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STRONGLY PERFECT PRODUCTS OF GRAPHS

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In this note we present some necessary and sufficient conditions for certain graph products (namely cartesian, tensor, symmetrical difference) to be strongly perfect.

1. INTRODUCTION

Only simple (i.e. finite, undirected, without loops or multiple edges) graphs will be considered. For such a graph $G = (V, E)$, we shall denote: its vertex set by $V = V(G)$; its edge set by $E = E(G)$; its complement by $\bar{G}$; the family of all its maximal (with respect to set inclusion) cliques by $\mathscr{C}(G)$. Instead of $(x, y) \in E$ we shall also use $x \sim y$. By K_n, $n \geqq 2$, we mean the complete graph on n vertices, and C_n, $n \geqq 3$, denotes the chordless cycle on n vertices. A set $S \subset V(G)$ such that $|S \cap Q| = 1$ for any $Q \in \mathscr{C}(G)$ is called a *stable transversal* of G, [1]. If G and all its induced subgraphs have a stable transversal, then G is called *strongly perfect*, [1]. Let us notice that the complete graphs, the bipartite graphs are strongly perfect; there are strongly perfect graphs whose complement is not strongly perfect (e.g. C_6), and a disconnected graph is strongly perfect if and only if all its connected components are strongly perfect. This is the reason for which we shall deal, in the sequel, with connected graphs. A graph is called *perfect*, [2], if the chromatic number of each of its induced subgraphs H equals the number of vertices in the largest clique of H. A strongly perfect graph is also perfect, [1].

2. STRONGLY PERFECT CARTESIAN PRODUCT OF GRAPHS

Let $G_i = (V_i, E_i)$, $i = 1, 2$, be any two graphs. Their cartesian product is the graph $G = G_1 \times G_2$ having $V = V_1 \times V_2$ as its vertex set, and $(x_1, y_1) \sim (x_2, y_2)$ if and only if either (a) $x_1 = x_2$ and $y_1 \sim y_2$, or (b) $x_1 \sim x_2$ and $y_1 = y_2$.

It is not difficult to verify: (i) this product is commutative, associative; (ii) $G_1 \times G_2$ is connected if and only if both G_1 and G_2 are connected; (iii) if G_1 and G_2 are bipartite, then $G_1 \times G_2$ is bipartite, too, [4].

Lemma. *If* $|V(G_i)| > 1$, $i = 1, 2$, *then* $G_1 \times G_2$ *is strongly perfect if and only if both* G_1 *and* G_2 *are bipartite.*

Proof. Let $G = G_1 \times G_2$ be strongly perfect; then both G_1 and G_2, as induced subgraphs of G, are strongly perfect and perfect, too, [1]. Therefore, none of them admits chordless cycles C_{2n+1}, $n \geqq 2$, as induced subgraphs. In addition to the above, taking into account that $K_2 \times K_3 = \bar{C}_6$, we get that G_1 and G_2 are free of C_{2n+1}, $n \geqq 1$, that is, they are both bipartite.

Conversely, if G_1 and G_2 are bipartite, then $G_1 \times G_2$ is bipartite and, consequently, strongly perfect.

Remarks. (a) If $|V(G_1)| = 1$ then, clearly, $G_1 \times G_2$ is strongly perfect if and only if G_2 itself has this property (since G_2 and $G_1 \times G_2$ are isomorphic).

(b) The cartesian product of two disconnected graphs, each having a non-empty edge set, is strongly perfect if and only if their connected components are bipartite.

Proposition. $G = G_1 \times G_2 \times \ldots \times G_n$, $n \geqq 3$, *is strongly perfect if and only if one of the following assertions holds*:

(i) *all* G_i*'s* $(i = \overline{1, n})$ *have* $|V(G_i)| = 1$;

(ii) *there exists* G_{i_0} *with* $|V(G_{i_0})| \geqq 2$ *which is strongly perfect, and* $|V(G_i)| = 1$, $i \in \{1, 2, \ldots, n\} \setminus \{i_0\}$;

(iii) *all* G_i *with* $|V(G_i)| \geqq 2$ *are bipartite, and they are at least two such graphs.*

Proof. Let $G = G_1 \times G_2 \times \ldots \times G_n$, $n \geqq 3$, be strongly perfect; if $|V(G)| = 1$, then each G_i has $|V(G_i)| = 1$ $(i = \overline{1, n})$.

If $|V(G)| > 1$, there must be some G_i with $|V(G_i)| > 1$.

We have two cases:

1. There is only one graph, which we denote by G_{i_0}, with $|V(G_{i_0})| > 1$; then, obviously, G and G_{i_0} are isomorphic and, consequently, G_{i_0} is strongly perfect.

2. Let $G_1, G_2, \ldots, G_k$, $1 < k \leqq n$, satisfy $|V(G_i)| > 1$ $(i = \overline{1, k})$ (otherwise, by virtue of commutativity, we may change the indexes); then G and $H_k = G_1 \times \times G_2 \times \ldots \times G_k$ are isomorphic and so H_k is strongly perfect, too. As the cartesian product is associative, we may write $H_k = (G_1 \times G_2 \times \ldots \times G_{k-1}) \times G_k = H_{k-1} \times G_k$ and, according to the above lemma, both H_{k-1} and G_k are bipartite. Thus, $H_{k-1} = H_{k-2} \times G_{k-1}$ is strongly perfect and hence, using again the same lemma, we get that H_{k-2} and G_{k-1} are bipartite, and so on.

Conversely, let us notice that (i) and (ii) ensure, evidently, the strong perfectness of G. For (iii) we get that $G = G_1 \times G_2 \times \ldots \times G_n$ is strongly prefect since, as is easily seen, it is a bipartite graph.

Corollary. $H = G \times G \times \ldots \times G = G^n$, $n \geqq 2$, *is strongly perfect if and only if either* (i) $|V(G)| = 1$, *or* (ii) G *is bipartite.*

3. STRONGLY PERFECT TENSOR PRODUCT OF GRAPHS

The tensor product of graphs $G_i = (V_i, E_i)$, $i = 1, 2$, is the graph $G = G_1 \wedge G_2$ with $V(G) = V_1 \times V_2$, and $(x_1, y_1) \sim (x_2, y_2)$ if and only if both $x_1 \sim x_2$ and $y_1 \sim y_2$. It is easy to find out:

(i) this product is associative;

(ii) $G_1 \wedge G_2$ is not always connected, even when both G_1 and G_2 are connected (e.g. $K_2 \wedge K_2$);

(iii) if G_1 or G_2 is bipartite, then $G_1 \wedge G_2$ is bipartite, too, [4];

(iv) $C_{2m+1} \wedge C_{2n+1}$, $m \geqq 1$, $n \geqq 2$, contains an induced odd cycle of length at least five, [4].

Lemma. *$G = G_1 \wedge G_2$ is strongly perfect if and only if either*

(a) *at least one of G_i's, $i = 1, 2$, has $|V(G_i)| = 1$, or*

(b) *G_1 or G_2 is bipartite.*

Proof. If $E(G) = \emptyset$ then either $E(G_1) = \emptyset$ or $E(G_2) = \emptyset$ or both of them are empty; hence we have (a). If $E(G) \neq \emptyset$, let us suppose neither G_1 nor G_2 are bipartite. Then, by virtue of the last of the above remarks, both G_1, G_2 must contain triangles. But $K_3 \wedge K_3$ is not strongly perfect (since $K_3 \times K_3$ contains C_6 as an induced subgraph, $\mathrm{cl}(K_3 \times K_3) = K_3 \wedge K_3$ contains $\bar{C}_6$ as an induced subgraph), which contradicts the strong perfectness of G; consequently, at least one of the graphs G_i does not contain C_{2n+1}, $n \geqq 1$, that is, one of them must be bipartite.

Conversely, if $|V(G_1)| = 1$, then $E(G_1 \wedge G_2) = \emptyset$ and, clearly, $G_1 \wedge G_2$ is strongly perfect; if G_1 or G_2 is bipartite, their tensor product is also strongly perfect, being a bipartite graph.

Corollary. *If G_1 or (and) G_2 is (are) disconnected, then $G_1 \wedge G_2$ is strongly perfect if and only if either* (i) *there is $i \in \{1, 2\}$ such that $E(G_i) = \emptyset$, or* (ii) *all connected components of G_1 or G_2 are bipartite.*

Proposition. *$G = G_1 \wedge G_2 \wedge \ldots \wedge G_n$, $n \geqq 3$, is strongly perfect if and only if there is $i_0 \in \{1, 2, \ldots, n\}$ such that either $|V(G_{i_0})| = 1$ or G_{i_0} is bipartite.*

Proof. If $E(G) = \emptyset$, then clearly there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $|V(G_{i_0})| = 1$. Suppose $E(G) \neq \emptyset$; since the tensor product is associative, we may consider $G = H_{n-1} \wedge G_n$, where $H_k = G_1 \wedge \ldots \wedge G_k$, $2 \leqq k \leqq n$. By the above lemma, we get that H_{n-1} or G_n is bipartite. If only $H_{n-1} = H_{n-2} \wedge G_{n-1}$ is bipartite, H_{n-1} is also strongly perfect and, applying again the above mentioned lemma, we obtain H_{n-2} or G_{n-1} is bipartite, and so on. Finally, arriving at the step: "$H_2 = G_1 \wedge G_2$ is strongly perfect", we get that G_1 or G_2 is bipartite.

Conversely, if there is $i_0 \in \{1, 2, \ldots, n\}$ such that:

(a) $|V(G_{i_0})| = 1$, then $E(G) = \emptyset$ and, evidently, G is strongly perfect;

(b) G_{i_0} is bipartite, then G itself is bipartite or has $E(G) = \emptyset$ and, anyway, G is strongly perfect.

Corollary. *$H = G \wedge G \wedge \ldots \wedge G = G^n$, $n \geqq 2$, is strongly perfect if and only if either $E(G) = \emptyset$ or G is bipartite (or, if G is disconnected, all its connected components are bipartite).*

4. STRONGLY PERFECT SYMMETRICAL DIFFERENCE OF GRAPHS

The symmetrical difference of graphs G_i, $i = 1, 2$, is the graph $G = G_1 \triangle G_2$ with $V(G) = V(G_1) \times V(G_2)$, and $(x_1, y_1) \sim (x_2, y_2)$ if either (a) $x_1 \sim x_2$, $y_1 \nsim y_2$ or (b) $x_1 \nsim x_2$, $y_1 \sim y_2$. Let us notice:

(i) this product is commutative and associative;

(ii) $G_1 \triangle G_2$ is disconnected if and only if both G_1 and G_2 are disconnected;

(iii) $K_2 \triangle K_3 = \bar{C}_6$;

(iv) $K_2 \triangle P_4$ is not strongly perfect, since it has no stable transversal or is not stochastic, [3] (by P_n, $n \geqq 3$, we mean the chordless chain on n vertices).

Lemma. *Let G_i, $i = 1, 2$, satisfy $E(G_i) \neq \emptyset$; then $G = G_1 \triangle G_2$ is strongly perfect if and only if both G_1 and G_2 are complete bipartite.*

Proof. If G is strongly perfect, then G_1, G_2, as induced subgraphs of G, are strongly perfect and, therefore, perfect, too, [1]. Consequently, they do not contain C_{2n+1}, $n \geqq 2$, as induced subgraphs; by the above remark (iii), neither do they admit induced complete subgraphs on $n \geqq 3$ vertices. So, they both are bipartite. In addition, since they are connected and P_4-free (see the last above stated remark), we conclude that they both must be complete bipartite.

Conversely, if G_1 and G_2 are complete bipartite, let $V_1 = V_{11} \cup V_{12}$ and $V_2 = V_{21} \cup V_{22}$ be, respectively, their bipartitions. Then G is also complete bipartite, with the following bipartition: $V = V' \cup V''$, where $V' = V_{11} \times V_{21} \cup V_{12} \times V_{22}$ and $V'' = V_{11} \times V_{22} \cup V_{12} \times V_{21}$. Consequently, G is strongly perfect.

Remark. Clearly, if $E(G_1) = \emptyset$, then $G_1 \triangle G_2$ is strongly perfect if and only if G_2 has this property. If G_1 and G_2 are disconnected, then $G_1 \triangle G_2$ is strongly perfect if and only if all their connected components are complete bipartite.

Proposition. *$G = G_1 \triangle G_2 \triangle \ldots \triangle G_n$, $n \geqq 3$, is strongly perfect if and only if either*

(i) *$E(G_i) = \emptyset$, $i = \overline{1, n}$, with the possible exception of one $i_0 \in \{1, 2, \ldots, n\}$ for which G_{i_0} is strongly perfect, or*

(ii) *all G_i's with non-empty edge sets are complete bipartite, and there are at least two such graphs.*

Proof. If $E(G) = \emptyset$, then $E(G_i) = \emptyset$, $i = 1, 2$. If G and some G_{i_0}, with $E(G_{i_0}) \neq \emptyset$, are isomorphic, then G_{i_0} is strongly perfect and obviously, $E(G_i) = \emptyset$, $i \in \{1, 2, \ldots, n\} - \{i_0\}$. Let $G_1, G_2, \ldots, G_k$, $1 < k \leqq n$, satisfy $E(G_i) \neq \emptyset$, $i = \overline{1, k}$; (otherwise, because of commutativity, we may change the indexes). Then G and $H_k = G_1 \triangle \ldots \triangle G_k = H_{k-1} \triangle G_k$ are isomorphic, since G_i with $i > k$ has $E(G_i) =$

$= \emptyset$. According to the above lemma, H_{k-1} and G_k are complete bipartite. Thus, $H_{k-1} = H_{k-2} \triangle G_{k-1}$ is strongly perfect and, using again the same lemma, we get that H_{k-2} and G_{k-1} are complete bipartite, and so on.

Conversely, G is strongly perfect since it is isomorphic, respectively, to: any G_i or G_{i_0} in the case (i); or $H_k = G_1 \triangle \ldots \triangle G_k$, which is complete bipartite, for the case (ii) (where $G_1, \ldots, G_k$, $1 < k \leqq n$, are all graphs with non-empty edge sets).

Corollary. $H = G \triangle G \triangle \ldots \triangle G = G^n$, $n \geqq 2$, *is strongly perfect if and only if either* $E(G) = \emptyset$ *or* G *is complete bipartite.*

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