr{G} = \{G_i : i = 1, 2, \ldots, r\}$ be a finite cozero cover of X. For each $i \in N$, let $\omega_i = \{0_\lambda : \lambda \in \Lambda_i\}$ be a locally finite collection of uniformly open sets of X with $\operatorname{Dim} 0_\lambda \leqq n$, $\Lambda_i \subset \Lambda_{i+1}$ and $\omega = \bigcup_{i=1}^{\infty} \omega_i$ a refinement of $\mathscr{G}$. For each λ in $\Lambda = \bigcup_{i=1}^{\infty} \Lambda_i$, let $f_\lambda : X \to I_\lambda$, where $I_\lambda = I$, be uniformly continuous with $f_\lambda^{-1}(0, 1] = 0_\lambda$.

We construct for each $\alpha \in \Lambda^*$

(1) a separable metric space Y_α and an open subset V_α of Y_α with $\dim V_\alpha \leqq n$,

(2) a uniformly continuous surjection $f_\alpha : X \to Y_\alpha$ with $f_\alpha^{-1}(V_\alpha) = 0_\alpha = \bigcap_{\lambda \in \alpha} 0_\lambda$, and

(3) a continuous surjection $\pi_{\alpha\beta} : Y_\alpha \to Y_\beta$ for $\beta < \alpha$ with $\pi_{\alpha\beta} \circ f_\alpha = f_\beta$ and $\pi_{\alpha\beta} : f_\alpha(A) \to f_\beta(A)$ a homeomorphism whenever $A \cap 0_\lambda = \emptyset$ for $\lambda \notin \beta$.

The construction is by induction on $|\alpha|$. Assuming the construction has been completed for all β with $|\beta| < |\alpha|$, where $|\alpha| > 1$, let $Y = \prod_{\beta < \alpha} Y_\beta$ and $f = \Delta_{\beta < \alpha} f_\beta$. By theorem 1, $\dim 0_\alpha \leqq n$ and, by theorem 4 and remark 2, there is a uniformly continuous $g : X \to Y \times I^N$ such that $\dim g(0_\alpha) \leqq n$, $f = \pi \circ g$, where π is the projection from $Y \times I^N$ onto Y, and $\pi : g(A) \to f(A)$ is a homeomorphism if $A \cap 0_\alpha = \emptyset$. Letting $Y_\alpha = g(X)$, $f_\alpha = g : X \to Y_\alpha$, $V_\alpha = f_\alpha(0_\alpha)$ and $\pi_{\alpha\beta} = \pi_\beta \circ \pi$, where $\pi_\beta : Y \to Y_\beta$ denotes canonical projection, it can be verified that (1), (2) and (3) hold. If $\alpha = \{\lambda\}$, $\lambda \in \Lambda$, we simply apply the same construction as above taking $f = f_\lambda : X \to I_\lambda$. Note that the restriction π_λ to $Y_{\{\lambda\}}$ of the projection $I_\lambda \times I^N \to I_\lambda$ satisfies $f_\lambda = \pi_\lambda \circ f_{\{\lambda\}}$.

At this point we can invoke [19, proposition 9] to deduce the existence of an ω-map from X into a metric space with $\dim \leqq n$, from which the result follows. For the sake of completeness, however, we give an outline of the rest of the proof.

For each i in N, let Y_i be the limit space of the inverse system $(Y_\alpha, \pi_{\alpha\beta}; \Lambda_i^*)$. Let $\pi_{i\alpha}: Y_i \to Y_\alpha$ and $\pi_{ij}: Y_i \to Y_j$, $j < i$, $\alpha \in \Lambda_i^*$, denote canonical projections, and $f_i: X \to Y_i$ the map induced by $\{f_\alpha: \alpha \in \Lambda_i^*\}$. Let M_i consist of all points of $\Pi(I_\lambda: \lambda \in \Lambda_i)$ with only a finite number of non-zero coordinates with metric d_i defined by

$$d_i(x, y) = \sup\{|x_\lambda - y_\lambda|: \lambda \in \Lambda_i\}.$$

The collections $\{f_\lambda: \lambda \in \Lambda_i\}$, $\{\pi_\lambda \circ \pi_{i\{\lambda\}}: \lambda \in \Lambda_i\}$ induce, respectively, $g_i: X \to M_i$ and $h_i: f_i(X) \to M_i$ with $h_i \circ f_i = g_i$. Because ω_i is locally finite, g_i is continuous, and h_i is continuous on $f_i(A)$ provided A intersects only a finite number of elements of ω_i. Henceforth, Z_i will denote the underlying set of $f_i(X)$ with topology generated by sets of the form $h_i^{-1}(G) \cap H$ with G open in M_i and H open in $f_i(X)$.

Fixing i in N, for each α in Λ_i^*, $P_\alpha = \{x \in M_i: x_\lambda \neq 0 \text{ for } \lambda \in \alpha\}$ is open in M_i with $g_i^{-1}(P_\alpha) = 0_\alpha$. For $k = 0, 1, 2, \ldots$, let E_k be the closed subset of M_i consisting of points with at most k non-zero coordinates. If $A \subset h_i^{-1}((E_k - E_{k-1}) \cap P_\alpha)$ and $|\alpha| = k \geqq 1$, it follows from (3) that A, both as a subspace of Y_i and Z_i, is homeomorphic with a subspace of 0_α in Y_α so that A is metric separable with $\dim \leqq n$. From the fact that $\{E_k - E_{k-1} \cap P_\alpha: \alpha \in \Lambda_i^*, |\alpha| = k\}$ is a discrete open cover of $E_k - E_{k-1}$, $\{E_k: k = 0, 1, 2, \ldots\}$ is a closed cover of the metric space M_i and $h_i: Z_i \to M_i$ is continuous it can be deduced that Z_i has a σ-discrete cozero cover with a σ-discrete closed shrinking $\mathscr{F}$ every element of which other than the singleton $h_i^{-1}(0)$ is contained in some $h_i^{-1}(E_k - E_{k-1}) \cap P_\alpha$ with $|\alpha| = k \geqq 1$. Hence every member of $\mathscr{F}$ is separable metric with $\dim \leqq n$. It follows in turn that each open subset of Z_i is cozero, Z_i is perfectly normal and $\dim Z_i \leqq n$.

For $j < i$, $\pi_{ij}: Z_i \to Z_j$ is continuous and $(Z_i, \pi_{ij}; N)$ is an inverse sequence with limit space a perfectly normal space Z with $\dim Z \leqq n$ [3, proposition 2]. We have continuous projections $\pi_i: Z \to Z_i$ and a continuous $f: X \to Z$ induced by $f_i: X \to Z_i$, $i \in N$. For $\lambda \in \Lambda$, let $P_\lambda = \{x \in M_i: x_\lambda \neq 0\}$ and $Q_\lambda = \pi_i^{-1}(h_i^{-1}(P_\lambda))$, where i is the first member of N with $\lambda \in \Lambda_i$. Then each Q_λ is open in Z with $f^{-1}(Q_\lambda) = 0_\lambda$. Let $\phi: \Lambda \to \{1, 2, \ldots, r\}$ be a function such that $0_\lambda \subset G_{\phi(\lambda)}$, $U_k = \bigcup(Q_\lambda: \phi(\lambda) = k)$ and $U = \bigcup_{k=1}^{r} U_k$. Then $\dim U \leqq \dim Z \leqq n$ and U has an open cover $\{V_k: k = 1, 2, \ldots, r\}$ of order $\leqq n$ with $V_k \subset U_k$. Finally, $\{f^{-1}(V_k): k = 1, 2, \ldots, r\}$ is an open shrinking of $\mathscr{G}$ of order $\leqq n$. Hence $\dim X \leqq n$.

Remark 4. The condition of theorem 6 is clearly equivalent to the requirement that each cozero set of X is the union of a σ-locally finite in X collection of uniformly open sets of X with $\operatorname{Dim} \leqq n$.

The following result will help to sharpen theorem 6. For the rest of this section, it is convenient to make use of the original notation of [2].

Theorem 7. *For every uniformly $\mathscr{U}$ on a topological space X there is a uniformity*

$\mathscr{V}$ on X finer than $\mathscr{U}$ such that $\mathscr{V}$-dim $Y \leqq \mathscr{U}$-dim Y for every subset Y of X and every clopen subset of a $\mathscr{U}$-open set of X is $\mathscr{V}$-open.

Proof. Let $\{(V_\alpha, U_\alpha): \alpha < \tau\}$ be the collection of all pairs (V, U) of subsets of X with U $\mathscr{U}$-open and V a clopen subset of U, where τ is an infinite cardinal. Let $f_\alpha: X \to I$ be a uniformly continuous function with respect to $\mathscr{U}$ such that $U_\alpha = f_\alpha^{-1}(0, 1]$ for each $\alpha < \tau$, and define a continuous function $g_\alpha: X \to R$ by

$$g_\alpha(x) = \begin{cases} f_\alpha(x) & \text{if } x \in V_\alpha \\ -f_\alpha(x) & \text{if } x \notin V_\alpha \end{cases}$$

Note that if g_α becomes uniformly continuous, then $V_\alpha = g_\alpha^{-1}(0, 1]$ becomes uniformly open.

For $\alpha < \tau$, let $\mathscr{V}_\alpha$ be the coarsest uniformity on X finer than $\mathscr{U}$ which makes uniformly continuous every continuous function $f: X \to M$ into a metric space M with $f \mid V_\beta$ and $f \mid X - V_\beta$ uniformly continuous with respect to $\mathscr{U}$ for some $\beta < \alpha$. Assume that for $Y \subset X$, $\mathscr{V}_\alpha$-dim $Y \leqq \mathscr{U}$-dim Y for all $\alpha < \beta$, where $\beta \leqq \tau$. If β is a limit ordinal, then V_β is the inverse limit of the uniformities $\mathscr{V}_\alpha$, $\alpha < \beta$, and since by hypothesis $\mathscr{V}_\alpha$-dim $Y \leqq \mathscr{U}$-dim Y for $Y \subset X$, by theorem 3, $\mathscr{V}_\beta$-dim $Y \leqq$ $\leqq \mathscr{U}$-dim Y. If $\beta = \alpha + 1$, then $\mathscr{V}_\beta$ is the uniformity on X whose uniform covers are precisely those that can be refined by a cover of the form $f^{-1}(\mathscr{C})$, where $f: X \to M$ is a continuous function into a metric space M with $f \mid V_\alpha$ and $f \mid X - V_\alpha$ uniformly continuous with respect to $\mathscr{V}_\alpha$ and $\mathscr{C}$ is a uniform cover of M. Clearly, $\mathscr{V}_\beta$ agrees with $\mathscr{V}_\alpha$ on both V_α and $X - V_\alpha$, g_α is uniformly continuous with respect to $\mathscr{V}_\beta$ and V_α is $\mathscr{V}_\beta$-open. Let $V_\alpha = \bigcup_{n=1}^{\infty} F_n$ where each F_n is $\mathscr{V}_\beta$-closed. For $Y \subset X$, if $Z =$ $= Y - V_\alpha$ or $Z = Y \cap F_n$, $n \in N$, then $\mathscr{V}_\beta$-dim $Z = \mathscr{V}_\alpha$-dim $Z \leqq \mathscr{U}$-dim $Z \leqq$ $\leqq \mathscr{U}$-dim Y, the last inequality being a consequence of the subset theorem for $\mathscr{U}$-dim. Hence, by the countable sum theorem for $\mathscr{V}_\beta$-dim, $\mathscr{V}_\beta$-dim $Y \leqq \mathscr{U}$-dim Y. Thus, transfinite induction readily implies that $\mathscr{V}_\alpha$-dim $Y \leqq \mathscr{U}$-dim Y for all $Y \subset X$ and all $\alpha \leqq \tau$. To complete the proof we need only to set $\mathscr{V} = \mathscr{V}_\tau$.

Remark 5. Let $\mathscr{U}_1$, $\mathscr{U}_2$ be uniformities on topological spaces X_1, X_2, and let $\mathscr{V}_1, \mathscr{V}_2$ be the corresponding uniformities constructed in the proof of theorem 7. If $f: (X_1, \mathscr{U}_1) \to (X_2, \mathscr{U}_2)$ is uniformly continuous, it is readily checked that f remains uniformly continuous as a function from $(X_1, \mathscr{V}_1)$ to $(X_2, \mathscr{V}_2)$.

The following result generalises theorem 6.

Theorem 8. *Let X be a uniform space satisfying the condition that every finite cozero cover of X can be refined by a σ-locally finite cover $\{G_\alpha: \alpha \in \Lambda\}$, where for each $\alpha \in \Lambda$, G_α is a clopen subset of a uniformly open set and* Dim $G_\alpha \leqq n$. *Then* dim $X \leqq n$.

Proof. Let $\mathscr{U}$ be the original uniformity on X and $\mathscr{V}$ the one provided by theorem 7. Now each G_α is uniformly open with respect to $\mathscr{V}$ and $\mathscr{V}$-dim $G_\alpha \leqq \mathscr{U}$-dim $G_\alpha \leqq n$. Clearly, theorem 6 applies to $(X, \mathscr{V})$ and we can conclude that dim $X \leqq n$.

Remark 6. Note that the condition of theorem 8 is equivalent to the requirement that every cozero set of X is the union of a σ-locally finite in X collection of sets with Dim $\leqq n$ each of which is a clopen set of some uniformly open set of X.

5. SUBSET, PRODUCT AND INVERSE LIMIT THEOREMS FOR COVERING DIMENSION

The following results are immediate corollaries of theorem 8. It appears that any result providing general conditions under which the inequality dim $\leqq$ Dim holds will imply corresponding results for the covering dimension of subsets, products and inverse limits. The fact that the product theorem for rectangular products was known as early as 1975 [17, theorem 1] while the corresponding result for inverse limits appeared in 1984 [26, Theorem] is a point in favour of Dim.

Proposition 7. If a subset X of a topological space Y is (n, d)-regular, then $\dim X \leqq \leqq n$ [20, theorem 1; 21, theorem 1].

Proof. If X is (n, d)-regular in Y, then an arbitrary cozero cover of X can be refined by a σ-locally finite in X cozero cover $\{V_\lambda : \lambda \in \Lambda\}$ of X such that, for each λ in Λ, there exists a cozero set U_λ of Y with $\dim U_\lambda \leqq n$ and V_λ clopen in $U_\lambda \cap X$. Let Y be equipped with its Stone-Čech uniformity. Since cozero sets are z-embedded, $\operatorname{Dim} U_\lambda = \dim U_\lambda \leqq n$ and by the subset theorem for Dim, $\operatorname{Dim} V_\lambda \leqq n$. Obviously, $U_\lambda \cap X$ is uniformly open in the subspace X of Y and theorem 8 applies to give $\dim X \leqq n$.

A finite topological product is called *rectangular* (resp. *piecewise rectangular*) if every finite cozero cover of it has a σ-locally finite refinement consisting of cozero rectangles (resp. clopen sets of cozero rectangles), a cozero rectangle being a product of cozero sets [17, 20]. An inverse system of topological spaces is called cylindrical (resp. piecewise cylindrical) if every finite cozero cover of its limit space has a σ-locally finite refinement consisting of cozero cylinders (resp. clopen subsets of cozero cylinders) [26].

Proposition 8. *If the topological product* $X = X_1 \times \ldots \times X_k$ *is piecewise rectangular and non-empty, then*

$$\dim X \leqq \dim X_1 + \ldots + \dim X_k$$

[17, theorem 1 and 20, theorem 4].

Proof. Let each X_i be equipped with its Stone-Čech uniformity and X with the resulting product uniformity. Then $\operatorname{Dim} X_i = \dim X_i$ and by the product theorem for Dim [4],

$$\operatorname{Dim} X \leqq \dim X_1 + \ldots + \dim X_k .$$

Noting that each cozero rectangle G is uniformly open in X and if $Y \subset X$, $\operatorname{Dim} Y \leqq$

$\leqq \operatorname{Dim} X$ by theorem 1, we see that the condition of theorem 8 is satisfied and hence $\dim X \leqq \dim X_1 + \ldots + \dim X_k$.

Proposition 9. *If an inverse limit system* $(X_\alpha, \pi_{\alpha\beta}; \Lambda)$ *of topological spaces with* $\dim X_\alpha \leqq n$ *for each* $\alpha \in \Lambda$ *is piecewise cylindrical, then its limit space* X *satisfies* $\dim X \leqq n$ [26, Theorem and 21, theorem 5].

Proof. For each α in Λ, let X_α carry its Stone-Čech uniformity and X the resulting inverse limit uniformity. Then $\operatorname{Dim} X_\alpha = \dim X_\alpha \leqq n$ for each α and by theorem 3, $\operatorname{Dim} X \leqq n$. Again, theorem 8 applies since a cozero cylinder G is uniformly open in X and $\operatorname{Dim} Y \leqq \operatorname{Dim} X \leqq n$ for $Y \subset X$, and hence $\dim X \leqq n$.

6. FURTHER APPLICATIONS

Further applications will follow from the following result.

Theorem 9. *Let* $f: X \to Y$ *be a perfect and uniformly continuous function and suppose that* Y *is paracompact and each cozero set of* Y *is the union of a* σ*-locally finite in* Y *collection of clopen subsets of uniformly open sets of* Y. *Then* $\dim X \leqq \operatorname{Dim} X$.

Proof. Firstly, in view of theorem 7 and remark 5, we may assume that each cozero set of Y is the union of a σ-locally finite in Y collection of uniformly open sets of Y. Secondly, if $\mathscr{U}$ is a uniformity on a topological space Z, the covers of Z that can be refined by finite $\mathscr{U}$-open covers of Z give rise to a precompact uniformity $\mathscr{V}$ on Z such that a subset of Z is $\mathscr{V}$-open iff it is $\mathscr{U}$-open [2, proposition 8]. We may therefore assume that the uniform covers of X (resp. Y) are those that can be refined by finite uniformly open covers of X (resp. Y).

Let $\hat{X}$ denote the completion of X, i and j the identity functions on X and $\hat{X}$ respectively and k the inclusion of X into $\hat{X}$. Then $g = (f \times j) \circ (i \Delta k): X \to Y \times \hat{X}$ is a uniformly continuous function and, since f is perfect, $g(X)$ is a closed subset of $Y \times \hat{X}$ homeomorphic with X. Also, since Y is paracompact and $\hat{X}$ is compact, then $Y \times \hat{X}$ is normal and $g(X)$ is z-embedded in $Y \times \hat{X}$. Hence, if G is a cozero set of X, there exists a cozero set H of $Y \times \hat{X}$ with $g(G) = g(X) \cap H$. Now, since the product $Y \times \hat{X}$ is rectangular [17, proposition 1], there exists a σ-locally finite in $Y \times \hat{X}$ collection $\{G_\alpha \times H_\alpha : \alpha \in A\}$ consisting of cozero rectangles whose union is H. Furthermore, for each α in A, there exists a σ-locally finite in Y collection $\{G_{\alpha\beta} : \beta \in B_\alpha\}$ of uniformly open sets of Y whose union is G_α. Now $\{G_{\alpha\beta} \times H_\alpha : \alpha \in A, \beta \in B_\alpha\}$ is a σ-locally finite in $Y \times \hat{X}$ oclection of uniformly open sets of $Y \times \hat{X}$ whose union is H and hence $\{g^{-1}(G_{\alpha\beta} \times H_\alpha) : \alpha \in A, \beta \in B_\alpha\}$ is a σ-locally finite in X collection of uniformly open sets of X whose union is G. Recalling that, by theorem 1, $\operatorname{Dim} Z \leqq \operatorname{Dim} X$ for every subset Z of X, we see that theorem 8 applies and gives $\dim X \leqq \operatorname{Dim} X$.

The following corollary of theorem 9 seems to be a new result.

Proposition 10. *Let $f: X \to X_0$ and $g: Y \to Y_0$ be perfect maps between non-empty topological spaces and suppose that the product $X_0 \times Y_0$ is piecewise rectangular and paracompact.*

Then
$$\dim X \times Y \leqq \dim X + \dim Y.$$

Proof. Let X_0, Y_0, X, Y be endowed with their Stone-Čech uniformities and $X_0 \times Y_0$, $X \times Y$ with the resulting product uniformities. Then each cozero set of the piecewise rectangular product $X_0 \times Y_0$ is the union of a σ-locally finite in $X_0 \times Y_0$ collection of clopen sets of cozero rectangles, which are uniformly open sets of $X_0 \times Y_0$. Hence theorem 9 applies to the perfect and uniformly continuous function $f \times g: X \times Y \to X_0 \times Y_0$ onto a paracompact space and gives $\dim X \times Y \leqq \leqq \operatorname{Dim} X \times Y$. The result follows since, by the product theorem for Dim [4], $\operatorname{Dim} X \times Y \leqq \operatorname{Dim} X + \operatorname{Dim} Y = \dim X + \dim Y$.

We describe below several situations where theorem 9 applies yielding mostly known results for dim.

Proposition 11. *If X and Y are non-empty paracompact p-spaces, then*
$$\dim X \times Y \leqq \dim X + \dim Y \qquad [9, 16].$$

Proof. If X, Y are paracompact p-space, there are perfect maps $f: X \to X_0$, $g: Y \to Y_0$ into metric spaces X_0, Y_0. Endow X_0, Y_0 with their metric uniformities, X, Y with the finest uniformities compatible with their topology and $X_0 \times Y_0$, $X \times Y$ with the resulting product uniformities. Then every cozero set of $X_0 \times Y_0$ is uniformly open and $f \times g: X \times Y \to X_0 \times Y_0$ is a perfect and uniformly continuous function into a paracompact space. Hence, by theorem 9, $\dim X \times Y \leqq \leqq \operatorname{Dim} X \times Y$. Now by the product theorem for Dim [4], $\operatorname{Dim} X \times Y \leqq \operatorname{Dim} X + \operatorname{Dim} Y = \dim X + \dim Y$. Hence $\dim X \times Y \leqq \dim X + \dim Y$.

Proposition 12. *Let $f: X \to Y$ be a perfect mapping into a completely paracompact space Y. Then there exist a completely paracompact space Z with $wZ \leqq wY$ and* $\dim Z \leqq \dim X$ *and perfect mappings $g: X \to Z$ and $h: Z \to Y$ such that $f = h \circ g$.*

Proof. Suppose $\tau = wY$ is infinite, consider Y as a subspace of the uniform space I^τ and endow X with its Stone-Čech uniformity. Then f is uniformly continuous and theorem 5 provides a subspace Z of I^τ and uniformly continuous $g: X \to Z$ and $h: Z \to Y$ such that $f = h \circ g$ and $\operatorname{Dim} g(X) \leqq \operatorname{Dim} X = \dim X$. We may clearly take Z to be $g(X)$, in which case g and h are perfect and Z is completely paracompact [22, proposition 2.5.9]. Finally, by [20, proposition 1], every cozero set of Y is the union of a σ-locally finite in Y collection of clopen sets of uniformly open sets of Y and we can apply theorem 9 to $h: Z \to Y$ to deduce $\dim Z \leqq \operatorname{Dim} Z \leqq \dim X$.

Remark 7. The above result can also be obtained by standard methods using the fact that completely paracompact spaces have the monotonicity property with respect to dim [3]. It is interesting to speculate whether the result holds when Y is merely paracompact.

Proposition 13. *For every paracompact space* X, *there is a space* Z *with* $\dim Z \leqq 0$ *and a perfect surjection* $f: Z \to X$ [24, *theorem* 2 *and* 22, *proposition* 6.3.15].

Proof. As βX can be embedded in a cube I^τ, there exists a continuous surjection $f: Y \to \beta X$, where Y is a closed subset of a product of copies of the Cantor discontinuum. Then $\dim Y \leqq 0$ and, if $Z = f^{-1}(X)$, $f: Z \to X$ is perfect and surjective. If we endow X, Y with their Stone-Čech uniformities, every cozero set of X is uniformly open, $\operatorname{Dim} Y = \dim Y \leqq 0$ and, by theorem 9, applied to $f: Z \to X$ and the subset theorem for Dim, $\dim Z \leqq \operatorname{Dim} Z \leqq \operatorname{Dim} Y \leqq 0$.

For a paracompact space X, $\Delta X \leqq n$ iff there exist a space Y with $\dim Y \leqq 0$ and a continuous and closed surjection $f: Y \to X$ of multiplicity $\leqq n + 1$ [22, proposition 6.3.8]. It was proved by Pears and Mack [23] that $\Delta X = \Delta\beta X$ for X paracompact, the inequality $\Delta\beta X \leqq \Delta X$ following from the fact that if a closed $f: X \to Y$ has multiplicity $\leqq n + 1$ then the same holds for its extension to Stone-Čech compactifications [22, proposition 6.4.9. and 19, proposition 3]. The following result more than establishes the reverse inequality.

Proposition 14. *Let* X, Y *be paracompact spaces with* $X \subset Y$. *Then* $\Delta X \leqq \Delta Y$ *if either* X *is z-embedded in* Y *or* X *is completely paracompact.*

Proof. Suppose $\Delta Y \leqq n$ and let $f: A \to Y$ be a closed surjection of multiplicity $\leqq n + 1$, where $\dim A \leqq 0$. Let $B = f^{-1}(X)$ and endow A and Y with their Stone-Čech uniformities. Then $\operatorname{Dim} A = \dim A \leqq 0$ and, by the subset theorem, $\operatorname{Dim} B \leqq \leqq 0$. Also, $f: B \to X$ is perfect and uniformly continuous and theorem 9 applies, giving $\dim B \leqq \operatorname{Dim} B \leqq 0$. Since also $f: B \to X$ has multiplicity $\leqq n + 1$, then $\Delta X \leqq n$.

The class of spaces that satisfy the conditions of the following result includes all finite-dimensional quotient spaces of locally compact groups [19, section 5]. A similar result was announced without proof by Leibo [11, theorem 3 and corollary 3].

Corollary. *If* $f: X \to M$ *is a closed continuous surjection from a paracompact space* X *into a metrisable space* M *with* $\dim f = 0$, *then* $\dim X = \operatorname{Ind} X = \Delta X$.

Proof. $\Delta X \leqq \Delta\beta X$ by proposition 14, $\Delta\beta X \leqq \dim X$ by [19, theorem 12] and the result follows since $\dim X \leqq \operatorname{Ind} X \leqq \Delta X$ for all paracompact spaces X [22].

Several other applications of theorem 9 exist and we will give two in which every cozero set of the range of f is uniformly open. For such situations it is sufficient for f to be continuous and closed and to have Lindelöf fibers. It is convenient to call such a function almost perfect. It is not hard to see that inverse images of Lindelöf spaces under almost perfect maps are Lindelöf and composites of almost perfect maps are almost perfect.

Theorem 10. *Let* $f: X \to Y$ *be an almost perfect uniformly continuous function into a (paracompact) space* Y *with the property that every open cover of* Y *has a* σ*-locally finite refinement consisting of clopen sets of uniformly open sets. Then* X *is paracompact and* $\dim X \leqq \operatorname{Dim} X$.

Proof. Let $\mathscr{G}$ be an open cover of X. For each y in Y, since $f^{-1}(y)$ is Lindelöf and uniformly open sets constitute a base for X, there are uniformly open sets $G_{1y}, G_{2y}, \ldots$ such that $f^{-1}(y) \subset G_y = \bigcup_{i=1}^{\infty} G_{iy}$ and each G_{iy} is contained in some member of $\mathscr{G}$. Since f is closed, there exists an open neighbourhood V_y of y with $f^{-1}(V_y) \subset G_y$. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a σ-locally finite refinement of the open cover $\{V_y : y \in Y\}$ of Y where each V_α is a clopen set of some uniformly open set of Y, and for each α fix a point $y(\alpha)$ of Y with $V_\alpha \subset V_{y(\alpha)}$. It is straightforward to verify that $\{f^{-1}(V_\alpha) \cap G_{iy(\alpha)} : \alpha \in \Lambda,\ i \in N\}$ is a σ-locally finite refinement of $\mathscr{G}$ each member of which is a clopen set of some uniformly open set of X. It follows that X is paracompact and the condition of theorem 8 is satisfied so that $\dim X \leqq \operatorname{Dim} X$.

The following result strengthens [16, corollary 3] and [10, corollary 1.2].

Proposition 15. *Let $(X_\alpha, \pi_{\alpha\beta}; \Lambda)$ be an inverse system of topological spaces with limit space X. If $\dim X_\alpha \leqq n$ for each α in Λ and, for some $\beta \in \Lambda$, X_β is paracompact and the canonical projection $\pi_\beta : X \to X_\beta$ is almost perfect, then $\dim X \leqq n$.*

Proof. Endow each X_α with its Stone-Čech uniformity and X with the resulting inverse limit uniformity. Then for each α, $\operatorname{Dim} X_\alpha = \dim X_\alpha \leqq n$ so that by theorem 3, $\operatorname{Dim} X \leqq n$. Now theorem 10 applies to $\pi_\beta : X \to X_\beta$ and gives $\dim X \leqq \leqq \operatorname{Dim} X \leqq n$.

We digress here to give a related result whose proof we base on proposition 9 and which generalises [10, theorem 1.1].

Proposition 16. *Let X be the limit space of an inverse system $(X_\alpha, \pi_{\alpha\beta}; \Lambda)$ of normal spaces with $\dim X_\alpha \leqq n$ and surjective canonical projection $\pi_\alpha : X \to X_\alpha$ for each α in Λ. Then X is normal and $\dim X \leqq n$ if, additionally, for some $\beta \in \Lambda$, X_β is $|\Lambda|$-paracompact and π_β is closed and satisfies*

(*) for each $x \in X_\beta$ and each open cover $\{G_\alpha : \alpha \in \Lambda\}$ of X with $G_{\alpha_1} \subset G_{\alpha_2}$ whenever $\alpha_1 < \alpha_2$, there exists some $\alpha \in \Lambda$ with $\pi_\beta^{-1}(x) \subset G_\alpha$.

Proof. Let $\{G_i : i \in M\}$ be a finite open cover of X and for each $i \in M$ and $\alpha \in \Lambda$, let $G_{i\alpha}$ be the biggest open set of X_α with $\pi_\alpha^{-1}(G_{i\alpha}) \subset G_i$, $G_\alpha = \bigcup (G_{i\alpha} : i \in M)$ and H_α the biggest open subset of X_β with $\pi_\beta^{-1}(H_\alpha) \subset \pi_\alpha^{-1}(G_\alpha)$. In view of (*) and the fact that π_β is closed, $\{H_\alpha : \alpha \in \Lambda\}$ is an open cover of the $|\Lambda|$-paracompact space X_β and so it has an open locally finite shrinking $\{V_\alpha : \alpha \in \Lambda\}$, which, since X_β is normal, has a closed shrinking $\{F_\alpha : \alpha \in \Lambda\}$. For $\beta < \alpha$, since π_α is surjective $\{\pi_{\alpha\beta}^{-1}(F_\alpha) \cap G_{i\alpha} : i \in M\}$ is an open cover of $\pi_{\alpha\beta}^{-1}(F_\alpha)$ which, since X_α is normal has a closed shrinking $\{E_{i\alpha} : i \in M\}$ so that we can insert a cozero set $U_{i\alpha}$ of X_α between $E_{i\alpha}$ and $\pi_{\alpha\beta}^{-1}(V_\alpha) \cap G_{i\alpha}$. It is readily verified that $\{\pi_\alpha^{-1}(U_{i\alpha}) : i \in M,\ \beta < \alpha\}$ is a locally finite refinement of $\{G_i : i \in M\}$ consisting of cozero cylinders. It follows that X is normal and the inverse system is cylindrical so that, by proposition 9, $\dim X \leqq n$.

Remark 8. The condition (*) holds if π_β is perfect or π_β is almost perfect and each countable subset of Λ has an upper bound in Λ.

The following example shows that the restrictions of paracompactness in proposition 15 and $|A|$-paracompactness in proposition 16 are not redundant.

Example. Let M be a subspace of $[0, \omega_1) \times I^n$ with $\dim M = n$ and $\operatorname{loc}\dim M = 0$ [22, proposition 5.4.5], where ω_1 is the first uncountable ordinal. For each $\alpha < \omega_1$, let M_α be the closure of $M \cap [0, \alpha] \times I^n$ in βM and $X_\alpha = M_\alpha \cup [0, \omega_1) \times \{0\}$. For $\beta < \alpha$, let $\pi_{\alpha\beta}: X_\alpha \to X_\beta$ be the unique map whose restriction to $M \cap X_\alpha$ sends (γ, x) to $(\gamma, 0)$ if $\beta < \gamma \leqq \alpha$ and to itself otherwise. Then $(X_\alpha, \pi_{\alpha\beta}; [0, \omega_1))$ is an inverse limit system, each X_α is countably paracompact and normal with $\dim X_\alpha = 0$ and each $\pi_{\alpha\beta}$ is perfect. For the limit space, however, $X = \bigcup_{\alpha < \omega_1} X_\alpha$ we have $\dim X = \dim M = n$ since $M \subset X \subset \beta M$.

We quote one last corollary of theorem 10. This generalises theorem 4 of [19], where the definitions of bwX, compact weight of X, and μwX, metric weight of X, were introduced. We call a space X an *almost paracompact p-space* if there is an almost perfect map from X onto a metric space.

Proposition 17. *Let $f: X \to Y$ be a continuous function into an almost paracompact p-space. Then there exists an almost paracompact p-space Z with $bwZ \leqq bwY$, $\mu wZ \leqq \mu wY$ and $\dim Z \leqq \dim X$ and continuous $g: X \to Z$ and $h: Z \to Y$ with $f = h \circ g$.*

Proposition 17 is a special case of [6, proposition 1]. Several other applications of results presented in this paper can be found in [5, 6].

References

[1] *A. Arhangel'skiĭ:* Factorization of mappings according to weight and dimension, Soviet Math. Dokl. *8* (1967) 731—734.

[2] *M. G. Charalambous:* A new covering dimension function for uniform spaces. J. London Math. Soc. (2), *11* (1975), 137—143.

[3] *M. G. Charalambous:* The dimension of inverse limits, Proc. Amer. Math. Soc. *58* (1976), 289—295.

[4] *M. G. Charalambous:* A note on the dimension of products, Proc. fourth Prague Topological Symposium (1977), 70—71.

[5] *M. G. Charalambous:* Universal spaces for locally finite-dimensional Tychonoff spaces, Proc. Amer. Math. Soc. *106* (1989), 507—514.

[6] *M. G. Charalambous:* The factorisation theorem for paracompact Σ-spaces, Comment. Math. Univ. Carolinae, *29*, 3 (1988), 411—420.

[7] *R. Engelking:* General Topology, Polish Scientific Publishers, Warsaw, 1977.

[8] *R. Engelking:* Dimension Theory, Polish Scientific Publishers, Warsaw, 1978.

[9] *V. V. Filippov:* On the dimension of normal spaces, Soviet Math. Dokl. *14* (1973), 547—550.

[10] *Y. Katuta:* On the covering dimension of inverse limits, Proc. Amer. Math. Soc. *84* (1982), 588—592.

[11] *I. M. Leibo:* On the dimension of certain spaces, Soviet Math. Dokl. *25* (1982), 20—22.

[12] *K. Morita:* Dimension of general topological spaces, Surveys in General Topology, Academic Press, New York (1980), 297—336.

[13] *K. Nagami:* Finite-to-one closed mappings and dimension II, Proc. Japan Acad. *35* (1959), 437—439.
[14] *K. Nagami:* Dimension of non-normal spaces, Fund. Math. *109* (1980), 113—121.
[15] *B. A. Pasynkov:* A factorization theorem for non-closed sets, Soviet Math. Dokl. *13* (1972), 292—295.
[16] *B. A. Pasynkov:* On the dimension of products of normal spaces Soviet Math. Dokl. *14*, (1973), 530—533.
[17] *B. A. Pasynkov:* On the dimension of rectangular products, Soviet Math. Dokl. *16* (1975), 344—347.
[18] *B. A. Pasynkov:* On the dimension of topological products and limits of inverse sequences, Soviet Math. Dokl. *22* (1980), 596—601.
[19] *B. A. Pasynkov:* Factorization theorems in dimension theory, Russian Myth. Surveys *36* (1981), 175—209.
[20] *B. A. Pasankov:* On the monotonicity of dimension, Soviet Math. Dokl. *26* (1982), 654—658.
[21] *B. A. Pasynkov:* On dimension theory, Aspects of Topology, Cambridge University Press, Cambridge, 1985, 227—250.
[22] *A. R. Pears:* Dimension Theory of General Spaces, Cambridge University Press, Cambridge, 1976.
[23] *A. R. Pears* and *J. Mack:* Closed covers, dimension and quasi-order spaces, Proc. London Math. Soc. (3) *29* (1974), 289—316.
[24] *V. Ponomarev:* On paracompact spaces and their continuous mappings, Soviet Math. Dokl. *3* (1962), 347—350.
[25] *P. Vopěnka:* Remarks on the dimension of metric spaces, Czechoslovak Math. J. *9* (1959), 519—522.
[26] *Y. Yajima:* On the dimension of limits of inverse systems, Proc. Amer. Math. Soc. *91* (1984), 461—466.
[27] *V. P. Zolotarev:* On the dimensions of subspaces, Moscow Univ. Math. Bull. *30* (1975), 7—8.

Author's address: 14, Philippou Street, Agios Dometios, Nicosia, Cyprus.