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COMBINATORIAL PROPERTIES OF PRODUCTS OF GRAPHS

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1. INTRODUCTION

Let $S = (X, \cdot)$ be a commutative semigroup and let $M \subseteq X$. Define the function $f_M: X \to \{0, 1, 2, \ldots\}$ as follows: $f_M(x)$ is the number of expressions of x in the form $x = m \,.\, n$, where $m, n \in M$.

We say that M is a *multiplicative basis* if $f_M(x) > 0$ for every $x \in X$.

In [1], P. Erdös proved the following theorem.

Theorem 1. *Let $S = (\mathbb{N}, \cdot)$ be the semigroup of all positive integers with the usual multiplication and let $M \subseteq \mathbb{N}$ be a multiplicative basis. Then for every positive integer p there exists $x \in \mathbb{N}$ such that $f_M(x) > p$.*

Erdös' proof of Theorem 1 was very complicated and had a purely number-theoretical character. Thus it gave no possibility to generalize Theorem 1 to other commutative semigroups. However, in [6], J. Nešetřil and V. Rödl gave another proof of Theorem 1, based on the theorem of Ramsey, which was very simple and provided a straightforward possibility of generalization to other structures.

In this paper we show how Theorem 1 can be generalized to other commutative semigroups and, in particular, prove analogues of Theorem 1 for direct, cartesian and strong product of finite simple graphs.

2. PRIME SEMIGROUPS

Nešetřil and Rödl's proof of Theorem 1 essentially uses the following property of the set P of all prime numbers.

Property (P). *For every finite set $\{p_1, p_2, \ldots, p_r\} \subseteq P$ the following holds: If $p_1 \cdot p_2 \cdot \ldots \cdot p_r = x \,.\, y$, where x, y are positive integers, then there exist sets $I, J \subseteq \{1, 2, \ldots, r\}$ such that $I \cup J = \{1, 2, \ldots, r\}$, $\prod_{i \in I} p_i = x$ and $\prod_{j \in J} p_j = y$.*

Theorem 1 can be easily derived from property (P) and the following lemma which is based on the theorem of Ramsey.

Lemma 1 (Nešetřil, Rödl, see [6]). *Let X be an infinite set. Denote by $\mathscr{F}(X)$ the set of all finite subsets of X. Let $M \subseteq \mathscr{F}(X)$ be a set of finite subsets of X such that the following holds:*

For every $F \in \mathscr{F}(X)$ there are $F_1, F_2 \in M$ such that $F_1 \cup F_2 = F$ and $F_1 \cap F_2 = \emptyset$.

Then for every positive integer p there is a set F which can be expressed in at least p different ways as a union of two disjoint elements of M.

Aplying Nešetřil and Rödl's method of the proof of Theorem 1, some immediate stronger versions of this theorem can be given. Let us state them.

Definition 1. Suppose that $S = (X, \cdot)$ is a commutative semigroup, M is a subset of X and $k \geqq 2$ is *an* integer. Define the function $f_{M,k}: X \to \{0, 1, 2, \ldots\}$ as follows: $f_{M,k}(x)$ is the number of expressions of x in the form $x = m_1 \,.\, m_2 \,.\, \ldots \,.\, m_k$, where $m_i \in M$ for $i = 1, 2, \ldots, k$.

We say that M is an *asymptotic multiplicative basis of order* k if $f_{M,k}(x) > 0$ for all but finitely many $x \in X$.

The following result can be obtained by the methods from [6] (see also [5]).

Theorem 2. *Let $k \geqq 2$ and let M be an asymptotic multiplicative basis of order k in the semigroup $(\mathbb{N}, \cdot)$. Then for every positive integer p there exists $x \in \mathbb{N}$ such that $f_{M,k}(x) > p$.*

To prove Theorem 2 we use the following stronger version of Lemma 1.

Lemma 2. *Let X be an infinite set and let $k \geqq 2$ be an integer. Suppose that M is a subset of $\mathscr{F}(X)$ such that all but finitely many sets $F \in \mathscr{F}(X)$ can be expressed in the form $F = \bigcup_{i=1}^{k} F_i$ where $F_i \in M$ for $i = 1, 2, \ldots, k$. Then for every positive integer p there is a set $F \in \mathscr{F}(X)$ and at least p pairwise disjoint sets $\{F_1, F_2, \ldots, F_k\}$ such that $F = \bigcup_{i=1}^{k} F_i$, $F_i \in M$ for $i = 1, 2, \ldots, k$ and $F_i \cap F_j = \emptyset$ for $i \neq j$.*

Now we show that Lemma 2 immediately enables us to generalize Theorem 2 to the class of those commutative semigroups which contain an infinite set P with property (P). First we give some definitions.

Definition 2. Let $S = (X, \cdot)$ be a commutative semigroup with the identity element 1. We say that

(i) x *divides* y (or x *is a divisor* of y), where $x, y \in X$, if there exists $z \in X$ such that $x \,.\, z = y$. We denote this by $x \mid y$;

(ii) $j \in X$ is a *unit* if $j \mid 1$;

(iii) elements $x, y \in X$ are *associated* if there is a unit j such that $x = y \,.\, j$. We denote this by $x \sim y$. (Clearly j is a unit iff $j \sim 1$. Remark that $\sim$ is an equivalence relation on X.)

(iv) $x \in X$ is *irreducible* if x is not a unit and, moreover, the following holds: if $x = a \, . \, b$ then $a \sim 1$ or $b \sim 1$.

All notions introduced in the above definition can be transferred to the semigroups without an identity element as follows.

Definition 3. Let $S = (X, \cdot)$ be a commutative semigroup without an identity element.

(i) We say that x divides y if either $x = y$ or there exists $z \in X$ such that $x \, . \, z = y$.
(ii) There is no unit in S.
(iii) $x \sim y$ iff $x = y$.
(iv) $x \in X$ is irreducible if it cannot be expressed in the form $x = a \, . \, b$, where $a, b \in X$.

Now, let $S = (X, \cdot)$ be a commutative semigroup and let $F = \{x_1, x_2, \ldots, x_k\}$ be a finite subset of X. We denote the product $x_1 \, . \, x_2 \, . \, \ldots \, . \, x_k$ of elements of F by $\prod F$. Further we put $\prod \emptyset = 1$, where 1 is the identity element.

Definition 4. We say that a set $P \subseteq X$ is a *prime set* if it contains no unit, if no two different elements of P are associated, and if for every finite (non-empty) set $F \subseteq P$ the following condition holds: if $\prod F = x_1 \, . \, x_2$ then there exist finite sets $F_1, F_2 \subseteq F$ (possibly empty) such that $F_1 \cup F_2 = F$, $x_1 \sim \prod F_1$ and $x_2 \sim \prod F_2$.

Definition 5. A commutative semigroup is said to be a *prime semigroup* if it contains an infinite prime set and has only finitely many units.

In the next theorem we show that the result stated in Theorem 2 for the semigroup $(\mathbb{N}, \cdot)$ holds for every prime commutative semigroup.

Theorem 3. *Let $S = (X, \cdot)$ be a prime semigroup and let $k \geqq 2$ be an integer. Suppose that M is an asymptotic multiplicative basis of order k in the semigroup S. Then for every positive integer p there exists $x \in X$ such that $f_{M,k}(x) > p$.*

Let us prove Theorem 3. In the proof we will use the fact that every prime set is „productively independent" in the sense of the following proposition.

Proposition 1. *Let $S = (X, \cdot)$ be a commutative semigroup and let $P \subseteq X$ be a prime set. Then for every two finite sets $P_1, P_2 \subseteq P$ the following condition holds*: if $\prod P_1 \sim \prod P_2$ then $P_1 = P_2$.

Proof. Let P_1, P_2 be finite subsets of P such that $\prod P_1 \sim \prod P_2$ and $P_2 \setminus P_1 \neq \emptyset$. Choose an arbitrary element $p \in P_2 \setminus P_1$. Since $p \mid \prod P_1$ and P is a prime set, there is a set $Q \subseteq P_1$ such that $p \sim \prod Q$. Clearly $p \notin Q$ and since p is not a unit, we have that $Q \neq \emptyset$. Let q be an arbitrary element of Q. Then $q \mid p$ and therefore $q \sim p$ by the definition of the prime set. Thus $q = p$, hence $p \in Q$, a contradiction.

Proof of Theorem 3. Denote by n the number of units in S. For $x \in X$ define $[x] = \{y \in X; \, y \sim x\}$ and for $Y \subseteq X$ put $[Y] = \bigcup_{y=Y} [y]$. Let $P \subseteq X$ be an infinite prime set in the semigroup S. Define a set $M' \subseteq \mathscr{F}(P)$ by $M' = \{F \in \mathscr{F}(P); \prod F \in [M]\}$.

By Proposition 1, the mapping $F \mapsto \prod F$ from $\mathscr{F}(P)$ to X is an injection and therefore for all but finitely many sets $F \in \mathscr{F}(P)$ there exist elements $m_1, m_2, \ldots, m_k \in M$ such that $\prod F = m_1 . m_2 . \ldots . m_k$. Suppose that the equality $\prod F = m_1 . m_2 . \ldots . m_k$ holds. Then, by the definition of the prime set, there are sets F_i for $i = 1, 2, \ldots, k$ such that $F = \bigcup_{i=1}^{k} F_i$ and $m_i \sim \prod F_i$. This further implies that $F_i \in M'$ for every i, hence the infinite set P and the set M' fulfil the assumptions of Lemma 2. Thus for every p there exists a set $F \in \mathscr{F}(P)$ and at least $p . n + 1$ pairwise disjoint sets $\{F_1, F_2, \ldots, F_k\}$ such that $F = \bigcup_{i=1}^{k} F_i$, $F_i \in M'$ for $i = 1, 2, \ldots, k$ and $F_i \cap F_j = \emptyset$ for $i \neq j$. If $\{F_1, F_2, \ldots, F_k\}$ is such a set then $\prod F = \prod_{i=1}^{k} (\prod F_i)$, where $\prod F_i \in [M]$. Hence there exists a unit j and a set $\{m_1, m_2, \ldots, m_k\} \subseteq M$ such that $j . \prod F = \prod_{i=1}^{k} m_i$ and $m_i \sim \prod F_i$. This yields by Proposition 1 that there exists a unit j such that $f_{M,k}(j . \prod F) > p$.

Clearly the proof of Theorem 3 that we have just presented provides the following stronger result.

Corollary 1. *Let $S = (X, \cdot)$ be a prime semigroup, $k \geqq 2$ an integer and M an asymptotic multiplicative basis of order k. Denote by $\binom{M}{k}$ the family of all subsets of M having the size k. Then for every positive integer p there exists $x \in X$ and a family $\mathscr{M} \subseteq \binom{M}{k}$ such that the following conditions hold:*

(i) $|\mathscr{M}| > p$;
(ii) *if* $A \in \mathscr{M}$ *then* $\prod A = x$;
(iii) *if* $A, B \in \mathscr{M}$, $A \neq B$, *then* $[A] \cap [B] = \emptyset$.

Example. Let $S = (\mathbb{Q}, \cdot)$ be the semigroup of all rational numbers with the usual multiplication. Then every $q \in \mathbb{Q}$ is a unit in S and so no non-empty prime set in S exists. Thus Theorem 3 cannot be applied to S. Actually, the following holds (see [7]): if a function $f: \mathbb{Q} \to \mathbb{N}$ and an integer $k \geqq 2$ are given, then there is a set $M \subseteq \mathbb{Q}$ such that $f_{M,k} \equiv f$.

3. PRIME SETS

We have seen that the existence of large prime sets in commutative semigroups essentially influences the combinatorial properties of the semigroup multiplication. Now we show some ways how large prime sets can be constructed. We begin with some definitions.

Definition 6. Let $S = (X, \cdot)$ be a commutative semigroup. We say that $p \in X$ is a *prime* if the following conditions hold:

(i) p is not a unit,

(ii) if $p \mid x \,.\, y$ then $p \mid x$ or $p \mid y$.

Let $S = (X, \cdot)$ be a commutative semigroup. A decomposition $x = x_1 \,. \ldots .\, x_n$ of an element $x \in X$ is said to be *irreducible* if all x_i's are irreducible. Two irreducible decompositions $x = a_1 \,. \ldots .\, a_k = b_1 \,. \ldots .\, b_l$ of x are *associated* if there is a one-to-one mapping $\varphi: \{1, \ldots, k\} \to \{1, \ldots, l\}$ such that $a_i \sim b_{\varphi(i)}$ for $i = 1, \ldots, k$.

Definition 7. We say that a commutative semigroup $S = (X, \cdot)$ has the *unique factorization property* if every element $x \in X$, which is not a unit, has an irreducible decomposition, and every two irreducible decompositions of x are associated.

The following proposition follows immediately from the definition of the prime set.

Proposition 2. *Let $S = (X, \cdot)$ be a commutative semigroup with the unique factorization property. Then every set of pairwise non-associated irreducible elements of S is a prime set.*

In semigroups without the unique factorization property, prime sets can be constructed from primes as follows.

Proposition 3. *Let $S = (X, \cdot)$ be a commutative semigroup fulfilling the following condition* (U):

(U) *if $x \,.\, y \sim x$ then $y \sim 1$, where 1 is the identity element.*

Then every set of pairwise non-associated primes of S is a prime set.

In the proof of Proposition 3 we will use the following proposition.

Proposition 4. *Suppose that $S = (X, \cdot)$ is a commutative semigroup fulfilling condition* (U), *and that $p_1, p_2, \ldots, p_k$, where $k \geqq 1$, are pairwise non-associated primes. Then the following holds: if $p_i \mid x$ for $i = 1, 2, \ldots, k$ then $p_1 \,.\, p_2 \,. \ldots .\, p_k \mid x$.*

Proof. We proceed by induction on k. For $k = 1$ the proposition holds trivially. So, let us suppose that $k \geqq 2$ and that $p_i \mid x$ for $i = 1, 2, \ldots, k$. By the induction hypothesis $p_1 \,. \ldots .\, p_{k-1} \mid x$, thus $x = p_1 \,. \ldots .\, p_{k-1} \,.\, y$, $y \in X$. Since, moreover, $p_k \mid x$ and p_k is a prime, we infer that either $p_k \mid p_1 \,. \ldots .\, p_{k-1}$ or $p_k \mid y$. We show that $p_k \mid p_1 \,. \ldots .\, p_{k-1}$ leads to a contradiction.

Indeed, suppose that $p_k \mid p_1 \,. \ldots .\, p_{k-1}$. Since p_k is a prime, we obtain that $p_k \mid p_i$ for some $i = 1, \ldots, k - 1$, i.e. $p_i = p_k \,.\, u$, $v \in X$. This implies that either $p_i \mid p_k$ or $p_i \mid u$. In the first case, $p_k = p_i \cdot v$, $v \in X$, hence $p_k = p_k \,.\, u \,.\, v$. According to (U), this implies that $u \,.\, v \sim 1$. Thus $u \sim 1$ and $p_i \sim p_k$, which is a contradiction. In the latter case, $u = p_i \,.\, v$, $v \in X$, and thus $p_i = p_k \,.\, p_i \,.\, v$. This yields that $p_k \,.\, v \sim 1$ and so $p_k \sim 1$, which is a contradiction, too.

Hence, necessarily $p_k \mid y$, and consequently $p_1 \,. \ldots .\, p_{k-1} \,.\, p_k \mid x$.

Proof of Proposition 3. Let $P \subseteq X$ be a set of pairwise non-associated primes of S and let F be a finite subset of P. Suppose that $\prod F = x_1 \,.\, x_2$. Then $p \mid x_1 \,.\, x_2$ for every $p \in F$, and so either $p \mid x_1$ or $p \mid x_2$ for every $p \in F$. Let us denote $F_1 =$

$= \{p \in F;\ p \mid x_1\}$ and $F_2 = F \setminus F_1$. By Proposition 4, $\prod F_1 \mid x_1$ and $\prod F_2 \mid x_2$. Thus $x_1 = y_1 . \prod F_1$ and $x_2 = y_2 . \prod F_2$, where $y_1, y_2 \in X$, which gives $\prod F = x_1 . x_2 = y_1 . y_2 . \prod F$. By condition (U), $y_1 . y_2 \sim 1$, hence $y_1 \sim 1$ and $y_2 \sim 1$. Thus we find that $x_1 \sim \prod F_1$ and $x_2 \sim \prod F_2$.

Remark 1. It can be easily shown that in semigroups fulfilling condition (U), every prime is irreducible. Moreover, it is well known that in every semigroup having the unique factorization property, x is a prime if and only if x is irreducible.

Now we show that Proposition 3 can be applied to the cardinal multiplication in locally finite categories.

Definition 8. Let $\mathscr{K}$ be a category and let X, Y be two objects of $\mathscr{K}$. The set of all morphisms from X to Y is denoted by $\mathrm{Hom}(X, Y)$.

The category $\mathscr{K}$ is said to be *locally finite* if $\mathrm{Hom}(X, Y)$ is finite for every pair X, Y of objects of $\mathscr{K}$.

$\mathscr{K}$ is said to be *connected* if $\mathrm{Hom}(X, Y)$ is non-empty for every pair X, Y of objects.

Remark 2. A very important property of locally finite categories is the following one (see [2]): Assume that there are monomorphisms f from X to Y and g from Y to X. Then both f and g are isomorphisms.

Definition 9. Let $\mathscr{K}$ be a category with finite products, where the product of objects X and Y is denoted by $X \times Y$. For an object X of $\mathscr{K}$ denote by $[X]$ the class of all objects isomorphic to X and put $|\mathscr{K}| = \{[X];\ X$ is an object of $\mathscr{K}\}$. Then it is correct to define a binary operation $\times$ on the class $|\mathscr{K}|$ as follows:

$$[X] \times [Y] = [X \times Y] .$$

Thus we obtain a commutative semigroup $(|\mathscr{K}|, \times)$, where $|\mathscr{K}|$ may possibly be a proper class. Let us remark that often we will not distinguish objects of $\mathscr{K}$ and of $(|\mathscr{K}|, \times)$.

Proposition 5. *If $\mathscr{K}$ is a connected, locally finite category with finite products, then the semigroup $(|\mathscr{K}|, \times)$ fulfils condition* (U).

Before proving Proposition 5, we give some auxiliary statements. Let us suppose that $\mathscr{K}$ is a connected, locally finite category with finite products. The notation $X \cong Y$ means that X and Y are isomorphic objects of $\mathscr{K}$.

Lemma 3. *The following conditions concerning an object Y of the category $\mathscr{K}$ are equivalent:*

(i) *there is an object X of $\mathscr{K}$ such that $X \times Y \cong X$,*

(ii) *Y is a terminal object of $\mathscr{K}$,*

(iii) *$[Y]$ is the identity element of the semigroup $(|\mathscr{K}|, \times)$, i.e. $X \times Y \cong X$ for every object X of $\mathscr{K}$.*

Proof. (i) $\Rightarrow$ (ii). Choose X such that $X \times Y \cong X$ and let Z be an arbitrary object of $\mathscr{K}$. Then $|\mathrm{Hom}(Z, X)| = |\mathrm{Hom}(Z, X \times Y)| = |\mathrm{Hom}(Z, X)| \cdot |\mathrm{Hom}(Z, Y)|$, hence $|\mathrm{Hom}(Z, Y)| = 1$ since $|\mathrm{Hom}(Z, X)| \neq 0$ by the assumption.

(ii) $\Rightarrow$ (iii). Let Y be a terminal object and X an arbitrary object of $\mathscr{K}$. We show that the projection $\pi_X: X \times Y \to X$ is a monomorphism (since, moreover, there is a monomorphism $(1_X, f): X \to X \times Y$, we get by Remark 2 that $X \times Y \cong X$). So, suppose that $r, s: Z \to X \times Y$ are arbitrary morphisms such that $\pi_X \circ r = \pi_X \circ s$. Since Y is a terminal object, we also have that $\pi_Y \circ r = \pi_Y \circ s$. Thus $r = s$.

(iii) $\Rightarrow$ (i). Obvious.

Proof of Proposition 5. Since every connected category contains, up to isomorphism, at most one terminal object, the equivalence (ii) $\Leftrightarrow$ (iii) implies that the only unit in $(|\mathscr{K}|, \times)$ is the identity element. Thus the relation $X \cong Y$ concerning objects of category $\mathscr{K}$ is equivalent to the relation $[X] \sim [Y]$ concerning objects of the semigroup $(|\mathscr{K}|, \times)$. We conclude that condition (U) is equivalent to the implication (i) $\Rightarrow$ (iii) in Lemma 3.

We say that an element X of $\mathscr{K}$ is a prime if $[X]$ is a prime in $(|\mathscr{K}|, \times)$. Now, Propositions 3 and 5 immediately imply the following statement.

Corollary 2. *If $\mathscr{K}$ is a connected, locally finite category containing an infinite set of pairwise non-isomorphic primes, then the semigroup $(|\mathscr{K}|, \times)$ is prime.*

4. PRODUCTS OF GRAPHS

The graphs we consider are simple, i.e. undirected and without loops and multiple edges. If G is a graph, then $V(G)$ is the set of vertices and $E(G)$ the set of edges of G. Denote by $\mathscr{G}$ the class of all *finite* simple graphs. For $G \in \mathscr{G}$ denote by $[G]$ the set of all graphs isomorphic to G and put $|\mathscr{G}| = \{[G]; G \in \mathscr{G}\}$.

Now suppose that $*$ is a binary operation on $\mathscr{G}$ fulfilling the following condition: if $G \cong G'$ and $H \cong H'$, then $G * H \cong G' * H'$. Then we can define $[G] * [H] = [G * H]$ and thus obtain a grupoid $(|\mathscr{G}|, *)$. Notice that $|\mathscr{G}|$ is a proper class.

There are many natural ways how to define products of simple graphs. We shall concentrate our attention only on three well-known products, namely on the *direct*, *cartesian* and *strong products*. Let us give the definitions of these products. Let G and H be simple graphs. We denote the direct product of G and H by $G \times H$, the cartesian product by $G \square H$ and the strong product by $G \boxtimes H$. All products have the set of vertices equal to $V(G) \times V(H)$ while the sets of edges are defined as follows: $\{(x, x'), (y, y')\}$ belongs to

$$E(G \times H) \text{ iff } \{x, y\} \in E(G) \text{ and } \{x', y'\} \in E(H),$$

$$E(G \square H) \text{ iff either } x = y \text{ and } \{x', y'\} \in E(H)$$
$$\text{or } \{x, y\} \in E(G) \text{ and } x' = y',$$

$$E(G \boxtimes H) = E(G \times H) \cup E(G \square H).$$

Obviously, the operations $\times$, $\square$ and $\boxtimes$ are commutative and associative, i.e. $(|\mathscr{G}|, \times)$, $(|\mathscr{G}|, \square)$ and $(|\mathscr{G}|, \boxtimes)$ are commutative semigroups. Algebraic properties of these semigroups have been studied by many authors. In particular, in [8], G. Sabidussi proved that the semigroup $(|\mathscr{G}|, \square)$ has the unique factorization property, and in paper [3] R. McKenzie proved the same result for the semigroup $(|\mathscr{G}|, \boxtimes)$. These results have, by virtue of Proposition 2, the following corollary.

Proposition 5. *The semigroups* $(|\mathscr{G}|, \square)$ *and* $(|\mathscr{G}|, \boxtimes)$ *are prime.*

We are going to show that the semigroup $(|\mathscr{G}|, \times)$ is prime, too. In this case, irreducible elements cannot be used for the construction of an infinite prime set, because the semigroup $(|\mathscr{G}|, \times)$ does not possess the unique factorization property. Namely, it is well known that in commutative semigroups with the unique factorization property, every irreducible element is prime. But D. J. Miller proved in [4] that the emigroup $(|\mathscr{G}|, \times)$ contains no prime except the trivial graph with one vertex. So, Proposition 3 also cannot be used to construct an infinite prime set. In spite of this, the semigroup $(|\mathscr{G}|. \times)$ is prime and an infinite prime set can be constructed for example as follows.

As usual, let $K_{n,m}$ denote the complete bipartite graph. Put $S_n = K_{1,n}$ and call this graph a *star*.

Proposition 6. *The set of all stars* S_p, *where* p *is a prime number, is a prime set in the semigroup* $(|\mathscr{G}|, \times)$.

Proof. By $\prod_{i\in I} G_i$ and $\sum_{i\in I} G_i$, respectively, we shall denote the direct product and the disjoint sum of the collection $\{G_i: i \in I\}$ of graphs. The disjoint sum of two graphs G and H is denoted by $G + H$. The obvious equality $K_{n,m} \times K_{r,s} \cong K_{nr,ms} + K_{ns,mr}$ will be used.

Let $S_{p_1} \times \ldots \times S_{p_k} \cong H_1 \times H_2$, where $2 \leqq p_1 < \ldots < p_k$ are prime numbers. The graph $S_{p_1} \times \ldots \times S_{p_k}$ will be denoted by G. It can be easily shown that

$$G \cong \sum_{\{Q,R\}} K_{\Pi Q, \Pi R}\,,$$

where $\{Q, R\}$ runs through the set of all disjoint partitions of the set $\{p_1, \ldots, p_k\}$. It follows from this that components of H_1 and H_2 are complete bipartite graphs. Indeed, let us suppose that one of the graphs H_1 and H_2 is non-bipartite. Then it contains a circuit C_ϱ of odd length $\varrho \geqq 3$ as a full subgraph, and so $H_1 \times H_2$ contains a circuit $C_\varrho \times K_2 \cong C_{2\varrho}$ of length $2\varrho \geqq 6$ as a full subgraph. But G does not contain any circuit of length greater then 4 as a full subgraph, a contradiction. Thus both H_1 and H_2 are bipartite graphs. Further, since every product of connected non-complete bipartite graphs has non-complete components, the components of H_1 and H_2 are complete. Moreover, one of the components of G is isomorphic to the star $S_{p_1 \ldots . p_k}$ hence there are components of graphs H_1 and H_2 isomorphic respectively to S_m and S_n, where $m \,.\, n = \prod_{i=1}^{k} p_i$. Let $\{Q, R\}$ be the partition of the set $\{p_1, \ldots, p_k\}$

such that $m = \prod Q$ and $n = \prod R$. Denote by q and r the sizes of Q and R, respectively. We shall show that $H_1 \cong \prod_{q \in Q} S_q$ and $H_2 \cong \prod_{r \in R} S_r$.

So, let $K_{u,v}$ be an arbitrary component of H_1. Since $K_{u,v} \times S_n \cong K_{u,nv} + K_{v,nu}$, we find that $K_{u,nv}$ and $K_{v,nu}$ are components in G. Consequently, $n \,.\, u \,.\, v = \prod_{i=1}^{k} p_i = n \,.\, m$ and so $u \,.\, v = m$. Hence $K_{u,v} \cong K_{\Pi Q_1, \Pi Q_2}$ where $\{Q_1, Q_2\}$ is a partition of Q. Moreover, the components of the graph H_1 are pairwise non-isomorphic because the graph G has this property. Analogous statements about the components of H_2 can be deduced.

To complete the proof it suffices to show that H_1 (and H_2) contain as a component the graph $K_{\Pi Q_1, \Pi Q_2}$ for every partition $\{Q_1, Q_2\}$ of Q (and R, respectively). Let us suppose the contrary. Then H_1 or H_2, respectively, has less then 2^{q-1} or 2^{r-1} (= the number of partitions of Q or R) components, and so $H_1 \times H_2$ has less then $2 \,.\, 2^{q-1} \,.\, 2^{r-1} = 2^{q+r-1}$ components. On the other hand, it can be easily seen that $S_{p_1} \times \ldots \times S_{p_k}$ has exactly 2^{k-1} components, thus $k < q + r = k$, a contradiction.

Corollary 3. *Let $k \geqq 2$ be a positive integer and let $*$ be some of the operations $\times$, $\square$ and $\boxtimes$. Suppose that $\mathcal{M} \subseteq \mathcal{G}$ is a set of finite graphs such that all but finitely many $G \in \mathcal{G}$ can be expressed in the form $G \cong G_1 * \ldots * G_k$ where $G_i \in \mathcal{M}$ for $i = 1, \ldots, k$. Then for every positive integer p there is a graph G which can be expressed as a product of k, not necessarily distinct, graphs of $\mathcal{M}$ in at least p different ways.*

Proof. Immediate corollary of Theorem 3, Lemma 3, Proposition 5 and Proposition 6.

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