Czechoslovak Mathematical Journal

Werner Georg Nowak

On the lattice rest of a convex body in $\mathbf{R}^s$, III

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 2, 359–367

Persistent URL: `http://dml.cz/dmlcz/102468`

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

ON THE LATTICE REST OF A CONVEX BODY IN $\mathbf{R}^s$, III

WERNER GEORG NOWAK, Wien

(Received April 2, 1990)

1. INTRODUCTION

Like in parts I and II of this work [10], [11], we consider a compact convex subset $\mathscr{B}$ of $\mathbf{R}^s$ ($s \geqq 2$) which contains the origin as an inner point and has a smooth boundary $\partial\mathscr{B}$ with finite nonzero Gaussian curvature throughout. We assume that the correspondance which maps every point of the unit sphere in $\mathbf{R}^s$ to the point of $\partial\mathscr{B}$ where the outward normal has the same direction, is one-one and of class C^∞. For a large real variable T, we denote by $A(T)$ the number of lattice points (of the standard lattice $\mathbf{Z}^s$) in the "blown up" body $(\sqrt{T})\,\mathscr{B} = \{\boldsymbol{x} \in \mathbf{R}^s\colon (1/\sqrt{T})\,\boldsymbol{x} \in \mathscr{B}\}$ and define the "lattice rest" in the usual way by $P(T) = A(T) - VT^{s/2}$ (V the volume of $\mathscr{B}$).

For the special case that $\mathscr{B}$ is an ($\mathbf{0}$-symmetric) ellipsoid, a wealth of deep and enlightening results is available. They usually either impose some "rationality" or "irrationality" condition on the coefficients of the quadratic form involved or are of a metric kind. The reader is in particular referred to the classic papers of the Czechoslovak school of lattice point theory, namely of Jarník, Diviš and B. Novák, listed in the textbooks of Fricker [4] and Krätzel [8].

For a quite general convex body $\mathscr{B}$ satisfying the conditions stated above, it was shown by Hlawka [6] that

$$(1.1)\qquad P(T) = O(T^{s(s-1)/2(s+1)})\,,$$

and

$$(1.2)\qquad P(T) = \Omega(T^{(s-1)/4})\,.$$

(For $s = 2$, this was obtained much earlier by Jarník [7]. For the O- and Ω-notation, the reader may consult [4] or [8].) Later on, Krupička [9] refined (1.2) by

$$(1.3)\qquad P(T) = \Omega_\pm(T^{(s-1)/4})$$

and

$$(1.4)\qquad \frac{1}{T}\int_0^T |P(t)|\,\mathrm{d}t \gg T^{(s-1)/4}\,.$$

This paper is part of a research project supported by the Austrian "Fonds zur Förderung der wissenschaftlichen Forschung" (Nr. P7514-PHY).

In parts I and II of this article [10], [11], we established the improvements

$$(1.5) \qquad P(T) = \Omega(T^{(s-1)/4}(\log T)^{1/4})$$

for every dimension $s \geqq 2$ and, for s incongruent to 1 modulo 4,

$$(1.6) \qquad P(T) = \Omega_*(T^{(s-1)/4}(\log T)^{(1/2)-(1/2s)}),$$

where $* = -$ for $s \equiv 2, 3, 4 \pmod 8$ and $* = +$ for $s \equiv 6, 7, 8 \pmod 8$.

In this note we extend this last result to the case $s \equiv 1 \pmod 4$, making the Ω-estimate two-sided at the same time, for arbitrary $s \geqq 4$.

Theorem 1. *For the lattice rest $P(T)$ of the convex body $(\sqrt{T})\mathscr{B}$ in $\mathbf{R}^s$, where $\mathscr{B}$ satisfies the conditions stated above, we have*

$$P(T) = \Omega_\pm(T^{(s-1)/4}(\log T)^{(1/2)-(1/2s)}),$$

for every dimension $s \geqq 4$.

It is worth noting that no sharper result than this one is known even for an arbitrary $\mathbf{0}$-symmetric ellipsoid (without any restriction on the coefficients of the quadratic form involved). For this case the estimate is implicitly contained in a paper of Hafner [5], its proof being based on the functional equation of the Epstein zeta-function.

In our argument we will borrow some tools of Hafner's, along with the classic methods of Berndt [2] and of Szegö and Walfisz [13], [14]. In all of these papers, however, the arithmetic functions under consideration are coefficients of Dirichlet series which satisfy a certain functional equation; therefore, the analysis is usually based on a contour integration argument. In our problem we do not have such a "neat" functional equation. To overcome this difficulty, one has to replace complex integration by an application of the Poisson summation formula along with Hlawka's [6] asymptotic formula for the Fourier transform of the indicator function of a general convex body $\mathscr{B}$ (cf. [11]).

In section 3, we give a quantitative refinement of Theorem 1 which gives some information how quickly the "oscillations" of the lattice rest take place (Theorem 2).

2. PROOF OF THEOREM 1

For a large real parameter t, we put

$$(2.1) \qquad X = X(t) = K_1(\log t)^{-2/s}$$

and

$$(2.2) \qquad k = k(t) = \frac{1}{4\pi^2}(a + t\,X(t)^{-1/2})^2,$$

with a positive constant K_1 and real a to be specified later. We consider the "Borel mean-value"

$$(2.3) \qquad M(t) =^{\mathrm{def}} \frac{1}{\Gamma(k+1)} \int_0^\infty u^k e^{-u} P(Xu)\,du.$$

In [11], we already proved (with X and k defined slightly different which does not affect the argument) that

$$M(t) = \frac{1}{\pi}(kX)^{(s-1)/4} \sum_{\substack{0<|\boldsymbol{m}|\leq \varkappa \\ \boldsymbol{m}\in \mathbf{Z}^s}} \alpha(\boldsymbol{m})\,|\boldsymbol{m}|^{-(s+1)/2} \cos\left(2\pi H(\boldsymbol{m})\sqrt{(kX)} - \frac{\pi}{4}(s+1)\right) \times$$

$$(2.4) \qquad \times \exp\left(-\frac{\pi^2}{2} X\, H^2(\boldsymbol{m})\right) + O(k^{(s-3/2)/4}).$$

Here $\varkappa = k^{\varepsilon} X^{-1/2}$, $\varepsilon > 0$ a small constant, H is the tac-function of the body $\mathscr{B}$ ("Stützfunktion", cf. Bonnesen and Fenchel [3]), hence a positive C^∞-function on $\mathbf{R}^s - \{\mathbf{0}\}$ which is homogeneous of degree 1. Finally $\alpha(\cdot)$ is a positive C^∞-function on $\mathbf{R}^s - \{\mathbf{0}\}$ homogeneous of degree 0 (in fact, $\alpha(\boldsymbol{w}) = 1/\sqrt{(K(\boldsymbol{w}/|\boldsymbol{w}|))}$ where $K(\boldsymbol{x})$ is the Gaussian curvature at that point of $\partial\mathscr{B}$ which corresponds to the point $\boldsymbol{x}$ of the unit sphere under the C^∞-map mentioned at the very beginning; see Hlawka [6]). $|\cdot|$ denotes the Euclidean norm throughout.

In order to extend the range of summation in (2.4) to $\boldsymbol{m} \in \mathbf{Z}^s$, $\boldsymbol{m} \neq \mathbf{0}$, we observe that[1])

$$\sum_{\substack{|\boldsymbol{m}|>\varkappa \\ \boldsymbol{m}\in \mathbf{Z}^s}} \alpha(\boldsymbol{m})\,|\boldsymbol{m}|^{-(s+1)/2} \exp\left(-\frac{\pi^2}{2} X H^2(\boldsymbol{m})\right) \ll$$
$$\ll \sum_{\substack{n>\varkappa^2 \\ n\in \mathbf{N}}} r_s(n)\, n^{-(s+1)/4} \exp(-c_1 n X) \ll$$
$$\ll \exp(-c_2 \varkappa^2 X) + \int_{\varkappa^2}^{\infty} \exp(-c_2 X u)\, du \ll$$
$$\ll \exp(-c_3 \varkappa^2 X)\left(1 + \int_{\varkappa^2}^{\infty} (Xu)^{-2}\, du\right) \ll k^{-1},$$

where $r_s(n)$ denotes the number of ways to write n as a sum of s squares of integers, and the c_j are positive constants (depending at most on $\mathscr{B}$), throughout the sequel. Consequently,

$$(2.5) \qquad M(t) = \frac{1}{\pi}(kX)^{(s-1)/4} \sum_{\substack{\boldsymbol{m}\in \mathbf{Z}^s \\ \boldsymbol{m}\neq \mathbf{0}}} \alpha(\boldsymbol{m})\,|\boldsymbol{m}|^{-(s+1)/2} \cos\left(2\pi H(\boldsymbol{m})\sqrt{(kX)} - \frac{\pi}{4}(s+1)\right) \times$$
$$\times \exp\left(-\frac{\pi^2}{2} X\, H^2(\boldsymbol{m})\right) + O(k^{(s-3/2)/4}).$$

The next step is to approximate this infinite series by an expression of the form

$$(2.6) \qquad f(X, a) = \sum_{\substack{\boldsymbol{m}\in \mathbf{Z}^s \\ \boldsymbol{m}\neq \mathbf{0}}}^{\infty} \beta(\boldsymbol{m})\, g(X, H(\boldsymbol{m}), a)$$

where, for short,

$$\beta(\boldsymbol{m}) = \alpha(\boldsymbol{m}) \left(\frac{|\boldsymbol{m}|}{H(\boldsymbol{m})}\right)^{-(s+1)/2},$$

---

[1]) By homogeneity of H, $c_4|\boldsymbol{m}| \leq H(\boldsymbol{m}) \leq c_5|\boldsymbol{m}|$.

(2.7) $$g(X, u, a) =^{\text{def}} \exp\left(-\frac{\pi^2}{2} Xu^2\right) u^{-(s+1)/2} \cos\left(au\sqrt{X} - \frac{\pi}{4}(s+1)\right).$$

(Note that the function β is again positive and homogeneous of degree 0 with continuous partial derivatives on $\mathbf{R}^s - \{\mathbf{0}\}$.)

We now need some information about the asymptotic behaviour of the summatory function of β.

Lemma 1. *For $u \to \infty$,*

$$S(u) =^{\text{def}} \sum_{\substack{0 < H(\boldsymbol{m}) \leq u \\ \boldsymbol{m} \in \mathbf{Z}^s}} \beta(\boldsymbol{m}) = c_0 u^s + R(u)$$

with

(2.8) $$c_0 = \int_{H(\boldsymbol{x}) \leq 1} \beta(\boldsymbol{x})\, d\boldsymbol{x}, \quad R(u) = O(u^{s-1}).$$

Proof. It is clear that

$$I(u) =^{\text{def}} \int_{H(\boldsymbol{x}) \leq u} \beta(\boldsymbol{x})\, d\boldsymbol{x} = c_0 u^s.$$

For any $\boldsymbol{m} = (m_1, \ldots, m_s) \in \mathbf{Z}^s$, define the unit cube

$$E(\boldsymbol{m}) = [m_1, m_1 + 1] \times \ldots \times [m_s, m_s + 1],$$

and denote by $\mathscr{K}(u)$ the body in $\mathbf{R}^s$ given by $H(\boldsymbol{x}) \leq u$. Then we easily see that

$$I(u) - S(u) = \sum_{\substack{\boldsymbol{m} \in \mathbf{Z}^s \\ E(\boldsymbol{m}) \subset \mathscr{K}(u)}} \int_{E(\boldsymbol{m})} (\beta(\boldsymbol{x}) - \beta(\boldsymbol{m}))\, dx + O(u^{s-1}),$$

because there are at most $O(u^{s-1})$ cubes $E(\boldsymbol{m})$ which intersect the boundary of $\mathscr{K}(u)$. (Note that there exists c_6 so that the boundaries $\partial\mathscr{K}(u + c_6)$ and $\partial\mathscr{K}(u - c_6)$ both have a minimal distance exceeding $\sqrt{s}$ from $\partial\mathscr{K}(u)$, and the volumes of these bodies differ only by $O(u^{s-1})$.) For $\boldsymbol{x} \in E(\boldsymbol{m})$ (and $|\boldsymbol{m}|$ sufficiently large),

$$|\beta(\boldsymbol{x}) - \beta(\boldsymbol{m})| \leq \sum_{j=1}^{s} \max_{\frac{1}{2}|\boldsymbol{m}| \leq |\boldsymbol{w}| \leq 2|\boldsymbol{m}|} \left|\frac{\partial \beta}{\partial w_j}(\boldsymbol{w})\right| \ll \frac{1}{|\boldsymbol{m}|},$$

since $\partial\beta/\partial w_j$ is homogeneous of degree -1. Thus

$$I(u) - S(u) \ll \sum_{\substack{\boldsymbol{m} \neq \boldsymbol{0} \\ H(\boldsymbol{m}) \leq u}} \frac{1}{|\boldsymbol{m}|} + u^{s-1} \ll \sum_{1 \leq n \leq c_7 u^2} r_s(n)\, n^{-1/2} + u^{s-1} \ll u^{s-1},$$

as asserted in Lemma 1.

We are now able to give an asymptotic formula for $f(X, a)$, as $X \to 0+$, a some real constant, in the spirit of Berndt [2], Lemma 3.1.

Lemma 2. *For $X \to 0+$,*

(2.9) $$f(X, a) = c_8\, F(a)\, X^{-(s-1)/4} + O_a(X^{-(s-3)/4})$$

with

(2.10) $$F(a) =^{\text{def}} \int_0^\infty v^{(s-3)/2} e^{-v^2} \cos\left(\frac{\sqrt{2}}{\pi} av - \frac{\pi}{4}(s+1)\right).$$

Proof. Writing (2.6) as a Stieltjes integral, we get

$$f(X, a) = \int_0^\infty g(X, u, a)\,\mathrm{d}S(u) =$$

$$= c_0 s \int_0^\infty u^{s-1} g(X, u, a)\,\mathrm{d}u - \int_0^\infty \frac{\partial g}{\partial u}(X, u, a)\,R(u)\,\mathrm{d}u\,.$$

By the substitution $u = (\sqrt{2}/\pi)\,X^{-1/2}v$, the first integral is just $c_9\,F(a)\,X^{-(s-1)/4}$, and the second integral is

$$\ll \int_0^\infty \left|\frac{\partial g}{\partial u}(X, u, a)\right| u^{s-1}\,\mathrm{d}u \ll$$

$$\ll \int_0^\infty u^{s-1} \exp\left(-\frac{\pi^2}{2} X u^2\right)\left(X u^{(-s+1)/2} + u^{(-s-3)/2} +\right.$$

$$\left. + (\sqrt{X})\,u^{(-s-1)/2}\right)\mathrm{d}u \ll X^{-(s-3)/4}\,.$$

This proves Lemma 2.

We now apply Dirichlet's approximation theorem (see e.g. [5]): Let A_1 be a large positive integer and $q \in \mathbf{N}$ a parameter to be fixed later (none of the c_j and order constants may depend on q in what follows). Then there exists[2]) a value of t in the interval

$$(2.11) \qquad A_1 \leqq t \leqq A_1 q^{c_{10} A_1^s}$$

so that

$$(2.12) \qquad \left\|\frac{1}{2\pi} H(\boldsymbol{m})\,t\right\| \leqq \frac{1}{q}$$

for all $\boldsymbol{m} \in \mathbf{Z}^s$ with $0 < H(\boldsymbol{m}) \leqq A_1$. (Here $\|\cdot\|$ denotes the distance from the nearest integer.) We infer from (2.11) that

$$A_1 \gg (\log t)^{1/s} (\log q)^{-1/s}$$

and define

$$A =^{\mathrm{def}} C(q)\,(\log t)^{1/s}\,, \quad C(q) = c_{11}(\log q)^{-1/s}\,,$$

with c_{11} so small that $A \leqq A_1$. Furthermore, by (2.12), the mean- value theorem and the definition of $k(t)$, we conclude that, for $0 < H(\boldsymbol{m}) \leqq A$,

$$\left|\cos\left(2\pi H(\boldsymbol{m})\sqrt{(k)}\sqrt{(X)} - \frac{\pi}{4}(s+1)\right) - \cos\left(a\,H(\boldsymbol{m})\sqrt{(X)} - \frac{\pi}{4}(s+1)\right)\right| \ll$$

$$\ll \left\|\frac{1}{2\pi} H(\boldsymbol{m})\,t\right\| \leqq \frac{1}{q}\,.$$

[2]) Note the order in the choice of the parameters: First A_1 is picked arbitrarily large, then t is chosen according to Dirichlet's theorem (thus $t \to \infty$ with A_1); thereby $X(t)$, $k(t)$ and A are defined.

Thus we can in fact approximate a finite portion of the series in (2.5) by

$$f_A(X, a) =^{\text{def}} \sum_{\substack{0 < H(\boldsymbol{m}) \leqq A \\ \boldsymbol{m} \in \mathbf{Z}^s}} \beta(\boldsymbol{m})\, g(X, H(\boldsymbol{m}), a)\,.$$

We obtain

$$(2.13) \qquad \Bigg| f_A(X, a) - \sum_{\substack{0 < H(\boldsymbol{m}) \leqq A \\ \boldsymbol{m} \in \mathbf{Z}^s}} \alpha(\boldsymbol{m})\, |\boldsymbol{m}|^{-(s+1)/2}\,.$$

$$. \cos\left(2\pi H(\boldsymbol{m}) \sqrt{(kX)} - \frac{\pi}{4}(s+1)\right) \exp\left(-\frac{\pi^2}{2} X H^2(\boldsymbol{m})\right)\Bigg| \ll$$

$$\ll \frac{1}{q} \sum_{0 < H(\boldsymbol{m}) \leqq A} \beta(\boldsymbol{m})\, H(\boldsymbol{m})^{-(s+1)/2} \exp\left(-\frac{\pi^2}{2} X H^2(\boldsymbol{m})\right) =$$

$$= \frac{1}{q} \int_0^{A+} \exp\left(-\frac{\pi^2}{2} X u^2\right) \mathrm{d}G(u) \ll$$

$$\ll \frac{1}{q} \left(\exp\left(-\frac{\pi^2}{2} A^2 X\right) A^{(s-1)/2} + X \int_0^A u^{(s+1)/2} \exp\left(-\frac{\pi^2}{2} X u^2\right) \mathrm{d}u\right) \ll$$

$$\ll \frac{1}{q} X^{-(s-1)/4} \left(\exp\left(-\frac{\pi^2}{2} C(q)^2 K_1\right) \left(C(q)^2 K_1\right)^{(s-1)/4} + 1\right) \ll \frac{1}{q} X^{-(s-1)/4}\,,$$

where

$$G(u) =^{\text{def}} \sum_{\substack{0 < H(\boldsymbol{m}) \leqq u \\ \boldsymbol{m} \in \mathbf{Z}^s}} \beta(\boldsymbol{m})\, (H(\boldsymbol{m}))^{-(s+1)/2} \ll u^{(s-1)/2}\,,$$

which in turn follows readily via integration by parts.

Similarly, using the definitions of A and k, we get

$$(2.14) \qquad |f(X, a) - f_A(X, a)| \leqq \sum_{H(\boldsymbol{m}) > A} \beta(\boldsymbol{m})\, (H(\boldsymbol{m}))^{-(s+1)/2} \exp\left(-\frac{\pi^2}{2} X H^2(\boldsymbol{m})\right) =$$

$$= \int_{A+}^{\infty} \exp\left(-\frac{\pi^2}{2} X u^2\right) \mathrm{d}G(u) \ll$$

$$\ll \exp\left(-\frac{\pi^2}{2} A^2 X\right) A^{(s-1)/2} + X \int_A^{\infty} u^{(s+1)/2} \exp\left(-\frac{\pi^2}{2} X u^2\right) \mathrm{d}u \ll$$

$$\ll X^{-(s-1)/4} \left(\exp\left(-\frac{\pi^2}{2} C(q)^2 K_1\right) \left(C(q)^2 K_1\right)^{(s-1)/4} + \int_{(\pi^2/2) C(q)^2 K_1}^{\infty} w^{(s-1)/4} e^{-w} \mathrm{d}w\right) =$$

$$= \varepsilon\left(C(q)^2 K_1\right) X^{-(s-1)/4}\,,$$

where $\varepsilon(W) \to 0$ for $W \to \infty$. In the very same way the infinite „tail" of the series in (2.5) can be estimated.

Collecting the results (2.5), (2.13), and (2.14), we thus arrive at

$$M(t) = \frac{1}{\pi} k^{(s-1)/4} \left(X^{(s-1)/4} f(X, a) + O\left(\frac{1}{q}\right) + O\left(\varepsilon\left(C(q)^2 K_1\right)\right)\right),$$

and by Lemma 2 (for $t \to \infty$),

$$(2.15) \qquad M(t) = c_{12} k^{(s-1)/4} \left(F(a) + O\left(\frac{1}{q}\right) + O(\varepsilon(C(q)^2 K_1)) + o(1) \right).$$

We now make use of a deep result due to Steinig [12] which provides necessary and sufficient conditions for functions like $F(a)$ to have a change of sign.

Proposition (Steinig). *For $a', B, \gamma \in \mathbf{R}$, $\gamma > -1$, let*

$$G_{\gamma,B}(a') =^{\text{def}} \int_0^\infty e^{-u^2} u^\gamma \cos(a'u + B\pi)\, du\,.$$

Then $G_{\gamma,B}(a')$ as a function of a' has a sign change if and only if

$$(2.16) \qquad \gamma > -2|B - [B + \tfrac{1}{2}]|\,.$$

Otherwise, $G_{\gamma,B}(a') \neq 0$ for all real values of a'.

For $s \geqq 4$, $\gamma = \frac{1}{2}(s-3) > 0$ (for our function $F(a)$ defined in (2.10)), hence there exist real numbers a_1 and a_2 and a positive constant c_{13} so that

$$(2.17) \qquad F(a_1) \leqq -c_{13}\,, \quad F(a_2) \geqq c_{13}\,.$$

We take once $a = a_1$, then $a = a_2$ in the definition (2.2), i.e. we put (for $i = 1$ or 2)

$$(2.18) \qquad k_i = k_i(t) = \frac{1}{4\pi^2}(a_i + t\, X(t)^{-1/2})^2\,,$$

define $M_i(t)$ by (2.3), with k replaced by k_i, and infer from (2.15) (choosing first q and then K_1 sufficiently large) that there exists a sequence of real numbers t tending to $+\infty$ with

$$(2.19) \qquad M_1(t) \leqq -c_{14} k_1^{(s-1)/4}\,,$$

and

$$(2.20) \qquad M_2(t) \geqq c_{14} k_2^{(s-1)/4}\,.$$

To complete the proof, let us suppose that, for some small positive constant K_3,

$$\pm P(T) \leqq K_3 T^{(s-1)/4} (\log T)^{(1/2)-(1/2s)}$$

for all sufficiently large T. By the definition of $M_i(t)$, this would imply that, for every large real t,

$$(-1)^i M_i(t) \leqq \frac{K_3(X(t))^{(s-1)/4}}{\Gamma(k_i(t)+1)} \int_0^\infty u^{k_i(t)+(s-1)/4} L(X(t)\,u)\, e^{-u}\, du + c_{15}\,,$$

where $L(w) = (\log w)^{(1/2)-(1/2s)}$ for $w \geqq 2$, $L(w) = 0$ else. Estimating this integral by Hafner's Lemma 2.3.6 in [5], p. 51, we obtain

$$(-1)^i M_i(t) \leqq c_{16} K_3 (k_i(t)\, X(t))^{(s-1)/4}\, L(X(t)\, k_i(t)) \leqq$$
$$\leqq c_{17} K_3 (k_i(t))^{(s-1)/4}$$

after a short computation, using the definitions (2.1) and (2.18). Together with (2.19) and (2.20), this yields a positive lower bound for K_3 (both for $i = 1$ and $i = 2$) and thus completes the proof of Theorem 1.

Slightly stronger than Theorem 1, one can give some information how quickly the "oscillations" of the lattice rest $P(T)$ take place, on the lines of Hafner's Theorem C in [5].

Theorem 2. *There exist positive constants c, c' and a sequence of real numbers y tending to $+\infty$ such that both of the inequalities*

$$\pm P(T) \geqq c' T^{(s-1)/4} (\log T)^{(1/2)-(1/2s)}$$

have a solution in each of the intervals

$$y \leqq T \leqq y + c y^{1/2} (\log y)^{(1/2)-(1/s)} .$$

Proof. For $i = 1$ or 2, let $k_i = k_i(t)$ and $M_i(t)$ be defined like before. By (2.19), (2.20), there exists a sequence of real numbers t tending to $+\infty$ such that

$$(3.1) \qquad (-1)^i M_i(t) \geqq c_{14} k_i^{(s-1)/4} .$$

We define

$$(3.2) \qquad N_i^{\pm} =^{\mathrm{def}} k_i \pm K_4 (k_i \log k_i)^{1/2} ,$$

(K_4 a suitable constant at our disposition), and put

$$I_i^+ = \frac{1}{\Gamma(k_i+1)} \int_{N_i+}^{\infty} , \quad I_i^- = \frac{1}{\Gamma(k_i+1)} \int_0^{N_i-} ,$$

with the integrand $u^{k_i} \mathrm{e}^{-u} P(Xu)$ in both cases.

Using (1.1) (with $\theta = (s(s-1))/(2(s+1))$ for short), we conclude that

$$I_i^+ \ll \frac{1}{\Gamma(k_i+1)} \int_{N_i+}^{\infty} u^{k_i+\theta} \mathrm{e}^{-u} \, \mathrm{d}u \ll \frac{1}{\Gamma(k_i+1)} (N_i^+)^{k_i+\theta+2} \exp(-N_i^+) \int_{N_i+}^{\infty} \frac{\mathrm{d}u}{u^2} \ll$$

$$\ll \frac{1}{\Gamma(k_i+1)} (N_i^+)^{k_i+\theta+1} \exp(-N_i^+)$$

(since $N_i^+ > k_i + \theta + 2$ for t sufficiently large). The same bound holds for I_i^-, with N_i^+ replaced by N_i^-. By Stirling's formula (see e.g. [1], p. 257),

$$I_i^{\pm} \ll \left(1 \pm K_4 \left(\frac{\log k_i}{k_i}\right)^{1/2}\right)^{k_i} k_i^{\theta+1/2} \exp(\mp K_4 (k_i \log k_i)^{1/2}) =^{\mathrm{def}} B_i^{\pm} .$$

Taking logarithms and using the Taylor expansion

$$\log(1+w) = w - \frac{w^2}{2} + O(|w|^3) ,$$

we get

$$\log B_i^{\pm} = (\theta + \tfrac{1}{2} - \tfrac{1}{2} K_4^2) \log k_i + O(k_i^{-1/2} (\log k_i)^{3/2}) .$$

If we choose K_4 so large that $K_4^2 > 2\theta + 1$, $\log B_i^{\pm} \to -\infty$ for $t \to \infty$, hence

$I_i^{\pm} = o(1)$. Together with (3.1) this implies that

$$(-1)^i \frac{1}{\Gamma(k_i+1)} \int_{N_i^-}^{N_i^+} u^{k_i} e^{-u} P(Xu)\, du \geqq \tfrac{1}{2} c_{14} k_i^{(s-1)/4}$$

(again for an unbounded sequence of values t and $i = 1, 2$). Since, by the very same reasoning,

$$\frac{1}{\Gamma(k_i+1)} \int_{N_i^-}^{N_i^+} u^{k_i} e^{-u}\, du = 1 + o(1),$$

there always exists a value u_i in the interval $[N_i^-, N_i^+]$ such that

$$(3.3) \qquad (-1)^i P(Xu_i) \geqq \tfrac{1}{4} c_{14} k_i^{(s-1)/4}.$$

We now put $T_i = Xu_i$, then it follows from the definitions given earlier that

$$(3.4) \qquad T_i \asymp Xk_i \asymp t^2, \quad k_i \asymp T_i (\log T_i)^{2/s},$$

consequently

$$|T_1 - T_2| \ll X\Big(\max_{i=1,2} (k_i \log k_i)^{1/2} + |k_1 - k_2|\Big) \ll$$
$$\ll tX^{1/2} (\log t)^{1/2} \ll T_i^{1/2} (\log T_i)^{(1/2)-(1/s)}.$$

Taking $y = \min\{T_1, T_2\}$, we complete the proof of Theorem 2, in view of (3.3) and (3.4).

References

[1] *Abramowitz, M.*, and *Stegun, I.:* Handbook of mathematical functions. New York 1964.

[2] *Berndt, B.:* On the average order of a class of arithmetical functions, II. J. Number Theory *3*, 288—305 (1971).

[3] *Bonnesen, T.*, and *Fenchel, W.:* Theorie der konvexen Körper. Berlin 1934.

[4] *Fricker, F.:* Einführung in die Gitterpunktlehre. Basel–Boston-Stuttgart 1982.

[5] *Hafner, J. L.:* On the average order of a class of arithmetical functions. J. Number Theory *15*, 36—76 (1982).

[6] *Hlawka, E.:* Über Integrale auf konvexen Körpern I. Monatsh. f. Math. *54*, 1—36 (1950).

[7] *Jarník, V.:* Sur les points à coordonnées entiéres dans le plan. Bull. int. de l'acad. des sciences de Bohême 1924, 12 p.

[8] *Krätzel, E.:* Lattice points. Berlin 1988.

[9] *Krupička, S.:* On the number of lattice points in multidimensional convex bodies (in Czech with German summary). Czech. Math. J. *7*, 524—552 (1957).

[10] *Nowak, W. G.:* On the lattice rest of a convex body in $\mathbf{R}^s$. Arch. Math. (Basel) *45*, 284—288 (1985).

[11] *Nowak, W. G.:* On the lattice rest of a convex body in $\mathbf{R}^s$, II. Arch. Math. (Basel) *47*, 232—237 (1986).

[12] *Steinig, J.:* On an integral connected with the average order of a class of arithmetical functions. J. Number Theory *4*, 463—468 (1972).

[13] *Szegö, P.*, and *Walfisz, A.:* Über das Piltzsche Teilerproblem in algebraischen Zahlkörpern (Erste Abhandlung). Math. Z. *26*, 138—156 (1927).

[14] *Szegö, P.*, and *Walfisz, A.:* Über das Piltzsche Teilerproblem in algebraischen Zahlkörpern (Zweite Abhandlung). Math. Z. *26*, 467—486 (1927).

Author's address: Institut für Mathematik, Universität für Bodenkultur, Gregor Mendel-Straße 33, A-1180 Wien, Austria.